Wave function of the inflationary universe

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(Received 19 September 1983)

If one is interested in cosmological models with inflation occurring at or near the Planck energy then the effects of quantum gravity have to be taken into account. We may still hope to use an approximation with the Einstein action as a starting point but the semiclassical approach breaks down, forcing us to use the superspace construction for the wave function of the Universe. Using a recently proposed formulation of the ground state of the Universe, we obtain a unique quantum cosmology for a given inflationary model. Besides recovering the semiclassical inflationary picture, the wave function has other consequences, such as the following: (1) The wave function is suppressed for small values of the scale factor and can therefore be interpreted as corresponding to the nucleation of a bubble universe. (2) The parameters of the wave function become uncorrelated at small values of the scale factor which means that the concept of time breaks down. (3) The wave function at later times possesses structure on scales smaller than the Planck scale which leads to a spacetime-foam picture of the Universe. We may therefore view the existence of the Universe itself as a consequence of inflation.

I. INTRODUCTION

Although a great deal is known about the equations which govern the evolution of the Universe, relatively little work has been done towards finding out what the initial conditions should be or equivalently why the Universe follows the particular cosmological model that it does. An important step forward has been the discovery of the inflationary models[1,2] in which a wide range of initial conditions leads to the same outcome of a universe closely approaching a spatially flat Robertson-Walker model.

These inflationary universes have so far been semiclassical in structure involving one-loop quantum effects only in the effective potential of some scalar Higgs field. We wish to proceed a little further in this paper to consider quantum-gravity corrections using a minisuperspace formalism[3–5] in which the quantum state of the Universe is described by a wave function depending upon the spatial geometry and matter fields and satisfying the Wheeler-DeWitt equation.[3,4] Then the problem of specifying the initial conditions is replaced by one of selecting a particular solution and for this we shall employ the suggestion of Hartle and Hawking[6] for defining the vacuum state of the Universe. Our results show that this gives a realistic quantum cosmology with grand unified, inflationary, and broken-symmetry eras.

In the classical version of the inflationary universe, the Universe starts out in a high-temperature phase of unbroken symmetry. This symmetry is broken when a Higgs field ϕ evolves away from the symmetric value zero towards the minimum of the Higgs potential. If this potential is sufficiently flat near zero, then the vacuum energy of the symmetric state will come to dominate the energy density of the Universe leading to an exponential expansion of the scale factor. As a consequence of this, the spatial hypersurfaces become almost flat which explains why the Universe today is homogeneous on large scales with a density close to the critical density. Furthermore, the energy density in the form of radiation becomes vanishingly small but more is recreated from the decay of oscillations in the Higgs field about the minimum of the potential after the inflationary period is over. We therefore learn very little about the nature of the Universe prior to the phase transition.

A variation of this picture, proposed by Vilenkin,[7] is that the Universe arose spontaneously in the inflationary phase through a quantum-tunneling or bubble-nucleation event from "nothing" or from a preexisting phase where the classical notation of spacetime breaks down (Fig. 1). The decay of the false vacuum would then lead to causally separate Friedmann universes lying in a de Sitter metauniverse[8] because of the rapid expansion of de Sitter space and the slow transition rate (Fig. 2).

In either picture the homogeneous inflationary universe has spatial fluctuations imposed upon it arising from quantum fluctuations in the matter fields.[9] Most inflationary models considered so far lead to too much inhomogeneity to be consistent with the observed isotropy of the cosmic-microwave-background radiation unless the gauge coupling constant is fine tuned. This problem can

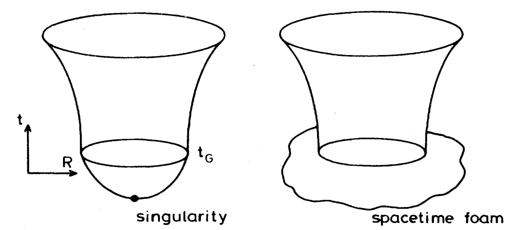

FIG. 1. The scale factor R is sketched as a function of time in the two versions of the inflationary universe.

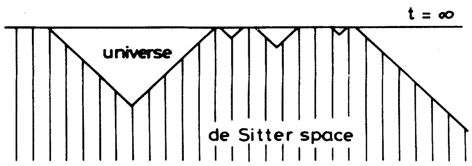

FIG. 2. This conformal diagram shows universes nucleating in a de Sitter metauniverse.

be less severe in primordial-inflation models[10] where the unification occurs at energies near the Planck energy of 10^{19} GeV at which quantum gravity becomes important. Furthermore, grand unified theories with larger symmetry groups usually involve more fields and consequently higher unification energies which can easily be close to the Planck energy.

We have chosen to approach this problem by constructing a wave function Ψ which represents a quantum cosmological model. We shall choose an inflationary Higgs potential

$$\tfrac{1}{6}R\phi^2+V(\phi)\ , \tag{1}$$

where

$$V(\phi)=\tfrac{1}{2}\alpha^2\phi^4\left[\ln\frac{\phi^2}{\phi_0{}^2}-\frac{1}{2}\right]+\tfrac{1}{4}\alpha^2\phi_0{}^4\ . \tag{2}$$

Here R is the Ricci scalar and α,ϕ_0 are coupling constants.

Inflation would be expected to take place in this model because the potential possesses a metastable minimum at $\phi=0$, a global minimum at $\phi=\phi_0$, and a low potential barrier between them (Fig. 3). The choice of the quadratic term in (1) was determined by our desire to include the effects of the dominant curvature term in the potential but with the coupling constant chosen to simplify our analysis. Although we do not claim that (1) is the true effective potential, it possesses most features that are required for a quantum inflationary model.

