Instanton chains with multimonopole limits: Lax pairs for non-axially-symmetric cases

A. Chakrabarti
Centre de Physique Théorique de l'Ecole Polytechnique, Plateau de Palaiseau, 91128 Palaiseau, Cedex, France
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A formalism is proposed for constructing sequences of instantons which give, in a simple scaling limit, separable multimonopole solutions. For this purpose essentially static techniques are adapted to the construction of finite-action solutions through a trick already used before for the axially symmetric case. For going beyond axial symmetry appropriate Lax pairs are constructed. Then, in the framework of Zakharov *et al.* for purely solitonic solutions, pole equations are solved and a suitable seed solution $\psi_{(0)}$ obtained. The restriction to axial symmetry in this framework is presented. The corresponding results in the monopole limit are shown to emerge trivially from those for instantons.

I. INTRODUCTION

Self-dual multimonopoles (corresponding to the Prasad-Sommerfield-Bogomolny limit) can be constructed *trivially* from instantons. The truth of this statement was demonstrated, for the axially symmetric case, in Ref. 1 [referred to hereafter as (I)]. Here I present a part of the results needed to establish it for the general case. The statement is also, to a certain extent, unjust. It is so because, in order to display *explicitly* the sequences ("chains") of instantons which yield the static multimonopole solutions in the infinite-action limit, I am profiting from the hindsight provided by certain methods recently used to construct multimonopole solutions. Let me add, however, that the monopole solutions, beyond indicating a general direction, do not automatically provide the prescriptions necessary for our generalization. At each step, crucial new ingredients have to be invented and the most obvious generalizations usually do not work. This is not surprising, since the monopole limit cannot contain the full information for the general case of which it is a limit. But once our formalism for instantons (possessing its own intrinsic interest even without direct reference to monopoles) is there, the monopoles indeed emerge trivially as a limit. (In the static limit A_t can be replaced by the Higgs scalar Φ. The term monopole will be used throughout in this, familiar, sense.)

In (I) the source of inspiration was the technique of Forgacs, Horvath, and Palla (FHP) who used Harrison-Neugebauer-type transformations (H-N) to construct nonlinear superpositions of monopoles.[2,3] To go beyond axial symmetry I am generalizing here, in a very particular fashion, the use FHP made[4,5] of the technique of Zakharov *et al.*[6–8] to construct separable multimonopole solutions. This technique is summarized in Appendix A and I will indicate in the concluding remarks what has been achieved for our case and what remains to be done. There are other remarkable approaches to the construction of multimonopole solutions.[9–13] Here I will be concerned only with the approach through Lax pairs.[5]

One point should be made clear immediately. A Lax-type linear pair in a time-dependent formalism has already been used for instantons by FHP.[14] But this is *not* suitable for my purpose. In order to get the particular sequences of instantons which yield directly the monopoles I adapt the *static* formalism [Harrison-type transformations in (I) and monopole Lax pairs here] for the finite-action case. I have explained the trick in several previous papers including (I). I present here again the essential features.

Starting with the flat Euclidean metric in the canonical spherical coordinates,

$$ds^2=dt^2+dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\ , \qquad (1.1)$$

the coordinate transformation

$$(t+ir)=\tan[\tfrac{1}{2}(\tau+i\rho)] \qquad (1.2)$$

gives

$$ds^2=(\cosh\rho+\cos\tau)^{-2}\times[d\tau^2+d\rho^2+\sinh^2\rho(d\theta^2+\sin^2\theta\, d\phi^2)]\ . \qquad (1.3)$$

