Thermal effects of acceleration for a classical dipole oscillator in classical electromagnetic zero-point radiation

Timothy H. Boyer
Department of Physics, City College of the City University of New York, New York, New York 10031
(Received 8 August 1983; revised manuscript received 16 December 1983)

In 1976 Unruh showed that a scalar quantum particle in a box accelerating through the vacuum of scalar quantum field theory responded as though it were in a thermal bath at temperature $T = \hbar a/2\pi ck$. Here we show an analogous result within classical electromagnetic theory. A classical electric dipole oscillator accelerating through classical electromagnetic zero-point radiation responds just as would a dipole oscillator in an inertial frame in classical thermal radiation with Planck's spectrum at temperature $T = \hbar a/2\pi ck$. In an earlier work it was shown that the electromagnetic field correlation functions for an observer accelerating through classical electromagnetic zero-point radiation correspond to a spectrum different from Planck's. The same spectrum is found in the quantum analysis of a vector field where the departure from Planckian form is assigned to the change in the number of normal modes associated with the event horizon of the accelerating observer. The present work shows that the relativistic radiation reaction for an accelerating classical charge contains a term which exactly compensates the departure of the electromagnetic spectrum from Planckian form so as to bring the oscillator's behavior into precise agreement with the usual Planckian thermal form.

INTRODUCTION

A physical system undergoing a uniform acceleration $\vec{a}$ relative to an inertial frame behaves as though it were immersed in a thermal bath at temperature

$$T = \hbar a/2\pi ck \ , \tag{1}$$

where $\hbar$ is Planck's constant and k is Boltzmann's constant. This profoundly simple result was introduced into physics by Unruh[1] and by Davies[2] as part of the analysis connected with black holes in relativistic astrophysics. The original arguments involve sophisticated work in both quantum field theory and general relativity, and touch on subtle questions distinguishing between real and virtual quanta and quantum fluctuations. This article treats the thermal effects of acceleration within the simpler context of classical electromagnetism and special relativity. The use of this simpler context may help a larger group of physicists to understand the matter, and may also serve to clarify the essential aspects of the phenomenon.

Newtonian mechanics does not show any thermal effects of acceleration. Hence in order to understand the matter we must go beyond classical mechanics to include classical electromagnetism and specifically random classical radiation with a Lorentz-invariant spectrum, classical electromagnetic zero-point radiation. This minimal classical extension of elementary physics which shows the thermal effects of acceleration is termed random electrodynamics or stochastic electrodynamics. The theory is simply the classical electron theory of Lorentz in which the homogeneous boundary conditions on Maxwell's equations have been chosen to include classical zero-point radiation. The theory gives quantitative explanations for several phenomena which are usually thought to require quantum analysis.[3]

This article is the second discussion of the thermal effects of acceleration within classical theory with classical zero-point radiation. In the first article[4] we considered the spectrum of random classical radiation seen by an observer undergoing uniform acceleration through the classical zero-point radiation. Our analysis for scalar zero-point radiation found that the accelerating observer saw exactly Planck's spectrum of scalar radiation with temperature as in (1). However, when we moved on to the electromagnetic case, we found that the observer undergoing uniform acceleration through classical electromagnetic zero-point radiation found field correlation functions corresponding to a spectrum different from Planck's spectrum. This seemed discouraging. However, upon investigating the quantum literature[5] for vector fields, we found that there too the resulting spectrum was not Planck's and was completely in agreement with our classical results. The quantum treatments within general relativity spoke of the spectrum as "thermal" but "non-Planckian."[6] Now a radiation spectrum which is "thermal" but non-Planckian makes uneasy at least those physicists who are not experts on general relativity. To be sure, the sophisticated analyses reassure one[5] that the departure from Planckian form involves a change in the counting of normal modes associated with the event horizon in the frame of the accelerating observer. Nevertheless the untutored physicist is likely to remain dissatisfied and to wonder why the effects of the event horizon appear in the vector-field case but are not seen for the scalar field.