We shall also employ the most stringent minisuperspace hypothesis, that is, we consider wave functions Ψ which depend only on a homogeneous field ϕ and the Robertson-Walker scale factor a. As usual with this approach, there is no dependence of Ψ upon time because a choice of the scale factor fixes the position of a spatial hypersurface within spacetime, but nevertheless all the large-scale information about the Universe, such as the amount of inflation and the entropy, should be obtainable from Ψ. Furthermore, one might expect our approach to quantum gravity to hold approximately even when the radius of curvature is of the order of the Planck length 10^{-33} cm or even smaller. Thus, one may ask, for example, what the probability is of finding a universe with a size less than the unification-epoch Hubble radius in order to test the validity of the Vilenkin picture for which this probability should be vanishingly small.

The equation satisfied by Ψ is the version of Schrödinger's equation known as the Wheeler-DeWitt equation,[3,4]

$$\left[\tfrac{1}{2}a^2a^{-p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a}-\frac{1}{2}\left[\frac{3m_P{}^2}{4\pi}\right]^2a^4+\frac{1}{2}\left[\frac{3m_P{}^2}{4\pi}\right]a^2\left[-\frac{\partial^2}{\partial\tilde{\phi}^2}+\tilde{\phi}^2\right]+\frac{3m_P{}^2}{4\pi}a^6V(a^{-1}\tilde{\phi})\right]\Psi=0\ , \tag{3}$$

where the dimensionless field $\tilde{\phi}=a\phi$ has been introduced in order to diagonalize the differential operator. The factor p represents the arbitrariness introduced by the factor-ordering problem. Fortunately, this affects the behavior of the wave function only when the scale factor is less than the Planck length and does not have a significant effect on the physical interpretation of Ψ. We shall therefore take $p=0$.

In order to realize an effective quantum cosmological model of this kind, one needs to specify which solution of (2) corresponds to the Universe. A way of uniquely defining the vacuum state of (2) using a Euclidean path integral has recently been suggested by Hartle and Hawking.[6] The expression they used is

$$\Psi(a,\phi)=\int[da][d\phi]e^{-S_E[a,\phi]}\ , \tag{4}$$

where S_E is the Euclidean action, i.e., $t\to it$, and the path integral is taken over paths $a(t)$ and $\phi(t)$ which go from zero to the end points a,ϕ. Unfortunately, taking the simplest models for $V(\phi)$ leads to empty universes. This can be avoided by choosing excited states but then the attractive uniqueness is lost. The inflationary model (1), on the other hand, releases latent heat which can give large energy densities even in the vacuum state. This encourages us to investigate the quantum cosmology which corresponds to the Hartle-Hawking definition of the vacuum state of the inflationary universe. One may contrast this with the usual approach to the initial conditions of cosmology where the Universe is assumed to leave the Planck era in thermal equilibrium with at least enough spatial flatness and homogeneity to reach the unification scale in the case of the inflationary universe, or considerably more flatness and homogeneity in a noninflationary scenario.

During the inflationary era, the spacetime within an observer's event horizon rapidly approaches the empty de Sitter spacetime[1,2] (a process known as cosmic baldness)

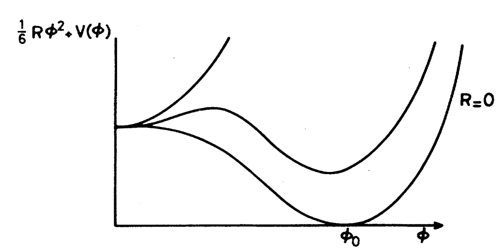

FIG. 3. The Higgs potential $\frac{1}{6}R\phi^2+V(\phi)$ is sketched for various values of the curvature R.

and consequently becomes homogeneous. It is also known that the effects of quantum fluctuations on inflation are dominated by homogeneous modes of the fields[11] because these can fluctuate most rapidly being essentially massless. One would therefore expect that neglecting inhomogeneous modes in the wave function should be an even more reasonable approximation for these inflationary models than for the usual minisuperspace models considered by previous authors.[3–6]. There should, however, be no obstacles in extending the model to include some effects of nonisotropy or nonhomogeneity.

When an inhomogeneity is larger than the event horizon then the region inside the horizon can be considered as if it were a separate, homogeneous universe. This is what happens in the metauniverse picture. We may therefore look at the metauniverse as an ensemble of nearly homogeneous universes. When we go over to the fully quantum-mechanical picture this is represented by the full wave function. If we restrict ourselves to homogeneous modes, this suggests that a quantity like $|\Psi(a,\tilde{\phi})|^2$ can be interpreted as the probability density of measuring the values a and $\tilde{\phi}$ in a random universe from the metauniverse. Comparing this with DeWitt's interpretation of the wave function[3] in terms of a many-worlds' picture we see no reason to distinguish between the two, that is, the Universes of the semiclassical-metauniverse picture should be included in the worlds of the many-world's picture under canonical quantization.

With this interpretation of $|\Psi|^2$ we do not expect any probability-conservation equation, but $|\Psi|^2$ should be normalized to unity over the whole $a,\tilde{\phi}$ plane. There does exist a conserved current corresponding to Eq. (3),[3] but we reject interpreting any of its components as a probability because this would be superspace-coordinate dependent and can lead to negative probabilities.

As might be expected, the solution of (3) and (4) is quite complicated. This has led us to consider a preliminary examination of a quartic potential with a negative coupling constant in Sec. II. We find that the wave function in this case is exponentially suppressed in the region where the scale factor a is less than the unification-epoch horizon size H^{-1} and can therefore be interpreted as the nucleation of a bubble universe. The scalar field also tunnels through the potential barrier from the metastable values near $\phi=0$, where the wave function resembles the Hartle-Hawking result,[6] to larger values where the symmetry is broken. This simple behavior of the wave function still occurs for small ϕ or a when the full potential (1) is used allowing us to integrate (3) numerically using the result for small a as initial data. These numerical results are given in Sec. IV.

