Electromagnetic energy and linear momentum radiated by two point charges

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We consider the linear four-momentum radiated by a system of two point charges during two proper-time infinitesimal intervals. As a simple generalization of Schild's result for a single charge, we show that the radiation rate is independent of the hypersurface which is used to integrate Maxwell's tensor and we give an exact covariant expression for it. Using the preceding result we give, in the framework of predictive relativistic mechanics, a definition of radiated linear four-momentum for a system of two interacting charges. We calculate this quantity at the lowest approximation in perturbation theory and we compare our result to the familiar one obtained in the framework of the slow-motion approximation.

I. INTRODUCTION

We consider the electromagnetic energy and momentum radiated by a system of moving point charges. To avoid complications in the notation, we shall restrict ourselves to the case of two charges. Nevertheless, the generalization of this work to N charges is an obvious one.

The definition of radiated energy and momentum we use is a trivial generalization of the usual one for a single charge: the four-momentum radiated by the two charges during the infinitesimal intervals of proper times $[\tau_a,\tau_a+d\tau_a]$, $a=1,2$, is the contribution to the total electromagnetic four-momentum which is present at future infinity due to the fields created by the charges in the considered intervals. The consistency of this definition is guaranteed by the fact that the aforementioned contribution is independent of the hypersurface which is used in the calculation of the electromagnetic four-momentum at future infinity. Therefore, this is a well-defined four-vector. The proof of this statement is almost identical to the one given by Schild[1] for one charge; it can be found in Sec. II.

This independence, moreover, allows us to choose, in Sec. III, a particular integration surface which makes relatively easy the calculation of the exact covariant four-momentum radiated by the charges during two infinitesimal proper-time intervals. This "radiation rate" depends only on accelerations, velocities, and the relative four-position of the charges in the considered configuration.

But, if we wish to calculate the four-momentum radiated during a finite interval of coordinate time or even simply—and this makes a big difference with respect to the case of a single charge—the momentum radiated per unit of coordinate time in a given spacelike configuration, we must know something more about the world lines, i.e., about the dynamics of the motion of the charges.

In Sec. IV, we shall analyze this problem in the framework of predictive relativistic mechanics[2] and show how, in principle, it can be calculated at any order in perturbation theory. At the same time we obtain a new definition of total four-momentum, as a function of the positions and four-velocities of the charges, as being the free four-momentum at past infinity minus the radiated four-momentum along the world lines coming from past infinity to the configuration which is being considered. This definition differs from that given in Ref. 3 and it is, we hope, free from the pathologies which beset the latter at the approximations which take into account effective electromagnetic radiation.

Finally, we perform some calculations at the first nonzero approximation of perturbation theory using the product of the charges as an expansion parameter. We check, in particular, that an additional expansion in powers of c^{-1} leads at its lowest order to the well-known expression for the dipole radiation of two charges that move according to Coulomb's law.

II. RADIATION RATE OF LINEAR FOUR-MOMENTUM

Let us consider two pointlike particles with charges e_a, $a=1,2$, and let their timelike world lines L_a be given in Minkowski space-time by the equations

$$x^\alpha=\phi_a^\alpha(\tau_a)\ , \tag{2.1}$$

where τ_a is the proper time of particle a:

$$\dot\phi_a^\alpha(\tau_a)\dot\phi_{a\alpha}(\tau_a)=-1\ . \tag{2.2}$$

We take the speed of light $c=1$ and the metric $\eta^{\alpha\beta}=\mathrm{diag}(-1,1,1,1)$. A dot indicates derivatives with respect to the proper time.

For an inertial observer with four-velocity n^α, the electromagnetic linear four-momentum corresponding to the value λ of its own proper time is

$$P^\alpha(n,\lambda)=-\int_{\Sigma(\lambda)}T^{\alpha\beta}d^3\sigma_\beta\ , \tag{2.3}$$

where $T^{\alpha\beta}$ is the electromagnetic Maxwell tensor, $\Sigma(\lambda)$ denotes the spacelike hyperplane of the equation

$$x^\alpha n_\alpha=-\lambda\ , \tag{2.4}$$

and $d^3\sigma^\alpha\equiv d^3\sigma\, n^\alpha$ is the volume element of $\Sigma(\lambda)$.

Since we suppose that the total electromagnetic field is the sum $\sum_{a=1}^{2} F_a^{\alpha\beta}$ of the retarded fields created by the two particles, the momentum (2.3) can be decomposed in the following form:

$$P^\alpha(n,\lambda)=\sum_{a=1}^{2} P_a^\alpha(n,\lambda)+P_{12}^\alpha(n,\lambda) . \tag{2.5}$$

Here the contribution due exclusively to the field of particle a is

$$P_a^\alpha(n,\lambda)\equiv-\int_{-\infty}^{\tau_a(n,\lambda)} d\tau_a \int_{\Sigma_a(\lambda,\tau_a)} T_a^{\alpha\beta} d^2\sigma_{a\beta} \tag{2.6}$$

with

$$T_a^{\alpha\beta}\equiv-\frac{1}{4\pi}(F_a^{\alpha\gamma}F_{a\gamma}{}^{\beta}+\tfrac{1}{4}\eta^{\alpha\beta}F_a^{\gamma\delta}F_{a\gamma\delta}) , \tag{2.7}$$

$$d^3\sigma^\alpha\equiv d\tau_a d^2\sigma_a^\alpha , \tag{2.8}$$

$\tau_a(n,\lambda)$ being the value of τ_a at the intersection of the world line L_a and the hyperplane $\Sigma(\lambda)$. $\Sigma_a(\lambda,\tau_a)$ is the intersection of $\Sigma(\lambda)$ with the future light cone $C_a^+(\tau_a)$ with vertex at the point $\phi_a^\alpha(\tau_a)$.