The situation has remained in this form since 1980. However, now a stumbling block in the simpler analysis has been removed. The ordinary physicist who may have had his enthusiasm dampened by the complications found at the end of the first article[4] is encouraged to look again at the matter because of the results we are reporting in this second article. The situation is as follows. The spectral calculations reported in the first article were based upon mathematical evaluation of the correlation functions for the electromagnetic field. However, measurements by an accelerating observer should refer not to purely mathematical functions but rather to the behavior of physical systems. In the present case we can imagine a uniformly accelerating observer carrying along a classical electric dipole oscillator which interacts with the random classical radiation, and the observer measuring the response of this physical oscillator. In this article we calculate the behavior of just such an oscillator undergoing uniform acceleration through classical electromagnetic zero-point radiation. Although, as found in the first article, the field correlation function which drives the oscillator is non-Planckian, there is also a new acceleration-related term in the relativistic radiation reaction force for the oscillator. The two departures from the usual inertial-frame behavior exactly cancel. The oscillator responds exactly with a Planckian distribution at the temperature given by (1). Thus the oscillator appears as though it were in a thermal bath of Planckian form and any measurement using the oscillator would never reveal the departure of the field correlation function from Planckian form. Again an untutored physicist will feel comfortable with the results. Mechanical systems indeed respond on acceleration as though they were in a thermal bath in the normal sense.

THE BASIC THEORY

Classical electron theory consists of Newton's second law of motion for the charged particles, Maxwell's equations for the electromagnetic fields, and boundary conditions on the associated equations of motion. In the traditional classical electron theory developed at the turn of the century, the homogeneous boundary conditions on Maxwell's equations were chosen so that there was no radiation in the far past.[7] Classical electron theory with classical electromagnetic zero-point radiation changes this homogeneous boundary condition to assume that in the far past there was random classical electromagnetic radiation with a Lorentz-invariant spectrum. The random radiation can be written as a sum over plane waves:

$$\vec{E}^{zp}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k},\lambda)H_{zp}(\omega)\times\cos[\vec{k}\cdot\vec{r}-\omega t-\theta(\vec{k},\lambda)]\ , \tag{2}$$

$$\vec{B}^{zp}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k[(\vec{k}\times\hat{\epsilon})/k]H_{zp}(\omega)\times\cos[\vec{k}\cdot\vec{r}-\omega t-\theta(\vec{k},\lambda)]\ , \tag{3}$$

where the $\theta(\vec{k},\lambda)$ are random phases distributed uniformly on $(0,2\pi)$, independently distributed for each $\vec{k},\lambda$. The assumption of Lorentz invariance determines the spectrum[8] up to a single multiplicative constant which is chosen as Planck's constant $\hbar$,

$$\pi^2 H_{zp}{}^2(\omega)=\tfrac{1}{2}\hbar\omega\ . \tag{4}$$

The introduction of Planck's constant at this point, and only at this point, allows this purely classical theory to give results beyond those of the traditional classical electron theory of the early part of the century. Van der Waals forces and diamagnetic results of the theory are in quantitative agreement with the quantum-theory predictions.[3] In this article we find a further advantage of the new choice of boundary condition over the traditional choice since classical electron theory with classical electromagnetic zero-point radiation shows the thermal effects of acceleration whereas the traditional classical electron theory does not.

RELATIVISTIC PARTICLE EQUATION OF MOTION

The mechanical system we consider in this article is a small harmonic electric-dipole oscillator. Although the system may be described in these general terms, we will picture and discuss the system as though it were a particle of mass m and charge e at the end of a massless spring of constant $K=m\omega_0{}^2$. In an inertial frame the equation of motion for the displacement $\vec{r}(t)$ of the particle from the spring equilibrium position can be written in the nonrelativistic dipole approximation as

$$md^2\vec{r}/dt^2=-m\omega_0{}^2\vec{r}+\tfrac{2}{3}(e^2/c^3)d^3\vec{r}/dt^3+e\vec{E}^{\,in}(\vec{R},t)\ . \tag{5}$$

Here the term $\frac{2}{3}(e^2/c^3)d^3\vec{r}/dt^3$ is the radiation damping and $e\vec{E}^{\,in}(\vec{R},t)$ is the force due to an electric field (not including the self-fields of the particle), evaluated at the fixed equilibrium position $\vec{R}$ of the oscillator at time t. We have assumed an isotropic oscillator. This equation (5) is an approximation to the relativistic equation of motion[9] which holds in any inertial frame,

$$m\ddot{x}^{\mu}=F^{\mu}+\tfrac{2}{3}(e^2/c^3)(\dddot{x}^{\mu}+c^{-2}\dot{x}^{\mu}\ddot{x}^{\nu}\ddot{x}_{\nu})+(e/c)\mathscr{F}^{in\nu\mu}\dot{x}_{\nu}\ , \tag{6}$$

where m is the rest mass, $x^{\mu}=(ct,\vec{r})$ is the four-vector displacement of the particle, F^{μ} is the four-force due to the spring, $\mathscr{F}^{in\nu\mu}$ is the electromagnetic field tensor (excluding the particle self-fields), and the overdots refer to differentiation with respect to the particle proper time. The term involving e^2/c^3 is the relativistic form for the radiation damping.