With the introduction of the inflationary potential in Sec. III, the wave function becomes suppressed at large values of ϕ but develops some interesting behavior around the global minimum of the Higgs potential near $\phi=\phi_0$. This is where, in the semiclassical picture of the inflationary universe, the vacuum energy gets transformed into Higgs-field oscillations which behave like pressure-free dust. It is possible to determine the amount of inflation from this or from the maximum size of the Universe. We find that the fall-off of the wave function with increasing radius is rather flat but a natural scale of inflation around $(m_P/H)^{3/2}$ emerges. This is insufficient to give rise to the present-day universe which requires a value of around 10^{23} unless H is very small, but this is to be expected because previous calculations have suggested that the conformally coupled mass term (1) is too steep. We expect that a smaller mass term will give the correct scale for our universe.

An interesting feature that turns up in the probability amplitude is the existence of a microstructure below the Planck length in the region around $\phi=\phi_0$. The Universe seems to jump between values of a rather than move smoothly. When the inhomogeneous modes are taken into account, these values of a will vary from place to place and one would see the spacetime-foam picture of Wheeler and Hawking emerging.[12]

II. THE CONFORMALLY COUPLED SCALAR FIELD

We shall consider first of all the case of a scalar-field potential which has the form

$$\tfrac{1}{6}R\phi^2+V(0)-\tfrac{1}{4}\lambda\phi^4 , \tag{5}$$

where λ and $V(0)$ are positive constants. If the dimensionless field $\tilde{\phi}=a\phi$ is used then the action can be expressed as the sum of two terms:

$$S_E[a,\tilde{\phi}]=S_E[a]+S_E[\tilde{\phi}] , \tag{6}$$

where

$$S_E[a]=-\int\left[\frac{1}{2}\left[\frac{3m_P{}^2}{4\pi}\right]a'^2+\frac{1}{2}\left[\frac{3m_P{}^2}{4\pi}\right]a^2 - V(0)a^4\right]d\eta ,$$

$$S_E[\tilde{\phi}]=\int\left[\tfrac{1}{2}\tilde{\phi}'^2+\tfrac{1}{2}\tilde{\phi}^2-\frac{\lambda}{4}\tilde{\phi}^4\right]d\eta . \tag{7}$$

Here η is the conformal time parameter which satisfies $a\,d\eta=dt$ and the prime denotes differentiation with respect to η. As a consequence, the Hartle-Hawking vacuum state defined by (4) decomposes into a product,

$$\Psi(a,\tilde{\phi})=\Psi(a)\Psi(\tilde{\phi}) , \tag{8}$$

where, in order to keep down the number of indices, the two separate functions $\Psi(a)$ and $\Psi(\tilde{\phi})$ will be distinguished by their arguments.

Now it is clear that the path integral for $\Psi(\tilde{\phi})$ will not converge because the potential (5) is not bounded below. This is a problem also encountered in statistical mechanical problems[13] where the resolution is to evaluate the integral for negative λ and then perform an analytic continuation in λ. Proceeding in this way, one therefore needs to calculate

$$\Psi(\tilde{\phi})=\int[d\tilde{\phi}]\exp\left[-\int(\tfrac{1}{2}\tilde{\phi}'^2+\tfrac{1}{2}\tilde{\phi}^2+\tfrac{1}{4}\hat{\lambda}\tilde{\phi}^4)d\eta\right] , \tag{9}$$

where $\hat{\lambda}=-\lambda$ and the paths $\tilde{\phi}(\eta)$ go from 0 to $\tilde{\phi}$. This integral can be evaluated in the steepest-descents approximation. Then the dominant contribution comes from a solution of

$$\tilde{\phi}''=\tilde{\phi}+\hat{\lambda}\tilde{\phi}^3 \tag{10}$$

such that $\tilde{\phi}(\eta)\to 0$ as $\eta\to-\infty$. We therefore take the solution

$$\tilde{\phi}(\eta)=\left[\frac{2}{\hat{\lambda}}\right]^{1/2}(\sinh\eta)^{-1} \tag{11}$$

and the corresponding first approximation to Ψ is

$$\Psi(\tilde{\phi})\approx\exp\left[-\frac{2}{3\hat{\lambda}}[(1+\tfrac{1}{2}\hat{\lambda}\tilde{\phi}^2)^{3/2}-1]\right] . \tag{12}$$

When we analytically continue $\hat{\lambda}$ we get

$$\Psi(\tilde{\phi})\approx\exp\left[\frac{2}{3\lambda}[(1-\tfrac{1}{2}\lambda\tilde{\phi}^2)^{3/2}-1]\right] , \tag{13}$$

which is sketched in Fig. 4(a). For small $\tilde{\phi}$ this wave function has the same form as a simple-harmonic-oscillator ground state, but at larger $\tilde{\phi}$ there is an oscillating tail where, because the analytic continuation in λ is complicated by a branch cut, we have an ambiguity in the sign of the exponent. One choice of sign can be associated with tunneling away from the metastable minimum of the potential but we shall return to this point later.

When one examines the integral for $\Psi(a)$, it has a very similar form to that for $\Psi(\tilde{\phi})$ except that the action has the opposite sign. This is a familiar problem in quantum gravity that the action is not positive definite in the conformal modes.[14] We shall make the usual ansatz that the contour of integration of the conformal fluctuations about a stationary path of the integral should be rotated into the complex plane to give a well-defined result. This must then be essentially identical in form to (12) but with the opposite overall sign in the exponent,

$$\Psi(a)\approx\exp\left[-\frac{m_P{}^2}{4\pi H^2}[(1-H^2a^2)^{3/2}-1]\right] , \tag{14}$$

where

$$H^2=\frac{8\pi V(0)}{3m_P{}^2} , \tag{15}$$

which is sketched in Fig. 4(b).