Similarly, the joint contribution due to both fields is

$$P_{12}^\alpha(n,\lambda)\equiv-\int_{-\infty}^{\tau_1(n,\lambda)} d\tau_1 \int_{-\infty}^{\tau_2(n,\lambda)} d\tau_2 \times \int_{\Sigma_{12}(\lambda,\tau_1,\tau_2)} T_{12}^{\alpha\beta} d\sigma_{12\beta} \tag{2.9}$$

with

$$T_{12}^{\alpha\beta}\equiv-\frac{1}{4\pi}(F_1^{\alpha\gamma}F_{2\gamma}{}^{\beta}+F_2^{\alpha\gamma}F_{1\gamma}{}^{\beta}+\tfrac{1}{2}\eta^{\alpha\beta}F_1^{\gamma\delta}F_{2\gamma\delta}) , \tag{2.10}$$

$$d^2\sigma_a^\alpha\equiv d\tau_a d\sigma_{12}^\alpha , \tag{2.11}$$

where $\Sigma_{12}(\lambda,\tau_1,\tau_2)$ is the intersection of $\Sigma_1(\lambda,\tau_1)$ with $\Sigma_2(\lambda,\tau_2)$.

The last integral in Eq. (2.6),

$$\frac{\partial P_a^\alpha}{\partial\tau_a}(n,\lambda,\tau_a)\equiv-\int_{\Sigma_a(\lambda,\tau_a)} T_a^{\alpha\beta}d^2\sigma_{a\beta} , \tag{2.12}$$

depends on n^α and λ, but as Schild[1] has proved the limit

$$\frac{dP_{ar}^\alpha}{d\tau_a}(\tau_a)\equiv\lim_{\lambda\to+\infty}\frac{\partial P_a^\alpha}{\partial\tau_a}(n,\lambda,\tau_a) \tag{2.13}$$

is in fact independent of n^α and can be understood in the single-particle case as the rate of linear momentum radiated by the particle which is being considered. Its value is the well-known relativistic generalization of the Larmor formula:

$$\frac{dP_{ar}^\alpha}{d\tau_a}(\tau_a)=\tfrac{2}{3}e_a{}^2\ddot\phi_a^\beta(\tau_a)\ddot\phi_{a\beta}(\tau_a)\dot\phi_a^\alpha(\tau_a) . \tag{2.14}$$

We shall prove that the last integral in Eq. (2.9),

$$\frac{\partial^2 P_{12}^\alpha}{\partial\tau_1\partial\tau_2}(n,\lambda,\tau_1,\tau_2)\equiv-\int_{\Sigma_{12}(\lambda,\tau_1,\tau_2)} T_{12}^{\alpha\beta}d\sigma_{12\beta} , \tag{2.15}$$

has a limit

$$\frac{\partial^2 P_{12r}^\alpha}{\partial\tau_1\partial\tau_2}(\tau_1,\tau_2)\equiv\lim_{\lambda\to+\infty}\frac{\partial^2 P_{12}^\alpha}{\partial\tau_1\partial\tau_2}(n,\lambda,\tau_1,\tau_2) , \tag{2.16}$$

which is also independent of the timelike unit four-vector n^α. Hence, we can interpret the quantity in Eq. (2.16) as the joint contribution—corresponding to the mixed terms of Maxwell's tensor—to the radiated linear momentum and energy of the retarded fields generated by the two charges per unit of proper times. Clearly, this contribution is zero if in the corresponding configuration the four-vector $\phi_1^\alpha(\tau_1)-\phi_2^\alpha(\tau_2)$ is timelike. The radiated linear four-momentum corresponding to the fields created by the charges up to proper times τ_1 and τ_2 is therefore

$$P_r^\alpha(\tau_1,\tau_2)=\sum_{a=1}^{2}\int_{-\infty}^{\tau_a} d\tau\frac{dP_{ar}^\alpha}{d\tau_a}(\tau) +\int_{-\infty}^{\tau_1} d\tau\int_{-\infty}^{\tau_2} d\tau'\frac{\partial^2 P_{12r}^\alpha}{\partial\tau_1\partial\tau_2}(\tau,\tau') . \tag{2.17}$$

Since the electromagnetic energy-momentum tensor is quadratic in the field and this field satisfies the superposition principle, in the case of N particles we would merely obtain as an obvious generalization of Eq. (2.18) the sum of N-particle contributions and the mixed contributions corresponding to the $N(N-1)/2$ pairs of charges.

The proof of the independence of the quantity (2.16) with respect to the four-vector n^α is very similar to the one given by Schild[1] for the single-particle case. The retarded electromagnetic field of particle a at the point x^α of space-time is

$$F_a^{\alpha\beta}=e_a r_a{}^{-1}\{[r_a{}^{-1}+(k_a\xi_a)](u_a^\alpha k_a^\beta-u_a^\beta k_a^\alpha) +(\xi_a^\alpha k_a^\beta-\xi_a^\beta k_a^\alpha)\} , \tag{2.18}$$

where

$$r_a=-(l_a u_a),\quad l_a^\alpha=x^\alpha-\phi_a^\alpha(\tau_a) ,$$
$$u_a^\alpha=\dot\phi_a^\alpha(\tau_a),\quad \xi_a^\alpha=\ddot\phi_a^\alpha(\tau_a) , \tag{2.19}$$
$$k_a^\alpha=r_a{}^{-1}l_a^\alpha .$$

Parentheses denote the scalar product of two four-vectors and τ_a is the proper time at the intersection of world line L_a with the past light cone of event x^α:

$$l_a^\alpha l_{a\alpha}=0,\quad l_a^0>0 . \tag{2.20}$$

The future-oriented null four-vector k_a^α is normalized by the relation

$$(k_a u_a)=-1 . \tag{2.21}$$

The volume element of the light cone $C_a^+(\tau_a)$ is

$$d^3c_a^\alpha=r_a{}^2 k_a^\alpha d^2\Omega_a dr_a , \tag{2.22}$$

where $d^2\Omega_a$ is the element of solid angle in the instantaneous rest frame of particle a for the value τ_a of its proper time. A straightforward calculation which makes use of Eqs. (2.10), (2.18), and (2.22) shows that

$$T_{12}^{\alpha\beta}dc_{a\beta}=\frac{e_1e_2}{4\pi r_{a'}}\{[r_{a'}{}^{-1}+(k_{a'}\xi_{a'})][(k_ak_{a'})u_{a'}^\alpha-(k_au_{a'})k_{a'}^\alpha] +(k_ak_{a'})\xi_{a'}^\alpha-(k_a\xi_{a'})k_{a'}^\alpha\}d^2\Omega_a dr_a \quad (a'\neq a) . \tag{2.23}$$