(The overall factor will play no role in our problem due to the conformal properties of the gauge fields.) Now the domain

$$-\infty < t < \infty, \quad 0 \le r < \infty \tag{1.4}$$

corresponds to

$$-\pi \le \tau \le \pi, \quad 0 \le \rho \le \infty \ . \tag{1.5}$$

Since the "time" τ has a finite domain (2π), it is possible to construct τ-static (τ-independent) solutions of finite action. Again let

$$t' = \alpha\tau, \quad r' = \alpha\rho \ ,$$

and let

$$\alpha \to \infty \ . \tag{1.6}$$

Then

$$(4\alpha^2)ds^2 \equiv ds'^2 = dt'^2 + dr'^2 + r'^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{1.7}$$

where $-\infty < t' < \infty$, $0 \le r' < \infty$, and (apart from the primes to avoid confusion) one has again (1.1), which is thus linked in two different ways to (1.3). These elementary considerations furnish the key to instanton sequences leading to monopoles in the simplest possible fashion. With correctly chosen τ-static solutions as starting points one has the following picture:

$$\begin{array}{l}
\text{finite-action } (t\text{-dependent}) \text{ instantons} \\
\quad\Big\uparrow \ [\text{coordinate transformation}(1.2)] \\
\text{finite-action } \tau\text{-static self-dual solutions} \\
\quad\Big\downarrow \ [\text{rescaling}(1.6)] \\
t\text{-static self-dual monopoles of finite energy} \ .
\end{array}$$

Thus monopoles are linked up with instantons in a most direct fashion. But this is not my *only* aim. The classes of instantons that can be rendered τ static should be studied for their many intrinsically interesting properties even without reference to monopoles. See, for example, in Refs. 15 and 16, the study of Green's functions and quantum fluctuation determinants for the simplest one-chain. It was also pointed out in (I) that our technique permits, for the first time, the explicit construction of instantons outside the 't Hooft-Jackiw-Nohl-Rebbi class. I will come back in Sec. VI to this aspect of the n-chain for $n \ge 2$.

So using our technique we study interesting structures and properties, in the space of instanton solutions, for their own sake and capture effortlessly, "en passant," the whole self-dual monopole sector as a limit.

In this paper I consider throughout the SU(2) gauge group and all matrices are 2×2. (Many aspects of the formalism can however be directly generalized to higher-dimensional groups.)

The context in which the Lax pairs are introduced is explained in detail in Appendix B. Here it is sufficient to mention the following facts. Starting from (1.2) I define (as in Sec. V of Ref. 17 and Sec. IV of Ref. 18)

$$\begin{aligned} z &= \tfrac{1}{2}(\rho + i\tau), \quad \bar z = \tfrac{1}{2}(\rho - i\tau) \ ; \\ y &= \tan\frac{\theta}{2}e^{i\phi}, \quad \bar y = \tan\frac{\theta}{2}e^{-i\phi} \ . \end{aligned} \tag{1.8}$$

The self-duality constraints now can be given as a single nonlinear matrix equation (Appendix B)

$$\sinh^2\rho(G_z G^{-1})_{\bar z} + (1 + y\bar y)^2(G_y G^{-1})_{\bar y} = 0 \ , \tag{1.9}$$

since now

$$ds^2 \approx (\sinh\rho)^{-2}dz\, d\bar z + (1 + y\bar y)^{-2}dy\, d\bar y \ . \tag{1.10}$$

The τ-static case ($\partial_z \approx \partial_{\bar z} \approx \partial_\rho$) is the one of main interest. The appropriate Lax pairs are presented in the following section.

II. τ-STATIC LAX PAIRS FOR INSTANTONS

The following nonlinear matrix equation is to be solved, in terms of the coordinates (1.8)

$$\sinh^2\rho(G_\rho G^{-1})_\rho + (1 + y\bar y)^2(G_y G^{-1})_{\bar y} = 0 \ . \tag{2.1}$$

I introduce the following system, linear in the matrix ψ:

$$\begin{aligned} \sinh\rho D_1\psi &\equiv [\sinh\rho\partial_\rho - \Lambda(1 + y\bar y)\partial_{\bar y} \\ &\quad - (\cosh\rho + y\Lambda)\Lambda\partial_\Lambda]\psi \\ &= \sinh\rho(G_\rho G^{-1})\psi \ , \end{aligned} \tag{2.2}$$

$$(1+y\bar{y})D_2\psi \equiv [\Lambda \sinh\rho\,\partial_\rho + (1+y\bar{y})\partial_y + (\Lambda\cosh\rho - \bar{y})\Lambda\partial_\Lambda]\psi = (1+y\bar{y})(G_y G^{-1})\psi \ . \tag{2.3}$$

One has

$$[D_1, D_2] = 0 \tag{2.4}$$

and the consistency condition

$$[D_1, D_2]\psi = D_1(G_y G^{-1}\psi) - D_2(G_\rho G^{-1}\psi) = 0 \tag{2.5}$$

can be shown to imply (2.1).