In this article we are interested in the situation when the dipole oscillator experiences a uniform acceleration through the random classical electromagnetic zero-point radiation. We assume that the equilibrium point of the spring moves with uniform proper acceleration $\vec{a}$ and that an external electric field $\vec{E}_0=m\vec{a}/e$ exists which causes a uniform acceleration for the charged particle in the absence of the spring. The massless spring provides a force $-m\omega_0{}^2\vec{r}$ in the instantaneous rest frame of the spring equilibrium position. For simplicity of calculation, we will restrict the oscillator to the yz plane perpendicular to the x direction of acceleration, $\vec{a}=\hat{i}a$.

In our analysis we will use a fixed inertial frame I_* and a set of inertial frames I_τ defined by the condition that the spring equilibrium position is instantaneously at rest in I_τ at the proper time τ measured by a clock at the spring equilibrium position. We assume that the equilibrium position moves along the positive x axis of I_*, coming instantaneously to rest when the coordinate X_* of the spring equilibrium position takes the value $X_*=0$ at time $t_*=0$. In each frame I_τ the spring equilibrium position labeled by $(X_\tau,Y_\tau,Z_\tau)=(X_\tau,0,0)$ moves with instantaneous acceleration $(d^2X_\tau/dt_\tau{}^2)_\tau=a$ at time τ. The acceleration as seen in I_* is related to that seen in I_τ by $d^2X_*/dt_*{}^2=\gamma_\tau{}^{-3}a$ where $\gamma_\tau=[1-(dX_*/dt_*)^2]^{-1/2}$ and dX_*/dt_* is the velocity of the spring equilibrium position and of I_τ relative to I_* at time τ. We can solve the acceleration differential equation for $X_*(t_*)$ and then relate the time t_* to the proper time τ measured at the equilibrium position giving[10]

$$X_*=[(c/a)^2+t_*{}^2]^{1/2}c \tag{7}$$

and

$$t_*(\tau)=(c/a)\sinh(a\tau/c)\ , \tag{8}$$

$$X_*(\tau)=(c^2/a)[\cosh(a\tau/c)-1]\ , \tag{9}$$

with the other spring equilibrium point coordinates

$$Y_*(\tau)=Z_*(\tau)=0 \tag{10}$$

by assumption of motion along the x_*-axis. Clearly in this notation the inertial frame I_* agrees with the frame I_τ at $\tau=0$.

The coordinate frame S carried along by the accelerating dipole will agree with the inertial frame I_τ when the proper time at the equilibrium position is τ. We choose the spring equilibrium position at the origin of S, and denote by $(c\tau,\vec{r})=(c\tau,0,y,z)$ the coordinates of the particle in the accelerating coordinate frame S. Also at time τ the spring equilibrium position is at the origin of I_τ at the time $t_\tau=\tau$.

EQUATION OF MOTION IN THE ACCELERATING FRAME

We are interested in evaluating the displacement $\vec{r}(\tau)$ of the particle from the spring equilibrium position as a function of the time τ read by a clock at the equilibrium position which is the origin of S. In order to obtain an equation of motion for $\vec{r}$ as a function of τ we go back to the relativistic equation of motion (6) for the displacement x^μ of the particle relative to an inertial frame. We will evaluate this equation in the fixed I_* frame, corresponding to putting a subscript asterisk on the quantities of Eq. (6), but then will rewrite the variables involving the particle displacement x_*^μ and particle proper time in terms of the displacement X_*^μ of the spring equilibrium position, and of the displacement $\vec{r}$ and time τ in the S frame.

As a start we note that the quantities $\dot{x}_*^\mu,\ddot{x}_*^\mu,\dddot{x}_*^\mu$, are each four-vectors because the particle proper time indicated by the differentiation overdot is a Lorentz scalar. Then four-vector transformations can be used to relate the quantities in the fixed inertial frame I_* to the quantities in the frame I_τ instantaneously at rest with respect to the spring equilibrium point at time τ. In the y and z directions transverse to the direction of acceleration, we see from the four-vector form of the Lorentz transformations that the displacements y_*, z_*, and four-force components F_{*y},F_{*z} are identical with the y_τ,z_τ and $F_{\tau y},F_{\tau z}$ seen in I_τ and hence with the y,z, and F_y,F_z in S.