Let us consider the four-volume $V(\lambda)$ bounded by $C_1^+(\tau_1)$, $C_1^+(\tau_1+d\tau_1)$, $C_2^+(\tau_2)$, $C_2^+(\tau_2+d\tau_2)$, $\Sigma(\lambda)$, and $\Sigma'(\lambda)$, the latter being the associated families of orthogonal hyperplanes corresponding to an arbitrary pair of timelike unit four-vectors n^α and n'^α. Since $T_{12}^{\alpha\beta}$ is divergenceless off the world lines of the charges, if we apply the Gauss theorem to this tensor in the four-volume $V(\lambda)$ and take the limit $\lambda\to+\infty$, we obtain

$$\lim_{\lambda\to+\infty}\left[\frac{\partial^2 P_{12}^\alpha}{\partial\tau_1\partial\tau_2}(n,\lambda,\tau_1,\tau_2)-\frac{\partial^2 P_{12}^\alpha}{\partial\tau_1\partial\tau_2}(n',\lambda,\tau_1,\tau_2)\right]=0\ , \tag{2.24}$$

because on those domains of the light cones that bound $V(\lambda)$ the quantity (2.23) falls off as $r_{a'}{}^{-1}\to 0$ when $\lambda\to+\infty$. In fact, the same argument holds using the appropriate limit at future infinity if we take in Eq. (2.15) any pair of hypersurfaces—spacelike or not—provided that they cut all the light rays generating the four light cones which we have considered.

III. MIXED CONTRIBUTION TO RADIATED FOUR-MOMENTUM

We now proceed to calculate the quantity (2.16) for spacelike configurations. As stated in Sec. II, this quantity is zero for timelike configurations and the result we obtain is also valid for the limit case corresponding to a null configuration.

Let us take two values, τ_1 and τ_2, of proper times. They are arbitrary with the restriction that (i) the relative position four-vector

$$x_{12}^\alpha\equiv\phi_1^\alpha(\tau_1)-\phi_2^\alpha(\tau_2)=l_2^\alpha-l_1^\alpha\ , \tag{3.1}$$

where l_a^α were defined in Eqs. (2.19), is spacelike,

$$x_{12}{}^2\equiv x_{12}^\alpha x_{12\alpha}>0\ , \tag{3.2}$$

and (ii) the velocities of the two particles are not equal,

$$(u_1u_2)\neq-1\ . \tag{3.3}$$

At the end though the final results will be valid also for parallel four-velocities.

Since the quantity (2.16) is independent of n^α, we take for simplicity

$$n^\alpha=u_1^\alpha\ . \tag{3.4}$$

Let us define

$$i^\alpha=-r_{21}{}^{-1}\epsilon^{\alpha\beta\gamma\delta}u_{1\beta}j_\gamma x_{12\delta}\ , \tag{3.5}$$

$$j^\alpha=-A^{-1}\epsilon^{\alpha\beta\gamma\delta}x_{12\beta}u_{1\gamma}u_{2\delta}\ , \tag{3.6}$$

$$m^\alpha=r_{21}{}^{-1}[x_{12}^\alpha+(x_{12}u_1)u_1^\alpha]\ , \tag{3.7}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the completely skew-symmetric Levi-Civita tensor with $\epsilon_{0123}=1$ and

$$r_{aa'}{}^2=x_{12}{}^2+(x_{12}u_{a'})^2\ , \tag{3.8}$$

$$A^2=x_{12}{}^2[(u_1u_2)^2-1]-[(x_{12}u_1)^2+2(u_1u_2)(x_{12}u_1)(x_{12}u_2)+(x_{12}u_2)^2]\ . \tag{3.9}$$

As is easily seen, in the orthonormal system $(u_1^\alpha,i^\beta,j^\gamma,m^\delta)$ we have the following decomposition:

$$l_a^\alpha=-(u_1l_a)u_1^\alpha+\rho r_{21}{}^{-1}(\cos\phi i^\alpha+\sin\phi j^\alpha)+r_{21}{}^{-1}[(x_{12}u_1)(u_1l_a)-\tfrac{1}{2}\eta_a x_{12}{}^2]m^\alpha\ , \tag{3.10}$$

where $\eta_a=(-1)^{a+1}$,

$$\rho^2=x_{12}{}^2[(l_1u_1)(l_2u_1)-\tfrac{1}{4}x_{12}{}^2]\ , \tag{3.11}$$

and where $\phi\in(-\pi,\pi]$ is the angle defined by

$$\rho\cos\phi=r_{21}(il_1)=r_{21}(il_2)\ , \tag{3.12}$$

$$\rho\sin\phi=r_{21}(jl_1)=r_{21}(jl_2)\ . \tag{3.13}$$

The geometrical meaning of the quantities just introduced is more easily seen in the rest frame of an inertial observer that moves with velocity u_1^α. In such a frame we have, for instance,

$$\cos\phi=-\frac{\vec{j}\cdot(\vec{l}_1\times\vec{l}_2)}{|\vec{l}_1\times\vec{l}_2|} \tag{3.14}$$

in three-dimensional notation.

As we prove in the Appendix, the volume element of any hyperplane orthogonal to u_1^α, at a point that lies in the intersection of light cones $C_1^+(\tau_1)$ and $C_2^+(\tau_2)$, is

$$d^3\sigma^\alpha=r_1r_2r_{21}{}^{-1}u_1^\alpha d\tau_1 d\tau_2 d\phi\ , \tag{3.15}$$

and thus, at such a point,

$$d\sigma_{12}^\alpha=r_1r_2r_{21}{}^{-1}u_1^\alpha d\phi\ . \tag{3.16}$$

From Eqs. (3.10) and (3.11) we find that

$$\lim_{r_1\to\infty} r_2r_1{}^{-1}=\alpha+\beta\cos\phi\ , \tag{3.17}$$

$$\lim_{r_1\to\infty} k_1^\alpha=x_{12}r_{21}{}^{-1}(\cos\phi i^\alpha+\sin\phi j^\alpha)+s^\alpha\ , \tag{3.18}$$

$$\lim_{r_1\to\infty} k_2^\alpha=(\alpha+\beta\cos\phi)^{-1}\lim_{r_1\to\infty} k_1^\alpha\ , \tag{3.19}$$

where

$$\alpha=-(su_2)=r_{21}{}^{-2}[(x_{12}u_1)(x_{12}u_2)-(u_1u_2)x_{12}{}^2]\ , \tag{3.20}$$