In the formalism of Refs. 7 and 8 (see Appendix A) one looks for solutions of the form

$$\psi = \left[I + \sum_k (\Lambda - \mu_k)^{-1} R_k \right] \psi_{(0)} \ , \tag{2.6}$$

where the poles μ_k and the matrices R_k do not depend on the spectral parameter Λ and $\psi_{(0)}$ is a known solution obtained for a suitably simple $G = G_{(0)}$. The poles must satisfy the equations (Appendix A)

$$\mathscr{D}_1\mu \equiv \sinh\rho\,\mu_\rho - \mu(1+y\bar{y})\mu_{\bar{y}} + (\cosh\rho + y\mu)\mu = 0 \ , \tag{2.7}$$

$$\mathscr{D}_2\mu \equiv \mu \sinh\rho\,\mu_\rho + (1+y\bar{y})\mu_y - (\mu\cosh\rho - \bar{y})\mu = 0 \ . \tag{2.8}$$

Defining

$$B_\pm(\mu) \equiv \left[\frac{\mu e^{\pm\rho} - \bar{y}}{1 + \mu e^{\pm\rho} y} \right] \ , \tag{2.9}$$

it is found that

$$[\sinh\rho\,\partial_\rho - \mu(1+y\bar{y})\partial_{\bar{y}}]B_\pm(\mu) = \frac{e^{\pm\rho}(1+y\bar{y})}{(1+\mu e^{\pm\rho}y)^2}\mathscr{D}_1\mu \ , \tag{2.10}$$

$$[\mu\sinh\rho\,\partial_\rho + (1+y\bar{y})\partial_y]B_\pm(\mu) = \frac{e^{\pm\rho}(1+y\bar{y})}{(1+\mu e^{\pm\rho}y)^2}\mathscr{D}_2\mu \ . \tag{2.11}$$

Hence

$$B_+(\mu) = c_+ \ \text{or} \ B_-(\mu) = c_- \ , \tag{2.12}$$

where c_+, c_- are constants, are solutions. Moreover one can now solve the pole equations generally by setting

$$H(B_+(\mu), B_-(\mu)) = 0 \ , \tag{2.13}$$

where H is an arbitrary (but "nice") function of the arguments $B_\pm$. Corresponding to the operators $D_{1,2}$ of (2.2) and (2.3) one has

$$D_1 B_\pm(\Lambda) = 0 \ , \tag{2.14}$$

$$D_2 B_\pm(\Lambda) = 0 \ . \tag{2.15}$$

An interesting combination is

$$\Gamma = \frac{B_+ + B_-}{B_+ - B_-} \ . \tag{2.16}$$

Thus

$$\Gamma(\Lambda) = [\sinh\rho(1+y\bar{y})]^{-1}[y\Lambda - \bar{y}\Lambda^{-1} + (1-y\bar{y})\cosh\rho] \tag{2.17}$$

$$= (\sinh\rho)^{-1}[\sin\theta\tfrac{1}{2}(\Lambda e^{i\phi} - \Lambda^{-1}e^{-i\phi}) + \cos\theta\cosh\rho] \ . \tag{2.18}$$