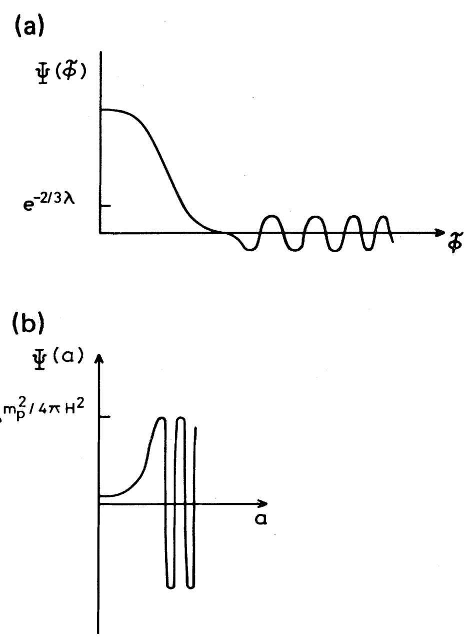

FIG. 4. The wave function is shown as a function of (a) $\tilde{\phi}$ and (b) a for the conformal theory.

The branch-cut ambiguity in (14) can be resolved by use of the Wheeler-DeWitt equation (3). For the separable wave function (8) we have

$$\left[-\frac{d^2}{d\tilde{\phi}^2}+\tilde{\phi}^2-\tfrac{1}{2}\lambda\tilde{\phi}^4\right]\frac{3m_P{}^2}{4\pi}\Psi(\tilde{\phi})=E\Psi(\tilde{\phi}) ,$$
$$\left[-\frac{d^2}{da^2}+\left[\frac{3m_P{}^2}{4\pi}\right]^2a^2-H^2\left[\frac{3m_P{}^2}{4\pi}\right]^2a^4\right]\Psi(a)=E\Psi(a) . \tag{16}$$

The solutions (13) and (14) are indeed approximate solutions of (16) provided that E is real, otherwise the exponents would contain a growing real part for large a or $\tilde{\phi}$. Consequently, Ψ must also be real as can be seen by examination of the conserved currents of Eq. (16). We therefore conclude that one must add together both branches of the wave functions. These may be thought of as corresponding to the expanding and contracting phases of the Universe.

We can now compare our results with those of Hartle and Hawking[6] who considered the case $\lambda=0$. Although they performed a path integration over the expansion rate rather than the scale factor in order to avoid ambiguities, their final result for $\Psi(a)$ is the same as our Eq. (14). Their wave function for $\tilde{\phi}$ also resembles Eq. (13) when $\tilde{\phi}$ is small but does not display the tunneling features, as one would expect in their model.

This suggests an alternative approach to finding the vacuum state for more complicated potentials such as (2) which avoids the difficulty of evaluating the path integral (4). This is to integrate the Wheeler-DeWitt equation directly using the path integral to provide the initial data only. When the initial data is required on $\tilde{\phi}=0$ then the Hartle-Hawking result (14) can be used. We shall perform such an integration numerically in Sec. IV.

In order to find a better approximation to the wave function we could integrate the quadratic fluctuations around the saddle-point path. We can, however, derive the same result by taking the JWKB approximation in Eq. (16) which gives

$$\Psi(a)\approx\frac{c_1}{Ha}(e^{im_P{}^2Ha^3/4\pi}+e^{-im_P{}^2Ha^3/4\pi}) ,$$
$$\Psi(\tilde{\phi})\approx\frac{c_2}{\tilde{\phi}}(e^{i\sqrt{\lambda/2}\tilde{\phi}^3/3}+e^{-i\sqrt{\lambda/2}\tilde{\phi}^3/3}) \tag{17}$$

for large a and $\tilde{\phi}$ where $c_1\approx 0.9\lambda^{1/4}\,(H/m_P)^2$ and $c_2\approx 0.6\lambda^{-1/4}$. We can therefore identify the factors multiplying the exponentials with the one-loop corrections to the path integrals. This result will be of use in Sec. III when we try to derive an analytic approximation to the quantum inflationary universe.

III. THE INFLATIONARY UNIVERSE

When the full potential (1) is used, we are no longer able to decompose the wave function into a product because the conformal invariance is lost. Nevertheless, there are regions in $\tilde{\phi}$-a space where the coupling between $\tilde{\phi}$ and a is small because the logarithm varies only slowly compared to the other terms in the potential. In these regions the logarithm acts as an effective negative λ term and we can be guided by our experience from Sec. II. We have found that the dominant contribution to the path integral comes from classical paths at the saddle points of the action

$$S_E=\int\left[\frac{1}{2}\frac{3m_P^2}{4\pi}a'^2-\frac{1}{2}\frac{3m_P^2}{4\pi}a^2-\tfrac{1}{2}\tilde{\phi}'^2-\tfrac{1}{2}\tilde{\phi}^2-a^4\hat{V}(a^{-1}\tilde{\phi})\right]d\eta\ , \tag{18}$$

where the replacement $\alpha=i\hat{\alpha}$ has been made in V. Of the analytic continuations which we have tried, only this one in α gives consistent results.

We therefore get the equations

$$\begin{aligned}\tilde{\phi}''&=\tilde{\phi}-2\hat{\alpha}^2\tilde{\phi}^3\ln\frac{\tilde{\phi}^2}{a^2\phi_0^2}\ ,\\ a''&=a-\frac{4\pi}{3m_P^2}\hat{\alpha}^2a^{-1}(\tilde{\phi}^4-a^4\phi_0^4)\end{aligned} \tag{19}$$

with the boundary conditions that $\tilde{\phi}$ and a vanish as $\eta\rightarrow-\infty$. We will find approximate solutions when $H<2\alpha\phi_0$. The alternative case $H>2\alpha\phi_0$, which can lead to inflation only when $\phi_0>m_P$, will be considered in a separate publication.