$$\beta=-x_{12}r_{21}{}^{-1}(iu_2)=Ax_{12}r_{21}{}^{-2}\ , \tag{3.21}$$

$$s^\alpha=u_1^\alpha-(x_{12}u_1)r_{21}{}^{-1}m^\alpha=-r_{21}{}^{-2}[(x_{12}u_1)x_{12}^\alpha-x_{12}{}^2u_1^\alpha]\ , \tag{3.22}$$

and we have $\alpha>\beta>0$. Note that in the limit the two null vectors k_1^α and k_2^α are parallel. This fact yields, after a direct calculation where use is made of Eqs. (2.10), (2.16), (2.18), and (3.16),

$$\frac{\partial^2 P_{12r}^\alpha}{\partial\tau_1\partial\tau_2}(\tau_1,\tau_2)=\int_{-\pi}^{\pi}d\phi\, p^\alpha(\phi,\tau_1,\tau_2) \tag{3.23}$$

with

$$p^{\alpha}(\phi,\tau_1,\tau_2)=\frac{e_1e_2}{2\pi}r_{21}{}^{-1}(\alpha+\beta\cos\phi)^{-1}\lim_{r_1\to\infty}\{[(u_1u_2)(k_1\xi_1)(k_2\xi_2)+(u_1\xi_2)(k_1\xi_1)+(u_2\xi_1)(k_2\xi_2)+(\xi_1\xi_2)]k_1^{\alpha}\}\ . \tag{3.24}$$

The quantity p^{α}—i.e., the "mixed" radiated linear momentum per unit of τ_1, τ_2, and ϕ—is a null four-vector whose direction lies in the asymptotic intersection of $C_1^+(\tau_1)$ and $C_2^+(\tau_2)$. Similar to what happens in the case of Larmor's formula, the only nonzero contribution to p^{α} comes from the radiation fields of the charges. Consequently, p^{α} vanishes if one of the charges moves at constant speed, as can also be seen from Eq. (3.24).

Using Eqs. (2.21), (3.23), and (3.24), we obtain

$$\frac{\partial^2P_{12r}^{\alpha}}{\partial\tau_1\partial\tau_2}=\frac{e_1e_2}{r_{21}}I^{\alpha}{}_{\beta\gamma}[(u_1u_2)\xi_1^{\beta}\xi_2^{\gamma}-(u_1\xi_2)\xi_1^{\beta}u_2^{\gamma}-(u_2\xi_1)u_1^{\beta}\xi_2^{\gamma}+(\xi_1\xi_2)u_1^{\beta}u_2^{\gamma}] \tag{3.25}$$

with

$$I^{\alpha\beta\gamma}=\lim_{r_1\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\frac{k_1^{\alpha}k_1^{\beta}k_1^{\gamma}}{(\alpha+\beta\cos\phi)^2}\ , \tag{3.26}$$

and from Eq. (3.18) we get

$$I^{\alpha\beta\gamma}=x_{12}{}^3r_{21}{}^{-3}I_1i^{\alpha}i^{\beta}i^{\gamma}+I_2s^{\alpha}s^{\beta}s^{\gamma}+S[x_{12}{}^2r_{21}{}^{-2}I_3j^{\alpha}j^{\beta}s^{\gamma}+x_{12}r_{21}{}^{-1}I_4i^{\alpha}s^{\beta}s^{\gamma}$$
$$+x_{12}{}^3r_{21}{}^{-3}(I_4-I_1)i^{\alpha}j^{\beta}j^{\gamma}+x_{12}{}^2r_{21}{}^{-2}(I_2-I_3)i^{\alpha}i^{\beta}s^{\gamma}]\ , \tag{3.27}$$

where S means cyclic summation over the indices α, β, γ, and where

$$I_1=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\frac{\cos^3\phi}{(\alpha+\beta\cos\phi)^2}=\frac{2\alpha}{\beta^3}\left[\frac{\alpha}{(\alpha^2-\beta^2)^{1/2}}-1\right]-\frac{\alpha^2}{\beta(\alpha^2-\beta^2)^{3/2}}\ , \tag{3.28}$$

$$I_2=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\frac{1}{(\alpha+\beta\cos\phi)^2}=\frac{\alpha}{(\alpha^2-\beta^2)^{3/2}}\ , \tag{3.29}$$

$$I_3=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\frac{\sin^2\phi}{(\alpha+\beta\cos\phi)^2}=\frac{1}{\beta^2}\left[\frac{\alpha}{(\alpha^2-\beta^2)^{1/2}}-1\right]\ , \tag{3.30}$$

$$I_4=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\frac{\cos\phi}{(\alpha+\beta\cos\phi)^2}=-\frac{\beta}{(\alpha^2-\beta^2)^{3/2}}\ . \tag{3.31}$$

Of course, though j^{α} is a pseudovector, $I^{\alpha\beta\gamma}$ is a tensor. Also, the final expression for Eq. (3.25) obtained using Eqs. (3.5)—(3.7), (3.9), and (3.20)—(3.22) is symmetric with respect to the exchange of the particle indices $a=1,2$.

We do not give this final expression because we do not need it in the following. In fact, from now on we assume that the whole motion of the charges lies in a timelike three-plane of Minkowski space-time. The reasons for this assumption will be briefly indicated in Sec. IV. In this case we can use the following decomposition,

$$\xi_a^{\alpha}=\eta_a a_a x_{12}^{\alpha}+b_{aa}u_a^{\alpha}+b_{aa'}u_{a'}^{\alpha},\quad \eta_a=(-1)^{a+1} \tag{3.32}$$

and $\partial^2P_{12r}/\partial\tau_1\partial\tau_2$ is then more easily obtained directly from Eqs. (3.23) and (3.24) by making use of Eqs. (3.29), (3.31),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\frac{1}{\alpha+\beta\cos\phi}=\frac{1}{(\alpha^2-\beta^2)^{1/2}}\ , \tag{3.33}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\frac{\cos\phi}{\alpha+\beta\cos\phi}=\frac{1}{\beta}-\frac{\alpha}{\beta(\alpha^2-\beta^2)^{1/2}}\ . \tag{3.34}$$