The significance of $\Gamma(\mu)$ and $\Gamma(\Lambda)$ will be better understood under restriction to axial symmetry (see Sec. III). In order to construct $\psi_{(0)}$ one must first choose a suitable $G_{(0)}$. In (I) the seed solution was

$$A_\tau = \alpha\frac{\tau_3}{2}, \quad A_\rho = A_\theta = A_\phi = 0 \ ,$$

where α is a constant. Such a zeroth step can also be utilized for the present, more general case. This means taking

$$G_{(0)} = \mathrm{diag}(e^{\alpha\rho}, e^{-\alpha\rho}) \ . \tag{2.19}$$

Substituting this in (2.2) and (2.3) one can show that

$$\psi_{(0)} = \mathrm{diag}(e^h, e^{-h}) \ , \tag{2.20}$$

where

$$e^h = e^{\alpha\rho}\left[\frac{\Lambda e^\rho - \bar{y}}{\Lambda e^{-\rho} - \bar{y}}\right]^{-\alpha/2} \mathscr{F}(B_+(\Lambda), B_-(\Lambda)) \tag{2.21}$$

and

$$\mathscr{F}(B_+(0), B_-(0)) = 1 \ , \tag{2.22}$$

so that

$$\psi_{(0)}(\Lambda = 0) = G_{(0)} \tag{2.23}$$

at a generic point. Simple examples are

$$\mathscr{F}=1,\quad \mathscr{F}=[B_-(\Lambda)B_+{}^{-1}(\Lambda)]^{-\alpha/2}\ . \tag{2.24}$$

The last choice gives

$$e^h=e^{\alpha\rho}\left[\frac{1+\Lambda e^{\rho}y}{1+\Lambda e^{-\rho}y}\right]^{-\alpha/2}\ . \tag{2.25}$$

The limiting form of h as $\rho\to 0$ is of interest in the monopole limit (Sec. IV). For $\mathscr{F}=1$, for example, one has

$$\begin{aligned} h&=\alpha\left[\rho-\tfrac{1}{2}\ln\left[\frac{\Lambda e^{\rho}-\bar{y}}{\Lambda e^{-\rho}-\bar{y}}\right]\right]\\ &=\alpha\left[-\frac{\rho\bar{y}}{\Lambda-\bar{y}}+O(\rho^2)\right]\ . \end{aligned} \tag{2.26}$$

The construction of the matrices R_k of (2.6) is described in detail in Appendix A. Finally this gives

$$G=\psi(\Lambda=0)\ . \tag{2.27}$$

III. RESTRICTION TO AXIAL SYMMETRY

Although our principal aim in this paper is in going beyond axial symmetry, the special features arising under this restriction are also of interest. The basic equations can be derived from those of Sec. II by redefining Λ and μ ($\Lambda e^{i\phi}\to\Lambda,\mu e^{i\phi}\to\mu$) which permits a consistent elimination of the derivative ∂_ϕ. The pole equations now admit only one-parameter family of solutions. To display such special features and also for comparing with the method of (I) the main results are given in an explicitly axially symmetric form. (This involves some other redefinitions, such as that of D_2. But this should cause no confusion. These results can easily be verified independently.)

The self-duality constraint is

$$\sinh^2\!\rho(G_\rho G^{-1})_\rho+(\sin\theta)^{-1}(\sin\theta\, G_\theta G^{-1})_\theta=0\ . \tag{3.1}$$

Let

$$\begin{aligned} \sinh\rho\, D_1\psi&\equiv[\sinh\rho\,\partial_\rho-\Lambda\partial_\theta\\ &\qquad-(\cosh\rho-\Lambda\cot\theta)\Lambda\partial_\Lambda]\psi\\ &=\sinh\rho(G_\rho G^{-1})\psi\ , \end{aligned} \tag{3.2}$$

$$\begin{aligned} D_2\psi&\equiv[\Lambda\sinh\rho\,\partial_\rho+\partial_\theta\\ &\qquad+(\Lambda\cosh\rho+\cot\theta)\Lambda\partial_\Lambda]\psi\\ &=(G_\theta G^{-1})\psi\ . \end{aligned} \tag{3.3}$$

One has

$$[D_1,(\sin\theta D_2)]=0 \tag{3.4}$$

and

$$[D_1,(\sin\theta D_2)]\psi=0 \tag{3.5}$$

leads to (3.1). The pole equations are now

$$\sinh\rho\,\mu_\rho-\mu\mu_\theta+(\cosh\rho-\mu\cot\theta)\mu=0\ , \tag{3.6}$$

$$\mu\sinh\rho\,\mu_\rho+\mu_\theta-(\mu\cosh\rho+\cot\theta)\mu=0\ . \tag{3.7}$$