At this point we emphasize the small-dipole approximation which we use. One should be quite clear about just what approximation in the usual inertial-frame analysis allows us to rewrite the relativistic oscillator equation (6) in the form (5). This is the small-oscillator approximation. In the inertial frame in which the oscillator (spring) potential is at rest, the magnitudes of all displacements and velocities depend upon the size of the oscillator and the characteristic frequency ω_0. If the size of the oscillator is small compared to the radiation wavelength $\lambda=2\pi c/\omega_0$, then the dipole approximation is justified in the inertial frame in which the oscillator potential is at rest. Also, if the size of the oscillator is small, then the characteristic speed $v=\omega_0 r$ will be small compared to c and hence a nonrelativistic approximation is justified in the inertial frame in which the oscillator potential is at rest.

We wish to make the small-oscillator approximation in the present calculation, but we must be careful to apply it in coordinate frames where the oscillator potential is at rest. Thus at time τ we will use the dipole approximation in the frame I_τ instantaneously at rest with respect to the spring equilibrium position; we neglect the magnetic force $(e/c)\vec{v}\times\vec{B}$ and evaluate the driving electric force $e\vec{E}_\tau(\vec{r}_\tau,t_\tau)$ not at the particle position $\vec{r}_\tau$ but at the equilibrium point of the oscillator as $e\vec{E}_\tau(0,\tau)$. Furthermore working in the approximation of a small oscillator so that the particle displacements and velocities in S are small, we can ignore the distinction between the proper time interval for the particle itself and the proper time τ measured by a clock at the origin of S. Thus all the derivatives which were denoted by overdots will now be regarded as derivatives with respect to the time τ in S.

From the comments above about four-vectors and the approximation that the particle proper time agrees with the time τ in S, we have

$$\dot{y}_*\cong dy/d\tau,\ \ \dot{z}_*\cong dz/d\tau\ , \tag{11}$$

and analogous equations involving second and third derivatives.

The final step in our analysis of the equation of motion for the oscillator involves the radiation reaction term $\frac{2}{3}(e^2/c^3)(\dddot{x}^\mu+c^{-2}\dot{x}^\mu\ddot{x}^\nu\ddot{x}_\nu)$ of (6). In the usual nonrelativistic approximation in an inertial frame the term $c^{-2}\dot{x}^\mu\ddot{x}^\nu\ddot{x}_\nu$ is omitted entirely because of the additional factor of c^{-2}. However, in the present work we are taking a relativistic point of view where all approximations are justified solely from the assumption of a small oscillator. Looking at the space components of the relativistic radiation reaction, we see that the first term $\dddot{\vec{r}}$ involves one factor in the size of the oscillator. The second term has $\dot{\vec{r}}$, which depends upon the oscillator size, multiplying $c^{-2}\ddot{x}^\nu\ddot{x}_\nu$. Now in the usual inertial-frame calculation

$\ddot{x}^{\nu}\ddot{x}_{\nu}$ is second order in the size of the oscillator making $c^{-2}\dot{\vec{r}}\ddot{x}^{\nu}\ddot{x}_{\nu}$ third order in size and hence negligible compared to $\dddot{\vec{r}}$. However, in the present work we find $\ddot{x}^{\nu}\ddot{x}_{\nu}$ includes a contribution from the uniform acceleration of the oscillator and this contribution does not depend upon the size of the oscillator. Thus both $\dddot{\vec{r}}$ and $c^{-2}\dot{\vec{r}}\ddot{x}^{\nu}\ddot{x}_{\nu}$ include terms first order in the oscillator size, and hence both terms must be included in the calculation. Retaining only those terms which do not vanish with the size of the oscillator, we write $x_* \cong X_*$ and have from (8) and (9)

$$\ddot{x}_*^{\nu}\ddot{x}_{*\nu} \cong (d^2ct_*/d\tau^2)^2-(d^2X_*/d\tau^2)^2 = a^2\sinh^2(a\tau/c)-a^2\cosh^2(a\tau/c) = -a^2 . \tag{12}$$

Thus the radiation reaction for the point charge involves an unfamiliar additional term which is not specific to the binding potential, but rather reflects the uniform acceleration $\vec{a}$. It is this term which is needed to compensate for the non-Planckian form of the field correlation functions seen in the accelerating frame S.