If we use the notation

$$\begin{aligned}&k=-(u_1u_2)\ ,\\&\Lambda^2=k^2-1\ ,\\&z_a=\eta_a\Lambda^{-2}[(x_{12}u_a)-k(x_{12}u_{a'})]\ ,\\&t_a^{\alpha}=u_a^{\alpha}-ku_{a'}^{\alpha}\ ,\\&h^{\alpha}=x_{12}^{\alpha}-z_1u_1^{\alpha}+z_2u_2^{\alpha}\ ,\\&l_{aa'}=b_{aa'}-z_{a'}a_a\end{aligned} \tag{3.35}$$

and the relations

$$\begin{aligned}&\xi_a^{\alpha}=\eta_a a_a h^{\alpha}+l_{aa'}t_{a'}^{\alpha}\ ,\\&A=\Lambda h\ ,\\&r_{aa'}=(h^2+\Lambda^2z_a{}^2)^{1/2}\ ,\\&x_{12}{}^2=h^2-z_1{}^2+2kz_1z_2-z_2{}^2\ ,\\&i^{\alpha}=-\Lambda^{-1}h^{-1}r_{21}{}^{-1}(\Lambda^2z_2h^{\alpha}+h^2t_2^{\alpha})\ ,\\&h^2\equiv h^{\alpha}h_{\alpha}\ ,\\&\alpha=r_{21}{}^{-2}(kh^2+\Lambda^2z_1z_2)\ ,\\&\alpha^2-\beta^2=r_{12}{}^2r_{21}{}^{-2}\ ,\\&s^{\alpha}=r_{21}{}^{-2}(z_1-kz_2)h^{\alpha}-\Lambda^{-2}t_1^{\alpha}\\&\qquad-\Lambda^{-2}r_{21}{}^{-2}(kh^2+\Lambda^2z_1z_2)t_2^{\alpha}\ ,\end{aligned} \tag{3.36}$$

we finally get, after a straightforward calculation,

$$\frac{\partial^2P_{12r}^{\alpha}}{\partial\tau_1\partial\tau_2}=e_1e_2\sum_{a=1}^{2}(A_av_a^{\alpha}-Bw_a^{\alpha})\ , \tag{3.37}$$

where

$$A_a=(kz_aa_a+l_{aa'})(z_aa_{a'}+l_{a'a})\ , \tag{3.38}$$

$$v_a^{\alpha}=\eta_ar_{aa'}{}^{-3}(kz_a-z_{a'})h^{\alpha}-\Lambda^{-2}r_{aa'}{}^{-3}(kh^2+\Lambda^2z_az_{a'})t_a^{\alpha}-\Lambda^{-2}r_{aa'}{}^{-1}t_{a'}^{\alpha}\ , \tag{3.39}$$

$$B=(h^2+2kz_1z_2)a_1a_2+2kl_{12}l_{21}+(z_1+kz_2)a_1l_{21}+(z_2+kz_1)a_2l_{12}\ , \tag{3.40}$$

$$w_a^\alpha=\eta_a h^{-2}r_{aa'}{}^{-1}z_a h^\alpha-\Lambda^{-2}r_{aa'}{}^{-1}t_a^\alpha\ . \tag{3.41}$$

It should be noted that in Eqs. (3.37)—(3.41) the symmetry with respect to the particle indices appears explicitly.

IV. RADIATED LINEAR MOMENTUM IN COPHASE SPACE

Hitherto we have assumed that the world lines of the particles were known but, in fact, to calculate $\partial^2P_{12r}/\partial\tau_1\partial\tau_2$ for fixed values of τ_1 and τ_2 we only need to know the positions, the velocities, and the accelerations of the charges corresponding to these proper times. On the other hand, to find the linear momentum radiated per unit of coordinate time we must know—unlike the case of a single charge—something more about the world lines of the charges. For instance, if the values of τ_1 and τ_2 are such that x_{12}^α is spacelike, we get from Eq. (2.17)

$$\frac{\partial P_r^\alpha}{\partial\tau_a}(\tau_1,\tau_2)=\frac{dP_{ar}^\alpha}{d\tau_a}(\tau_a)+\int_{\hat\tau_{aa'}}^{\tau_{a'}}d\tau\left.\frac{\partial^2P_{12r}^\alpha}{\partial\tau_1\partial\tau_2}\right|_{\tau_{a'}=\tau}\ , \tag{4.1}$$

where $\hat\tau_{aa'}$ is the value of proper time $\tau_{a'}$ at the intersection of world line $L_{a'}$ and the past light cone with vertex at the point $\phi_a^\alpha(\tau_a)$. To handle this problem we shall use the hypotheses and results of predictive relativistic mechanics.[2]

Let us assume that the world lines of the charges we are considering are the solution, for given initial conditions, of an invariant predictive system, i.e., of an ordinary differential system of type

$$\frac{dx_a^\alpha}{d\tau}=u_a^\alpha,\quad \frac{du_a^\alpha}{d\tau}=\xi_a^\alpha(x_b^\beta,u_c^\gamma)\ , \tag{4.2}$$

where the functions $\xi_a^\alpha(x_b^\beta,u_c^\gamma)$ are Poincaré four-vectors which satisfy the orthogonality condition

$$u_{a\alpha}\xi_a^\alpha(x_b^\beta,u_c^\gamma)=0 \tag{4.3}$$

and the Droz-Vincent equations[4]

$$u_{a'}^\beta\frac{\partial\xi_a^\alpha}{\partial x_{a'}^\beta}+\xi_{a'}^\beta\frac{\partial\xi_a^\alpha}{\partial u_{a'}^\beta}=0\ . \tag{4.4}$$

If we assume also the invariance of ξ_a^α under space reflections, the following decomposition holds:

$$\xi_a^\alpha=\eta_a a_a h^\alpha+l_{aa'}t_{a'}^\alpha\ . \tag{4.5}$$

We are using, of course, for obvious reasons the same notation ξ_a^α that we used in the preceding paragraph to represent here the functions of (x_b^β,u_c^γ) which define the dynamical system (4.2).