First of all, consider the paths to a point P with small values of the coordinate $\tilde{\phi}_P$. In this case the logarithm can be approximated by

$$l=-\ln\frac{\phi_0^2}{H^2} \tag{20}$$

and the equations of motion (19) decouple. A typical path in this region is shown in Fig. 5(a). As in Sec. II, the Euclidean action for this path splits up into the sum of two terms and the wave function can therefore be given by the product of Eqs. (13) and (14) with

$$\begin{aligned}\lambda&=2\alpha^2(l+\tfrac{1}{2})\ ,\\ H^2&=\frac{2\pi\alpha^2\phi_0^4}{3m_P^2}\ .\end{aligned} \tag{21}$$

Now let us turn to a general path which approaches the minimum of the potential at $\tilde{\phi}=a\phi_0$ as shown in Fig. 5(b). At first this path behaves as in the case above but then, at some value a_n of a, the path shoots across to oscillate around $\tilde{\phi}=a\phi_0$. In general there are several paths which reach P and may therefore contribute to the wave function. We shall enumerate these by the number of times n that they cross the line $\tilde{\phi}=a\phi_0$. We need to calculate the action of each of these paths which we shall do by dividing up the motion into two parts.

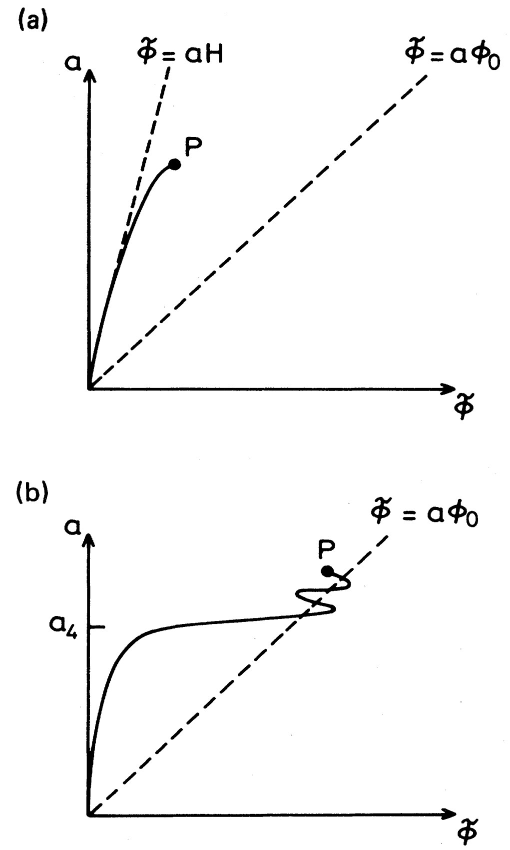

FIG. 5. Saddle-point paths are shown for (a) small $\tilde{\phi}_P$ and (b) $\tilde{\phi}_P$ near the minimum of the potential.

First of all, let us analyze the oscillatory part of the paths. This we may do by using the perturbed field $\psi=\tilde{\phi}-a\phi_0$ in the energy equation,

$$\frac{1}{2}\left[\frac{3m_P^2}{4\pi}\right]a'^2-\frac{1}{2}\left[\frac{3m_P^2}{4\pi}\right]a^2-\tfrac{1}{2}\tilde{\phi}'^2+\tfrac{1}{2}\tilde{\phi}^2+a^4\hat{V}(a^{-1}\tilde{\phi})=0\ , \tag{22}$$

which gives, up to order ψ^3,

$$\begin{aligned}&\frac{1}{2}\left[\frac{3m_P^2}{4\pi}\right]a'^2-\frac{1}{2}\left[\frac{3m_P^2}{4\pi}\right]a^2=E(\eta)\ ,\\ &\tfrac{1}{2}\psi'^2-\tfrac{1}{2}\psi^2+2\hat{\alpha}^2a^2\phi_0^2\psi^2=E(\eta)\ ,\end{aligned} \tag{23}$$

where $E(\eta)$ is slowly varying with respect to the ψ oscillations. We can therefore use adiabatic approximations

$$\begin{aligned}\psi&=\frac{1}{\sqrt{8}}\phi_0a_n^{3/2}a^{-1/2}\cos m_Ht\ ,\\ a&=\tfrac{1}{2}H^2a_n^3(\cosh\eta-1)\ ,\end{aligned} \tag{24}$$

where the Higgs mass m_H has been introduced,

$$m_H = 2\hat{\alpha}\phi_0 \,, \tag{25}$$

and the time t is given by

$$t - t_n = \int_{\eta_n}^{\eta} a\, d\eta \,. \tag{26}$$

We are now able to evaluate a_n from these formulas by using the fact that increasing n by 1 corresponds to including an extra half oscillation of ψ. Thus,

$$m_H \int_{\eta_n}^{\eta_P} a\, d\eta = n\pi + \delta \,, \tag{27}$$

where δ is a phase which depends upon ψ_P.

The motion (24) can now be substituted into the action (18) together with the earlier stage where the approximation (20) can be used. When the total result is analytically continued to imaginary values of $\hat{\alpha}$ one finds that the Lorenzian action S_n is wholly real for $a_n > a_c$ where

$$H a_c = (H a_P)^{1/3} \,. \tag{28}$$

On the other hand, the action contains an imaginary part for $a_n < a_c$ and consequently such paths are exponentially suppressed. These paths correspond to classical Friedmann models which recollapse before reaching the size a_P. For the remaining paths we can use (27) to obtain a simple expression for a_n,

$$a_n = \left[1 + \frac{n}{n_1} \right]^{-2/3} a_P \,, \tag{29}$$

where

$$n_1 = \frac{2}{3\pi^{5/4}} \left[\frac{\alpha m_P}{H} \right]^{1/2} . \tag{30}$$