The y-component of the relativistic equation of motion for the particle as seen in the I_* frame follows from (6) as

$$m\ddot{y}_* = F_{*y}^{(\mathrm{osc})}+F_{*y}^{(E_0)}+\tfrac{2}{3}(e^2/c^3)(\dddot{y}_*+c^{-2}\dot{y}_*\ddot{x}_*^{\nu}\ddot{x}_{*\nu}) +(e/c)\mathscr{F}_*^{\mathrm{zp}\nu y}\dot{x}_{*\nu} \tag{13}$$

with a similar equation for the z component. Now, inserting the considerations of this section and particularly Eq. (12), these equations can be written together as a vector equation for an oscillator isotropic in the yz plane:

$$m\frac{d^2\vec{r}}{d\tau^2} = -m\omega_0{}^2\vec{r}+\frac{2}{3}\frac{e^2}{c^3}\left[\frac{d^3\vec{r}}{d\tau^3}-\frac{a^2}{c^2}\frac{d\vec{r}}{d\tau}\right] +e\vec{E}_{\tau}^{\mathrm{zp}}(0,\tau) , \tag{14}$$

where $\vec{r}$ is the vector displacement of the particle in the S frame, τ is the time at the origin of this frame, and $\vec{E}_{\tau}^{\mathrm{zp}}(0,\tau)$ is the zero-point radiation field seen in the inertial frame I_τ at the oscillator equilibrium point at time τ.

The equation of motion (14) for the uniformly accelerating oscillator in its own frame S is similar in form to the equation of motion (5) for a small isotropic oscillator at rest in an inertial frame. The differences between the equations reside in an additional term $\frac{2}{3}(e^2/c^3)(-a^2/c^2)d\vec{r}/d\tau$ in the radiation reaction force and a change in the random driving field $\vec{E}^{\mathrm{in}}$. We will find that these two changes in Eq. (14) combine in just such a way as to give oscillator behavior identical to that found for the inertial oscillator in (5) when the driving spectrum is not zero-point radiation but rather thermal radiation with Planck's spectrum.

SOLUTION OF THE EQUATION OF MOTION

The equation of motion (14) is a linear stochastic differential equation in $\vec{r}(\tau)$ with a random driving term $e\vec{E}_{\tau}^{\mathrm{zp}}(0,\tau)$. The driving term involves the electric field seen at the equilibrium point of the oscillator in the inertial frame I_τ instantaneously at rest with respect to the equilibrium point. This field may be found by Lorentz transformation from the random field $\vec{E}_*^{\mathrm{zp}}(\vec{R}_*,t_*)$ found in the inertial frame I_* at the known position $\vec{R}_*(\tau)$ of the spring equilibrium point at time $t_*(\tau)$. Thus from the zero-point electromagnetic field of Eqs. (2) and (3), the coordinate equations (8), (9), and the known Lorentz transformation equations for the electromagnetic fields, we find[4,8]

$$\begin{aligned}\vec{E}_{\tau}^{\mathrm{zp}}(0,\tau) &= \sum_{\lambda=1}^{2}\int d^3k\{\hat{i}\epsilon_x+\hat{j}\gamma_\tau[\epsilon_y-\beta_\tau(\hat{k}\times\hat{\epsilon})_z]+\hat{k}\gamma_\tau[\epsilon_z+\beta_\tau(\hat{k}\times\hat{\epsilon})_y]\} \\ &\qquad\times H_{\mathrm{zp}}(\omega)\cos[\vec{k}\cdot\vec{R}_*(\tau)-\omega t_*(\tau)-\theta(\vec{k},\lambda)] \\ &= \sum_{\lambda=1}^{2}\int d^3k\{\hat{i}\epsilon_x+\hat{j}\cosh(a\tau/c)[\epsilon_y-\tanh(a\tau/c)(\hat{k}\times\hat{\epsilon})_z] \\ &\qquad+\hat{k}\cosh(a\tau/c)[\epsilon_z+\tanh(a\tau/c)(\hat{k}\times\hat{\epsilon})_y]\}H_{\mathrm{zp}}(\omega) \\ &\qquad\times\cos[k_x(c^2/a)\cosh(a\tau/c)-\omega(c/a)\sinh(a\tau/c)-\theta(\vec{k},\lambda)] .\end{aligned} \tag{15}$$

The zero-point radiation spectrum (4) is Lorentz invariant, taking the same form in each inertial frame I_τ. Accordingly there is no preferred inertial frame or preferred time for the uniformly accelerating dipole, and hence the displacement will be a stationary random process.