According to Eq. (4.1), we define the radiated linear momentum of the dynamical system (4.2) as the function $P_r^\alpha(x_b^\beta,u_c^\gamma)$ defined in cophase space[5] by the evolution equation

$$\mathscr{L}(\vec{\mathrm{H}}_a)P_r^\alpha(x_b^\beta,u_c^\gamma)=I_a^\alpha(x_b^\beta,u_c^\gamma)\equiv G_a^\alpha(x_b^\beta,u_c^\gamma)+\int_{\hat\tau_{aa'}}^0 d\tau\,\phi_{a'\tau}^* G_{12}^\alpha(x_b^\beta,u_c^\gamma) \tag{4.6}$$

for spacelike configurations (for timelike or null configurations we simply drop the integral), and the asymptotic condition

$$\lim_{\tau\to-\infty} R_1(\tau)R_2(\tau)P_r^\alpha(x_b^\beta,u_c^\gamma)=0\ . \tag{4.7}$$

In Eq. (4.6), $\mathscr{L}(\vec{\mathrm{H}}_a)$ denotes the Lie derivative with respect to the vector field,

$$\vec{\mathrm{H}}_a=u_a^\alpha\frac{\partial}{\partial x_a^\alpha}+\xi_a^\alpha(x_b^\beta,u_c^\gamma)\frac{\partial}{\partial u_a^\alpha}\ , \tag{4.8}$$

and

$$G_a^\alpha(x_b^\beta,u_c^\gamma)\equiv\tfrac{2}{3}e_a{}^2\xi_a{}^2(x_b^\beta,u_c^\gamma)u_a^\alpha\ , \tag{4.9}$$

$$G_{12}^\alpha\equiv e_1e_2\sum_{a=1}^2(A_av_a^\alpha-Bw_a^\alpha)\ , \tag{4.10}$$

where A_a, v_a^α, B, and w_a^α are defined in Eqs. (3.38)—(3.41). The retarded proper time $\hat\tau_{aa'}(x_b^\beta,u_c^\gamma)$ is defined by

$$[x_a^\alpha-\phi_{a'}^\alpha(x,u;\hat\tau_{aa'})][x_{a\alpha}-\phi_{a'\alpha}(x,u;\hat\tau_{aa'})]=0\ ,\quad x_a^0-\phi_{a'}^0(x,u;\hat\tau_{aa'})>0\ , \tag{4.11}$$

$\phi_a^\alpha(x,u;\tau)$ being the solution of Eqs. (4.2) for the initial conditions (x_b^β,u_c^γ):

$$\ddot\phi_a^\alpha(x,u;\tau)=\xi_a^\alpha[\phi_b^\beta(x,u;\tau),\dot\phi_c^\gamma(x,u;\tau)]\ ,\quad \phi_a^\alpha(x,u;0)=x_a^\alpha,\ \ \dot\phi_a^\alpha(x,u;0)=u_a^\alpha\ . \tag{4.12}$$

$\phi_{a'\tau}^*$ denotes the dual map of the map

$$\phi_{a'\tau}:(x_a^\alpha,x_{a'}^\beta,u_a^\gamma,u_{a'}^\delta)\to(x_a^\alpha,\phi_{a'}^\beta(x,u;\tau),u_a^\gamma,\dot\phi_{a'}^\delta(x,u;\tau))\ ,$$

and $R_a(\tau)$ is the shift operator defined by

$$R_a(\tau)f(x_a^\alpha,x_{a'}^\beta,u_a^\gamma,u_{a'}^\delta)\equiv f(x_a^\alpha+\tau u_a^\alpha,x_{a'}^\beta,u_a^\gamma,u_{a'}^\delta)\ . \tag{4.13}$$

The quantity $P_r^\alpha(x_b^\beta,u_c^\gamma)$ so defined is the linear four-momentum radiated by the system along its motion corresponding to initial conditions (x_b^β,u_c^γ) from past infinity to this configuration, whereas the total four-momentum radiated in the whole evolution is given by[6]

$$P_{tr}^\alpha(x_b^\beta,u_c^\gamma)=\lim_{\tau\to+\infty}\phi_{1\tau}^*\phi_{2\tau}^*P_r^\alpha(x_b^\beta,u_c^\gamma)\ . \tag{4.14}$$

We define the linear four-momentum of the dynamical system (4.2) corresponding to a given configuration as the free linear four-momentum the system had at past infinity minus the four-momentum radiated up to the configuration we are considering, i.e.,

$$P^\alpha(x_b^\beta,u_c^\gamma)\equiv P_{-\infty}^\alpha(x_b^\beta,u_c^\gamma)-P_r^\alpha(x_b^\beta,u_c^\gamma)\ , \tag{4.15}$$

where

$$P_{-\infty}^\alpha(x_b^\beta,u_c^\gamma)\equiv\sum_{a=1}^2\lim_{\tau\to-\infty}m_a\dot\phi_a^\alpha(x_b^\beta,u_c^\gamma;\tau)\ , \tag{4.16}$$

m_a being the mass of the particle a.

Once the functions $I_a^\alpha(x_b^\beta,u_c^\gamma)$ are known, the differential equation (4.6), together with the asymptotic condition (4.7), can be put in an equivalent integrodifferential form which in turn can be formally solved within the framework of perturbation theory by means of known techniques.[2] In order to calculate I_a^α or (4.14), the explicit dependence in Eq. (4.6) on the solution of the predictive invariant system is not a real problem because the general solution of such a system can be found with some inversions of functions and with simple algebraic calculations.[7] To illustrate this, we shall now proceed to calculate in perturbation theory the first significant order of I_a^α.

In classical electrodynamics the evolution equations of an isolated system of two charges interacting through the Lorentz force, or according to the Lorentz-Dirac equation, are of hereditary type, i.e., they are delay differential equations in which the accelerations depend on the past history of the two charges. But, so far as one assumes that the accelerations can be expanded as power series of the charges, it can be proved[6,8] that there is one and only one invariant predictive system associated to the hereditary one, in the sense that the functions ξ_a^α in Eq. (4.2) coincide with the accelerations of the particles moving according to the hereditary equations when the relative position four-vector is null. In the case of the Lorentz force, as well as in the case corresponding to the Lorentz-Dirac equation, the associated invariant predictive system can be written as follows[8]:

$$\xi_a^\alpha = e_a e_{a'}(\eta_a a_a^{(1,1)} h^\alpha + l_{aa'}^{(1,1)} t_{a'}^\alpha) + O(e^4) , \tag{4.17}$$

where $O(e^4)$ denotes terms vanishing at least as e^4 when $e=e_1=e_2\to 0$ and where

$$\begin{aligned} a_a^{(1,1)} &= m_a{}^{-1} k r_{aa'}{}^{-3} , \\ l_{aa'}^{(1,1)} &= -m_a{}^{-1} z_a r_{aa'}{}^{-3} . \end{aligned} \tag{4.18}$$

Furthermore, because of "spontaneous predictivization,"[9] (4.17) holds not only for all the solutions of the associated invariant predictive system—which are always solutions of the corresponding hereditary system—but also for any solution of the Lorentz equation, or of the order reduced Lorentz-Dirac equation, constructed by the methods of steps, provided that we are beyond the first step and that the distance between the charges does not become too small.