The dependence upon ψ_P coming from δ can be ignored but there is another coming from the condition that the amplitude of the oscillations be greater than ψ_P. Thus,

$$\psi_P < \frac{1}{\sqrt{8}} \frac{n_1}{n} a_P \phi_0 \,. \tag{31}$$

The calculations presented above lead to a wave function of the form

$$\Psi = \sum_n A_n e^{iS_n} \tag{32}$$

If the integral over fluctuations about the stationary-phase paths is ignored, so that $A_n = 1$, then the wave function is very nearly zero because of the large variation in the phases S_n. We can, however, make use of the fact that S_n is large by averaging the probability

$$|\Psi|^2 = \left[\sum_n A_n \right]^2 - 2 \sum_{n \neq m} A_n A_m \sin^2[(S_n - S_m)/2] \tag{33}$$

over some scale w for which A_n is nearly constant. Thus,

$$|\Psi|_{\rm AV}^2 = \sum A_n^2 \,. \tag{34}$$

The one-loop term A_n comes principally from fluctuations about the initial section of the path up to $a \approx a_n$, $\tilde{\phi} \simeq a_n \phi_0$ where we know from the JWKB approximation (17) that

$$A_n \approx c_3 (H a_n)^{-2} \tag{35}$$

with $c_3 = c_1 c_2$. Using the expression (28) for a_n gives

$$|\Psi|_{\rm AV} \approx c_3^2 (H a_P)^{-4} \sum_0^{n_c} \left[\frac{n}{n_1} \right]^{8/3} , \tag{36}$$

where the upper limit $n_c - (H a_P) n_1$ follows from the lower limit a_c on a_n (28). If, however, ψ_P is too large to satisfy inequality (31) at $n = n_c$ then the sum must be reduced accordingly.

Using the Euler-Maclaurin approximation we get

$$|\Psi|_{\rm AV}^2 \approx \frac{3c_3^2}{11} (H a_P)^{-1/3} n_1 \,. \tag{37}$$

This is our main result for the probability distribution of the size of the Universe. It is rather flat and does not lead to an average size but scales $H a_P \gg n_1^3$ are suppressed. The Universe today has $m_P a_P \gtrsim 10^{23}$ which would have a small probability by (28) unless H was quite small. It is more likely that the potential (1) is not sufficiently flat for a prolonged period of inflation and a smaller quadratic term must be used in a more realistic model.

The classical interpretation of the wave function is quite clear. Values of the scale factor $a < H^{-1}$ are improbable, indicating that the Universe arises like a bubble. Classical paths contributing to the path integral contain an inflationary period prior to $a = a_n$ and those that contribute most to the wave function come from values of n around n_c in which the scale factor inflates by an amount n_1^3. The part of the wave function around $\phi = \phi_0$ represents a Friedmann universe with energy density in the form of homogeneous scalar-field oscillations which have the same equation of state as pressure-free dust. From the width of this part of the wave function we can read off the energy density as a function of the scale factor. Indeed, the semiclassical approximation becomes very good from this point on. In a more realistic model, however, these oscillations would decay into radiation as in the usual inflationary universe.

Returning to Eq. (33) for the unaveraged probability, one sees that the probability density is very irregular on small scales. In fact, the order-of-magnitude approximation for S_n,

$$S_n \sim \left[\frac{m_P}{H} \right]^2 \left[\frac{n_1}{n} \right] (H a_P)^3 \,, \tag{38}$$

shows that the irregularity exists on wavelengths w such that

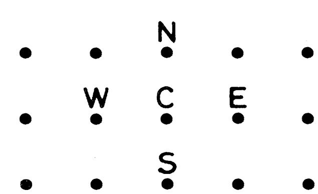

FIG. 6. The difference scheme for the numerical integration is given on the lattice shown here.

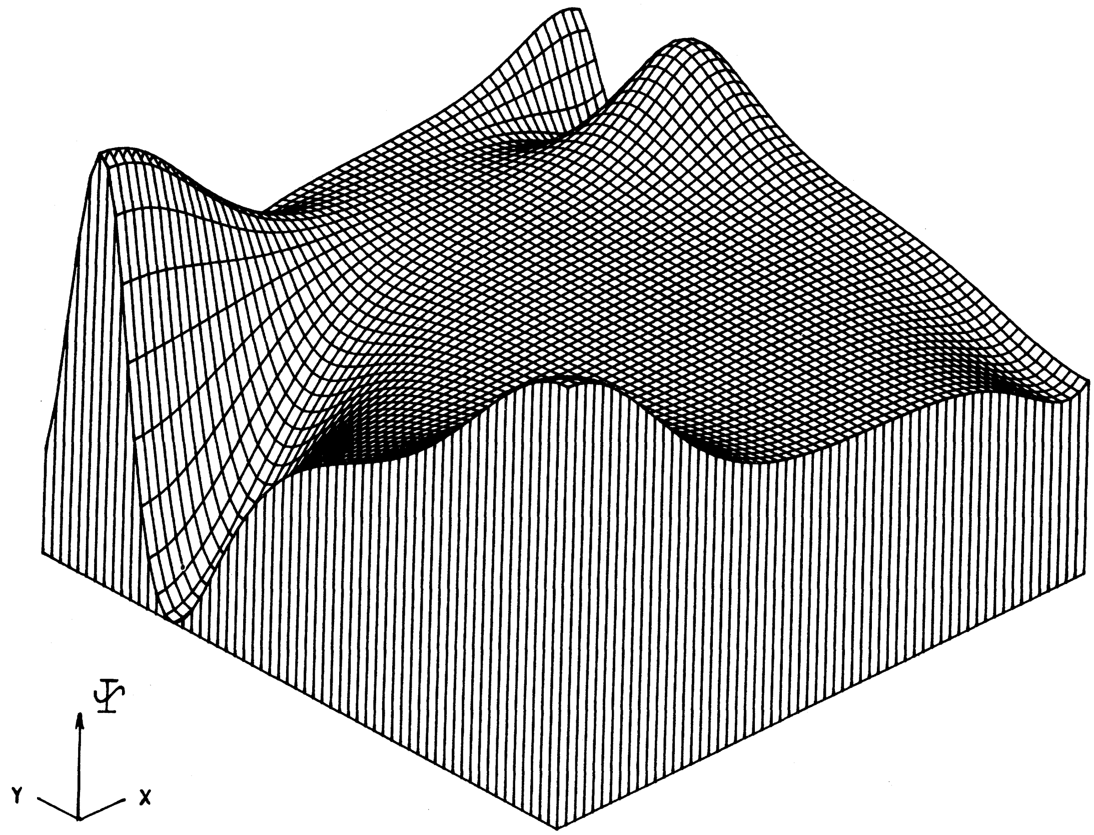

FIG. 7. The wave function is plotted here for the conformal theory with $\Lambda=0.12$, $\lambda=0.07$.