It is natural to solve a linear differential equation with a stationary stochastic driving term by means of Fourier transforms. Writing

$$\vec{r}(\tau)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\Omega\,\vec{\eta}(\Omega)\exp[-i\Omega\tau] \tag{16}$$

and

$$\vec{E}_{\tau}^{\mathrm{zp}}(0,\tau)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\Omega\,\vec{\Sigma}(\Omega)\exp[-i\Omega\tau] , \tag{17}$$

the differential equation (14) becomes

$$\left\{m(-i\Omega)^2-\frac{2}{3}\frac{e^2}{c^3}\left[(-i\Omega)^3-(-i\Omega)\frac{a^2}{c^2}\right]+m\omega_0{}^2\right\}\vec{\eta}(\Omega)=e\vec{\Sigma}(\Omega) \tag{18}$$

with the solution

$$\vec{r}(\tau)=\frac{1}{(2\pi)^{1/2}}\int d\Omega\,\frac{e}{m}\frac{\vec{\Sigma}(\Omega)\exp[-i\Omega\tau]}{[\omega_0{}^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}\ , \tag{19}$$

where the damping parameter Γ is

$$\Gamma=\tfrac{2}{3}e^2/mc^3\ . \tag{20}$$

Rather than working with an isotropic two-dimensional oscillator we can obtain the more familiar one-dimensional oscillator by projecting out the component $x(\tau)$ of $\vec{r}(\tau)$ in an arbitrary coordinate direction $\hat{n}$ within the yz plane,

$$x(\tau)=\hat{n}\cdot\vec{r}(\tau)\ . \tag{21}$$

Here we denote this coordinate by $x(\tau)$ even though it will actually be perpendicular to the direction of acceleration; previously we had called the direction of acceleration the x direction.

The average value of $x^2(\tau)$ follows from (19) and (21) as

$$\langle x^2(\tau)\rangle=\frac{1}{2\pi}\int d\Omega\int d\Omega'\frac{e^2}{m^2}\frac{e^{-i\Omega\tau}e^{i\Omega'\tau}\langle\Sigma_x(\Omega)\Sigma_x^*(\Omega)\rangle}{[\omega_0{}^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)][\omega_0{}^2-\Omega'^2+i\Gamma(\Omega'^3+\Omega' a^2/c^2)]}\ , \tag{22}$$

where we have used the fact that $x(\tau)$ is real to write $x^2(\tau)$ as $x(\tau)x^*(\tau)$. The average value $\langle\Sigma_x(\Omega)\Sigma_x(\Omega')\rangle$ needed in the integrand is related to the average value of the fields through the inverse Fourier transform from (17),

$$\langle\Sigma_x(\Omega)\Sigma_x^*(\Omega')\rangle=(2\pi)^{-1}\int d\tau\int d\tau'\exp[i\Omega\tau]\exp[-i\Omega'\tau']\langle E_{\tau x}(0,\tau)E_{\tau' x}(0,\tau')\rangle\ . \tag{23}$$

Now the correlation functions $\langle E_{\tau i}(0,\tau)E_{\tau' j}(0,\tau')\rangle$, $\langle E_{\tau i}(0,\tau)B_{\tau' j}(0,\tau')\rangle$, $\langle B_{\tau i}(0,\tau)B_{\tau' j}(0,\tau')\rangle$ for the electromagnetic fields seen at a point undergoing uniform acceleration were precisely what was evaluated in the first article[4] of this series. Using the expression (15) we showed that the correlation function $\langle E_{\tau i}(0,\tau)E_{\tau' j}(0,\tau')\rangle$ was the same irrespective of the direction of acceleration and that[11]

$$\begin{aligned}\langle E_{\tau i}(0,\tau)E_{\tau' j}(0,\tau')\rangle&=\delta_{ij}\frac{4\hbar}{\pi c^3}\left[\frac{a}{2c}\right]^4\operatorname{csch}^4\left[\frac{a(\tau-\tau')}{2c}\right]\\&=\delta_{ij}\frac{2}{3}\frac{\hbar}{\pi c^3}\int_0^\infty d\omega\,\omega^3\left[1+\left[\frac{a}{c\omega}\right]^2\right]\coth\left[\frac{\pi c\omega}{a}\right]\cos[\omega(\tau-\tau')],\quad i,j=1,2,3\ .\end{aligned} \tag{24}$$