The calculation of contribution G_a^α to I_a^α is readily made using Eqs. (4.9) and (4.17). The result is

$$\begin{aligned} G_a^\alpha = &-\tfrac{2}{3} e_a{}^4 e_{a'}{}^2 m_a{}^{-2} r_{aa'}{}^{-6} (k^2 \Lambda^{-2} h^2 + z_a{}^2) \\ &\times (t_a^\alpha + k t_{a'}^\alpha) + O(e^8) . \end{aligned} \tag{4.19}$$

To calculate the integral contribution to I_a^α, we note that[2]

$$\phi_{a'}^\alpha(x_b^\beta, u_c^\gamma; \tau) = x_{a'}^\alpha + \tau u_{a'}^\alpha + O(e^2) , \tag{4.20}$$

$$\hat{\tau}_{aa'}(x_b^\beta, u_c^\gamma) = \zeta_{aa'} + O(e^2) \tag{4.21}$$

with

$$\zeta_{aa'}(x_b^\beta, u_c^\gamma) = -(x_{aa'} u_{a'}) - r_{aa'} . \tag{4.22}$$

Consequently,

$$\begin{aligned} J_a^\alpha(x_b^\beta, u_c^\gamma) &\equiv \int_{\hat{\tau}_{aa'}}^0 d\tau\, \phi_{a'\tau}^* G_{12}^\alpha(x_b^\beta, u_c^\gamma) \\ &= e_1{}^3 e_2{}^3 \int_{\zeta_{aa'}}^0 d\tau\, R_{a'}(\tau) G_{12}^{\alpha(3,3)} + O(e^8) , \end{aligned} \tag{4.23}$$

where $G_{12}^{\alpha(3,3)}$ is the function that we obtain when in Eqs. (4.10) and (3.38)–(3.41) the $a_a^{(1,1)}$ and $l_{aa'}^{(1,1)}$ given in Eqs. (4.18) are substituted for a_a and $l_{aa'}$.

It is easily seen that if we use the notation given in Eqs. (3.35), Eq. (4.23) can be written as

$$J_a^\alpha(x_b^\beta, u_c^\gamma) = e_1{}^3 e_2{}^3 \int_{\hat{z}_{a'}}^{z_{a'}} du\, G_{12}^{\alpha(3,3)} \big|_{z_{a'}=u} + O(e^8) , \tag{4.24}$$

where

$$\hat{z}_{a'} \equiv R_{a'}(\zeta_{aa'}) z_{a'} = k z_a - r_{aa'} . \tag{4.25}$$

After a lengthy but straightforward calculation, we get from Eqs. (3.38)–(3.41), (4.10), (4.18), and (4.24)

$$J_a^\alpha = e_1{}^3 e_2{}^3 m_1{}^{-1} m_2{}^{-1} r_{aa'}{}^{-3} (\eta_a \rho_a h^\alpha + \sigma_{aa} t_a^\alpha + \sigma_{aa'} t_{a'}^\alpha) + O(e^8) \tag{4.26}$$

with

$$\begin{aligned} \rho_a = \Bigg[&\frac{z_a}{\Lambda^3} \left[\frac{\Lambda^2}{r_{aa'}{}^3} + \frac{k^2}{h^2 r_{aa'}} \right] \ln(\Lambda u + R) + \frac{k^2}{2\Lambda^2 h^2} \left[z_a{}^2 - \frac{h^2}{\Lambda^2} \right] R^{-2} + \frac{k z_a}{\Lambda^3 h^3} \left[\frac{\Lambda^2}{4} + k^2 \right] \tan^{-1} \left[\frac{\Lambda u}{h} \right] \\ &- \frac{k^2}{\Lambda^4 h^2} \ln R + \frac{2 k z_a{}^2}{\Lambda^2} \left[\frac{\Lambda^2}{r_{aa'}{}^3} + \frac{k^2}{h^2 r_{aa'}} \right] R^{-1} - \frac{k z_a}{2} u R^{-4} - z_a \left[\frac{k^2+1}{r_{aa'}{}^3} + \frac{k^2}{\Lambda^2 h^2 r_{aa'}} - \frac{k^2 z_a{}^2}{h^4 r_{aa'}} \right] u R^{-1} \\ &+ \frac{k z_a}{\Lambda^2 h^2} \left[\frac{\Lambda^2}{4} - k^2 \right] u R^{-2} + \frac{\Lambda^2 z_a{}^2 - k^2 h^2}{4\Lambda^2} R^{-4} \Bigg]_{\hat{z}_{a'}}^{z_{a'}} , \end{aligned} \tag{4.27}$$

$$\begin{aligned} \sigma_{aa} = \Bigg[&\left[\left[\frac{z_a{}^2}{\Lambda r_{aa'}{}^3} - \frac{k}{\Lambda^5 r_{aa'}} \right] \ln(\Lambda u + R) - \frac{k}{2\Lambda^3 h} \tan^{-1} \left[\frac{\Lambda u}{h} \right] \right] + \frac{k z_a}{\Lambda^2} \left[\frac{\Lambda^2 z_a{}^2}{r_{aa'}{}^3} - \frac{h^2}{r_{aa'}{}^3} - \frac{2k^2}{\Lambda^2 r_{aa'}} \right] R^{-1} \\ &- \left[z_a{}^2 \frac{k^2+1}{r_{aa'}{}^3} - k^2 \frac{\Lambda^2 h^2 + h^2 - \Lambda^2 z_a{}^2}{\Lambda^4 h^2 r_{aa'}} \right] u R^{-1} - \frac{z_a}{2\Lambda^2} R^{-2} + \frac{k}{2\Lambda^2} u R^{-2} \Bigg]_{\hat{z}_{a'}}^{z_{a'}} , \end{aligned} \tag{4.28}$$