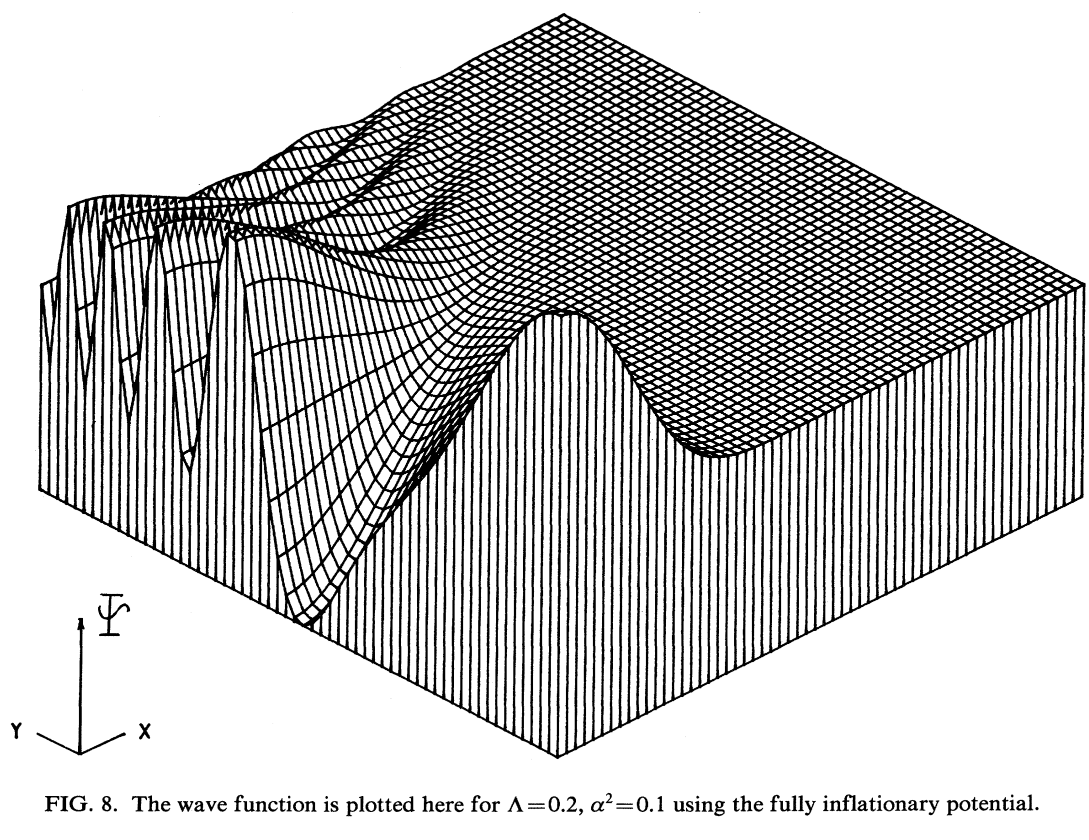

FIG. 8. The wave function is plotted here for $\Lambda=0.2$, $\alpha^2=0.1$ using the fully inflationary potential.

$$\left[\frac{H\alpha}{m_P}\right]^{3/2}(\alpha Ha)^{-2}m_P^{-1}\lesssim w\lesssim\left[\frac{H\alpha}{m_P}\right]^{1/2}m_P^{-1}\;, \tag{39}$$

which are all smaller than the Planck length. Presumably, when the inhomogeneous modes are taken into account, this leads to a foamy structure of spacetime.

It is interesting to note that the intrinsic nature of time has not hindered us at all in interpreting wave function. We can, however, consider the scalar field as a clock which is read by examining its correlations with the rest of the Universe. During the Friedmann era around $\phi=\phi_0$ the scale factor is closely correlated with the amplitude of the oscillations which is, in turn, related to the energy density of the scalar field. During the earlier era when the wave function is nearly a product, these correlations disappear altogether and the concept of time breaks down.

IV. NUMERICAL RESULTS

In order to check that the wave function behaves as predicted in Sec. III, we have solved Schrödinger's equation (3) numerically. After rescaling the variables, the equation takes the form

$$\Psi_{yy}-\Psi_{xx}+\omega^2\Psi=0\;, \tag{40}$$

where

$$y=\tilde{\phi},\quad x=\left[\frac{3m_P^{\,2}}{4\pi}\right]^{1/2}, \tag{41}$$

and ω^2 depends upon the form of $V(\tilde{\phi})$. This equation was solved on a lattice (Fig. 6) using a finite-difference scheme,

$$\begin{aligned}\Psi_{yy}-\Psi_{xx}&=(\Psi_N+\Psi_S-\Psi_E-\Psi_W)/2h\;,\\ \Psi&=\beta\Psi_C+(1-\beta)(\Psi_E+\Psi_W)/2\end{aligned} \tag{42}$$

with the parameter β chosen to give stability. For initial data we assumed that the wave function starts out as a direct product similar to Eq. (8) where $\Psi(x)$ is the lowest energy eigenstate of the separated-x equation.