The correlation functions depend only upon the time difference $\tau-\tau'$, just as required for a stationary random process. Rewriting the integrals in (23) to take advantage of the $\tau-\tau'$ dependence, recognizing the integrals leading to δ functions, and integrating over ω, we have

$$\begin{aligned}\langle\Sigma_x(\Omega)\Sigma_x^*(\Omega')\rangle&=(2\pi)^{-1}\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d(\tau'-\tau)\exp[-i(\Omega-\Omega')\tau]\exp[i\Omega'(\tau'-\tau)]\\&\qquad\times\frac{2}{3}\frac{\hbar}{\pi c^3}\int_0^\infty d\omega\,\omega^3\left[1+\left[\frac{a}{c\omega}\right]^2\right]\coth\left[\frac{\pi c\omega}{a}\right]\cos[\omega(\tau-\tau')]\\&=\frac{2}{3}\frac{\hbar}{c^3}\delta(\Omega-\Omega')2\Omega^3\left[1+\left[\frac{a}{c\Omega}\right]^2\right]\coth\left[\frac{\pi c\Omega}{a}\right]\ .\end{aligned} \tag{25}$$

A factor of 2 arises from equal contributions from $\omega=\pm\Omega'$.

Next we substitute this expression (25) back into (22), integrate over Ω' to remove the δ function, and find

$$\langle x^2(\tau)\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega\frac{e^2}{m^2}\frac{2}{3}\frac{\hbar}{c^3}\frac{2\Omega^3[1+(a/c\Omega)^2]\coth(\pi c\Omega/a)}{(\omega_0{}^2-\Omega^2)^2+\{\Gamma\Omega^2[1+(a/c\Omega)^2]\}^2}\ . \tag{26}$$

If $\Gamma=\frac{2}{3}e^2/mc^3$ is small, corresponding to an oscillator interacting weakly with the electromagnetic field, then the integrand of (26) is sharply peaked at $\Omega=\omega_0$. Accordingly we may write $\Omega=\omega_0$ in each of the terms not involving $\Omega-\omega_0\equiv z$. The integral takes the form

$$\int_{-\infty}^{\infty}dz\frac{1}{z^2+1}=\pi\ . \tag{27}$$

Thus in the narrow-linewidth approximation (26) becomes

$$\langle x^2(\tau)\rangle=\tfrac{1}{2}(\hbar/m\omega_0)\coth(\pi c\omega_0/a)\ . \tag{28}$$

The expectation value for the square of the velocity may also be easily evaluated from the Fourier transform expression (19). The derivative with respect to time τ merely brings down a factor of Ω in (19), which is then converted to a factor of ω_0 in the narrow-linewidth approximation.[12] Thus the average value for the velocity squared involves two factors of ω_0 compared to (28):

$$\langle (dx/d\tau)^2 \rangle = \tfrac{1}{2}(\hbar\omega_0/m)\coth(\pi c\omega_0/a) \ . \qquad (29)$$

The forms for $\langle x^2(\tau) \rangle$ and $\langle (dx/d\tau)^2 \rangle$ are indeed those of stationary random processes, independent of the time τ. They have a familiar form. The average values are identical with those found[13] for classical oscillators in an inertial frame bathed in Planck's spectrum[14] of random classical radiation at temperature

$$T = \hbar a/2\pi ck \ . \qquad (30)$$

We have now followed to completion the calculation required for our report at the beginning of this article. The correlation functions (24) for the fields involve a spectrum which departs from Planckian form by an additional factor of $[1+(a/c\omega)^2]$, but the final oscillator behavior in the accelerating frame takes exactly Planckian form. Indeed our evaluation of the equation of motion for $\vec{r}(\tau)$ and of the average value of $x^2(\tau)$ is very similar to that used for an oscillator in an inertial frame. In the familiar integral, corresponding (26) when the acceleration a vanishes, the term in Γ^2 in the denominator of the resonant integrand removes the factor of $\Gamma = \frac{2}{3}(e^2/mc^3)$ in the numerator. In the integral (26) needed here the term in $\{\Gamma[1+(a/c\omega)^2]\}^2$ in the resonant denominator cancels the factor of $\Gamma[1+(a/c\omega)^2]$ in the numerator. The radiation reaction correction due to the acceleration cancels the density of states correction in the correlation function for the random fields.

CLOSING REMARKS

Why does an accelerating observer find a state different from that holding for a inertial observer? The quantum analysis, in which the thermal effects of acceleration were first discovered, views the quantum electromagnetic vacuum as containing fluctuations but no photons.[15] However, a thermal spectrum does indeed involve photons. Thus in the quantum description there is a problem as to how by acceleration one goes from a situation of no photons to a situation where there are photons, or at least seem to be photons. The usual quantum analysis speaks of virtual photons in the vacuum being made real by acceleration, or of other such ideas. Indeed there is some uncertainty[16] as to whether an accelerating quantum detector records a virtual photon, making it real, or just what. In this connection authors from the quantum viewpoint often look back to the classical situation and refer to an article by Mould[17] dealing with the detection of signals by an accelerating classical system.