$$\sigma_{aa'}=\left[\frac{4k^2(h^2-h^2\Lambda^{-2}-z_a{}^2)+\Lambda^2 z_a{}^2-kh^2}{8\Lambda^3h^3}\tan^{-1}\left[\frac{\Lambda u}{h}\right]-\frac{z_a}{\Lambda^2 r_{aa'}}R^{-1}-\frac{kz_a{}^2}{h^2 r_{aa'}}uR^{-1}\right.$$
$$\left.+\frac{kz_a}{\Lambda^2}\left[\frac{1}{2}-\frac{k^2}{\Lambda^2}\right]R^{-2}+\frac{4k^2(h^2+h^2\Lambda^{-2}-z_a{}^2)+\Lambda^2 z_a{}^2-kh^2}{8\Lambda^2h^2}uR^{-2}-\frac{kh^2 z_a}{2\Lambda^2}R^{-4}+\frac{kh^2-\Lambda^2 z_a{}^2}{4\Lambda^2}uR^{-4}\right]_{\hat{z}_{a'}}^{z_{a'}}, \tag{4.29}$$

where

$$R\equiv R(u)=(h^2+\Lambda^2u^2)^{1/2} \tag{4.30}$$

and thus

$$R(z_{a'})=r_{a'a}\ ,\qquad R(\hat{z}_{a'})=R_{a'}(\zeta_{aa'})r_{a'a}=kr_{aa'}-\Lambda^2 z_a\ . \tag{4.31}$$

From these results one can easily obtain, for instance, the energy radiated per unit of coordinate time,[10]

$$\frac{dE}{dt}(t,\vec{x}_b,\vec{v}_c)=\sum_{a=1}^{2}(\bar{u}_a^0)^{-1}(\bar{G}_a^0+\bar{J}_a^0)+O(e^8)\ , \tag{4.32}$$

where $\bar{G}_a^0$ and $\bar{J}_a^0$ mean the values of functions G_a^0 and J_a^0 when their arguments take the values

$$\bar{x}_1^0=\bar{x}_2^0=t,\quad \bar{x}_a^i=x_a^i\ ,\qquad \bar{u}_a^0=(1-v_a^iv_{ai})^{-1/2},\quad \bar{u}_a^i=(1-v_a^iv_{ai})^{-1/2}v_a^i\ . \tag{4.33}$$

Note that, in fact, dE/dt depends only on $\vec{x}_1-\vec{x}_2$, $\vec{v}_1$, and $\vec{v}_2$.

Finally, let us say that, if desired, an additional expansion of (4.32) in terms of powers of c^{-1} would allow a comparison with the results obtained with the slow-motion conventional method. In particular, to lowest order one obtains after a straightforward but lengthy calculation

$$\frac{dE}{dt}=\frac{2}{3}\frac{e_1{}^2e_2{}^2}{c^3}\left[\frac{e_1}{m_1}-\frac{e_2}{m_2}\right]^2\frac{1}{r^4}\ , \tag{4.34}$$

where

$$r^2\equiv(x_1^i-x_2^i)(x_{1i}-x_{2i})\ . \tag{4.35}$$

Equation (4.34) is the well-known[11] expression for the dipole radiation of two point charges moving according to Coulomb's law.

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APPENDIX

Let us consider two four-vectors, n^α and j^α, satisfying the conditions

$$n^\alpha n_\alpha=-1,\quad n^0\geq 1\ ,\qquad j^\alpha j_\alpha=1,\quad (nj)=0\ , \tag{A1}$$

and let (y^α) be a coordinate system for which $n^\alpha=(1,\vec{0})$. Let us consider also the curvilinear coordinates $(\lambda,\tau_1,\tau_2,\phi)$ defined by Eqs. (2.4), (2.20), and

$$\rho\cos\phi=-\epsilon_{\alpha\beta\gamma\delta}n^\alpha l_1^\beta l_2^\gamma j^\delta\ ,\qquad \rho\sin\phi=\{[(nl_1)(jl_2))-(nl_2)(jl_1)]^2-2(l_1l_2)(jl_1)(jl_2)\}^{1/2} \tag{A2}$$

with

$$\rho^2=-2(l_1l_2)(nl_1)(nl_2)-(l_1l_2)^2\ . \tag{A3}$$

Note that the latter is equivalent to Eq. (3.11) because of the identity $x_{12}{}^2=-2(l_1l_2)$.

Taking into account that $y^0=\lambda$, the volume element of the hypersurface $\Sigma(\lambda)$ is

$$d^3\sigma^\alpha=d^3\sigma\, n^\alpha,\quad d^3\sigma=dy^1dy^2dy^3 \tag{A4}$$

or equivalently

$$d^3\sigma=\left|\frac{\partial(y^1,y^2,y^3)}{\partial(\tau_1,\tau_2,\phi)}\right|d\tau_1d\tau_2d\phi=\left|\frac{\partial(y^0,y^1,y^2,y^3)}{\partial(\lambda,\tau_1,\tau_2,\phi)}\right|d\tau_1d\tau_2d\phi\ . \tag{A5}$$

The Jacobian can be easily calculated by making use of (A2) and

$$\frac{\partial\lambda}{\partial y^\alpha}=\delta_\alpha^0=-n_\alpha\ ,\qquad \frac{\partial\tau_a}{\partial y^\alpha}=-r_a{}^{-1}l_{a\alpha}\ ,\qquad \frac{\partial l_a^\alpha}{\partial y^\beta}=\delta_\beta^\alpha+r_a{}^{-1}u_a^\alpha l_{a\beta} \tag{A6}$$

to find

$$\frac{\partial(\lambda,\tau_1,\tau_2,\phi)}{\partial(y^0,y^1,y^2,y^3)}=-\frac{(l_1l_2)[(jl_1)+(jl_2)]+[(nl_1)-(nl_2)][(jl_1)(nl_2)-(jl_2)(nl_1)]}{r_1r_2\rho\sin\phi}\ . \tag{A7}$$

We now choose two fixed values τ_1 and τ_2, a given timelike vector $n^\alpha = u_1^\alpha$, and the vector j^α defined by (3.6). Hence, property (3.13) is satisfied in the intersection of $\Sigma(\lambda)$, $C_1^+(\tau_1)$, and $C_2^+(\tau_2)$ and at such a point we get

$$\frac{\partial(\lambda,\tau_1,\tau_2,\phi)}{\partial(y^0,y^1,y^2,y^3)} = r_{21}{r_1}^{-1}{r_2}^{-1} . \tag{A8}$$

Finally, using (A4), (A5), and (A8) we obtain Eq. (3.15) for any point of the aforementioned intersection.

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