First of all, we solved for the simplified potential of Sec. II,

$$\omega^2=x^2-\frac{\lambda}{2}x^4-y^2+\frac{\Lambda}{2}y^4\;, \tag{43}$$

where

$$\Lambda=\frac{8\pi H^2}{3m_P^{\,2}}\;. \tag{44}$$

The choice $\beta=\frac{1}{2}$ in (42) is required for stability against short-wavelength perturbations, but long-wavelength perturbations can cause problems when ω^2 is large and positive. We avoided these by introducing an extra boundary at a large value of x.

The wave function is plotted in Fig. 7 for $\lambda=0.07$ and $\Lambda=0.12$. The x axis is across the bottom and the y axis runs up the left-hand side. As we expect, the wave function continues to have the form of a product. The region

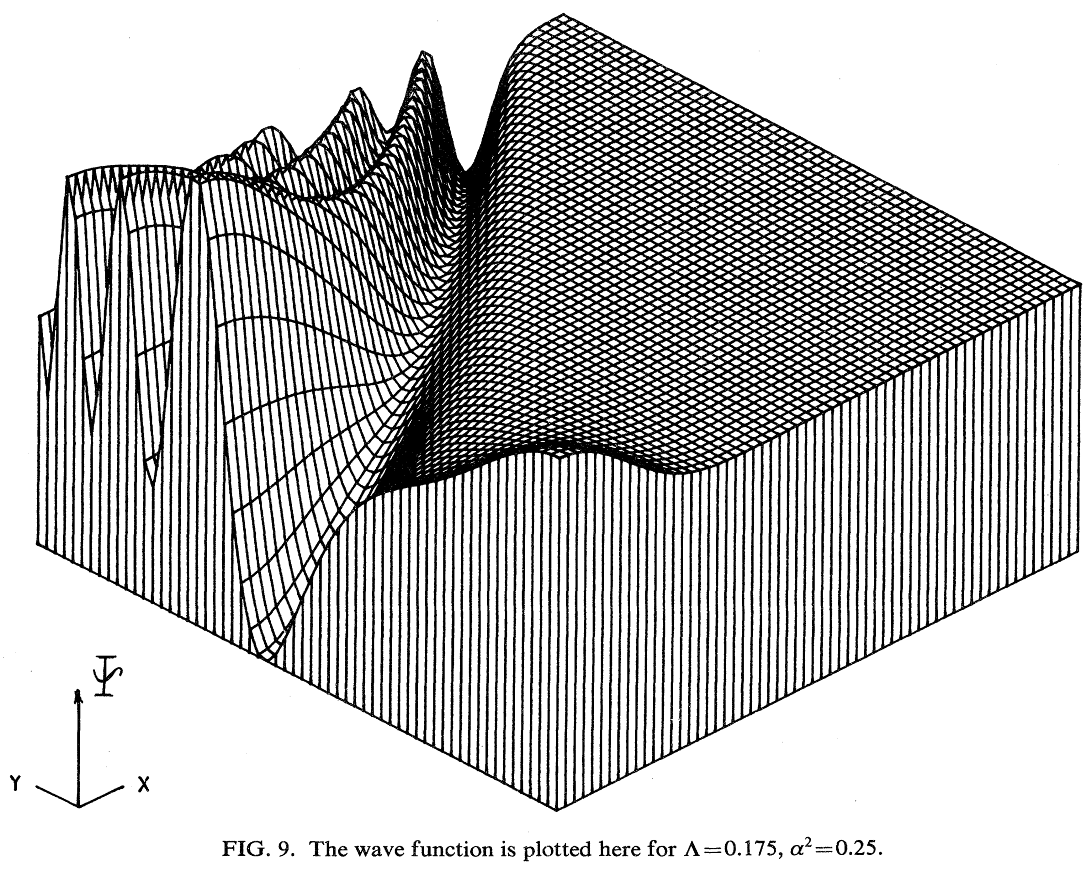

FIG. 9. The wave function is plotted here for $\Lambda=0.175$, $\alpha^2=0.25$.

near the y axis represents a situation of unbroken symmetry from which the Universe tunnels to larger, broken-symmetry values of x.

We next considered the full inflationary potential (2),

$$\omega^2=x^2+\alpha^2x^4\left[\ln\frac{c^2x^2+\gamma}{y^2+\gamma}-\frac{1}{2}\right]-y^2+\frac{\Lambda}{2}y^4\,, \quad (45)$$

where $c^4=\alpha^2/\Lambda$ and γ is a small constant chosen to prevent the logarithm from diverging on the x and y axis. The condition $H<2\alpha\phi_0$ of Sec. III becomes $c>1$.

In Fig. 8 we have plotted Ψ for $\Lambda=0.2$ and $\alpha^2=0.1$ to show what happens when $c<1$. The Universe remains in the symmetric state because of the large-effective-mass term in the potential.

In Fig. 9 we have plotted Ψ for $\Lambda=0.175$ and $\alpha^2=0.25$. The wave function is initially featureless and empty. At larger values, the oscillations in the symmetric region near the y axis give rise to waves which move across to the right. These waves separate themselves from the symmetric vacuum to congregate near the line $y=cx$, corresponding to our Friedmann universe with $\phi=\phi_0$. The region near the y axis resembles the $\lambda\phi^4$ theory above and the wave crests moving across correspond to the paths contributing to the path integral of Sec. III. This brings us to the end of the inflationary era and we find that the agreement with our analytic discussion is good. The subsequent evolution of the Universe follows essentially classical lines.

ACKNOWLEDGMENTS

We would like to express our gratitude to Stephen Hawking and Jim Hartle who each came to Newcastle to tell us about their ideas on the wave function of the Universe. We would also like to thank Stephen Hawking and Paul Davies for reading over the manuscript. Finally, we are grateful to the Science and Engineering Research Council for their financial support of this research.

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