This puzzlement over the correct quantum description stands in sharp contrast to the present classical analysis for the thermal effects of acceleration. Classical electron theory with classical electromagnetic zero-point radiation regards the zero-point radiation as real, just as real as thermal radiation which is always treated as radiation above the zero-point energy level. Thus on acceleration the change in the spectrum is not a change from something virtual to something real. In this article we have carried through the calculation for the behavior of a small classical oscillator undergoing uniform acceleration through classical electromagnetic zero-point radiation. We find that the oscillator responds exactly as it would in an inertial frame when bathed by Planck's spectrum of thermal radiation including zero-point radiation.

ACKNOWLEDGMENT

I wish to thank Daniel Cole for his assistance in proofreading the manuscript.

[1]W. G. Unruh, Phys. Rev. D 14, 870 (1976).

[2]P. C. W. Davies, J. Phys. A 8, 609 (1975).

[3]See the reviews of stochastic electrodynamics by T. H. Boyer, in *Foundations of Radiation Theory and Quantum Electrodynamics,* edited by A. O. Barut (Plenum, New York, 1980), and by L. De la Peña, in *Proceedings of the Latin American School of Physics, Cali, Colombia, 1982,* edited by B. Gomez *et al.* (World Scientific, Singapore, 1983).

[4]T. H. Boyer, Phys. Rev. D 21, 2137 (1980).

[5]See, for example, P. Candelas and J. S. Dowker, Phys. Rev. D 19, 2902 (1979).

[6]See, for example, Ref. 5 or P. Candelas and D. Deutsch, Proc. R. Soc. London A354, 79 (1977). In a footnote on p. 93 of their article Candelas and Deutsch write that the radiation they find is "precisely thermal" "in the sense that if thermal radiation of temperature $(2\pi\xi)^{-1}$ were added, the resulting state would be indistinguishable, as $\xi/a \rightarrow \infty$ from the Minkowski vacuum . . . ."

[7]See the discussion by T. H. Boyer, Phys. Rev. D 11, 790 (1975).

[8]T. W. Marshall, Proc. Cambridge Philos. Soc. 61, 537 (1965); T. H. Boyer, Phys. Rev. 182, 1374 (1969).

[9]See, for example, S. Coleman, in *Electromagnetism: Paths to Research,* edited by D. C. Teplitz (Plenum, New York, 1982). This is the publication of the unpublished Rand Corporation report of 1961.

[10]See the discussion of hyperbolic motion on p. 2139 of Ref. 4, or the discussion by F. Rohrlich, in *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965), Sec. 6-11, pp. 169—172.

[11]See Ref. 4 and combine Eqs. (68), (73), (77), and (78).

[12]The narrow-linewidth approximation gives the valid result despite the logarithmic divergence of the integral for $\langle (dx/d\tau)^2 \rangle$. The integral is actually cut off by relativistic particle behavior.

[13]See, for example, T. H. Boyer, Phys. Rev. D 11, 809 (1975).

[14]D. W. Sciama, P. Candelas, and D. Deutsch, in Adv. Phys. 30, 327 (1981), have emphasized that the random radiation seen in the accelerating frame takes Gaussian form. This can

be proved in the present classical analysis by considering the higher correlation functions

$$\langle E^{zp}_{\tau i}(0,\tau)E^{zp}_{\tau' i'}(0,\tau') \cdots E^{zp}_{\tau'' i''}(0,\tau'')\rangle \ .$$

Using the random phases appearing in Eq. (15), we can carry through an analysis parallel to that of Ref. 13. The higher correlation functions of odd order vanish and those of even order break up into products of two-field correlation functions. Also the higher correlation functions $\langle x^n(\tau)\rangle$ from (19) and (21) can be evaluated exactly as indicated in Ref. 13. Working in a narrow-linewidth approximation we see that the $\langle x^n(\tau)\rangle$ indicate a Gaussian random process.

[15]See, for example, the discussion by D. W. Sciama, in Ann. N.Y. Acad. Sci. 302, 161 (1977).
[16]See, for example, the comments by N. Sanchez, in the Introduction of Phys. Rev. D 24, 2100 (1981).
[17]R. A. Mould, Ann. Phys. (N.Y.) 27, 1 (1964).