The general solution can be shown to be given in terms of γ which played a crucial role in (I) [Eqs. (3.29)—(3.36) of (I)] and is

$$\begin{aligned} \gamma=\tan^{-1}[&\sinh c\sin\theta(\cosh c\sinh\rho\\ &-\sinh c\cosh\rho\sin\theta)^{-1}]\ , \end{aligned} \tag{3.8}$$

c being a constant $[\gamma=\gamma(c)]$. The general solution is

$$\begin{aligned} \mu=(\sinh c\sin\theta)^{-1}\{&(\cosh c\sinh\rho-\sinh c\cosh\rho\cos\theta)\\ &\mp[(\cosh c\sinh\rho-\sinh c\cosh\rho\cos\theta)^2+(\sinh c\sin\theta)^2]^{1/2}\} \end{aligned} \tag{3.9}$$

$$=\begin{cases}-\tan(\gamma/2) & \text{for upper sign ,}\\ \cot(\gamma/2) & \text{for lower sign .}\end{cases} \tag{3.10}$$

This corresponds to

$$\begin{aligned} \Gamma(\mu)&\equiv(\sinh\rho)^{-1}[\tfrac{1}{2}\sin\theta(\mu-\mu^{-1})\\ &\qquad+\cosh\rho\cos\theta]\\ &=\coth c,\ \text{a constant}\ . \end{aligned} \tag{3.11}$$

This should be compared to (2.18). As a consequence of the restriction $\partial_\phi\approx 0$, instead of $B_\pm(\Lambda)$ only the combination $\Gamma(\Lambda)$ (with $\Lambda e^{i\phi}\to\Lambda$) is now annihilated by $D_{1,2}$ of this section and instead of (2.13) one has now a general solution

$$\Gamma(\mu)=\coth c,\ \text{a constant}\ . \tag{3.12}$$

Setting again

$$G_{(0)}=\mathrm{diag}(e^{\alpha\rho},e^{-\alpha\rho}) \tag{3.13}$$

one now obtains

$$\psi_{(0)}=\mathrm{diag}(e^h,e^{-h})\ ,$$

where

$$e^h = e^{\alpha\rho}\left[\frac{\Lambda e^{\rho}-\tan\theta/2}{\Lambda e^{-\rho}-\tan\theta/2}\right]^{-\alpha/2}\mathscr{F}(\Gamma(\Lambda)) \ , \tag{3.14}$$

with $\mathscr{F}(\Gamma(0))=1$. This should be compared to (2.21). Choosing, for example,

$$\mathscr{F}(\Gamma(\Lambda)) = \left[\frac{\Gamma(\Lambda)-1}{\Gamma(\Lambda)+1}\right]^{-\alpha/2}$$
$$= \left[\frac{(\Lambda e^{-\rho}-\tan\theta/2)(\Lambda e^{\rho}+\cot\theta/2)}{(\Lambda e^{\rho}-\tan\theta/2)(\Lambda e^{-\rho}+\cot\theta/2)}\right]^{-\alpha/2} , \tag{3.15}$$

one obtains the form analogous to (2.25). Setting $\mathscr{F}=1$ and letting $\rho\to 0$, one has now corresponding to (2.26)

$$h = \alpha\left[-\rho\tan\frac{\theta}{2}\left[\Lambda-\tan\frac{\theta}{2}\right]^{-1}+O(\rho^2)\right] . \tag{3.16}$$

This will lead, in the monopole limit, to the corresponding axially symmetric result.

IV. THE MONOPOLE LIMIT

As explained in the Introduction, this limit is obtained very simply by setting $\rho=\alpha^{-1}r$ and letting $\alpha\to\infty$ (with correspondingly $A_\rho\to\alpha A_r, A_\tau\to\alpha A_t$). In (I) this limit led to axially symmetric monopoles in the "spherical R gauge," in which only this passage becomes direct. Here I obtain the Lax pairs for separable monopoles in the same formalism. As in (I) the following results should be compared throughout with the corresponding ones of FHP.[4,5] Since the limiting procedure is trivial, I briefly present the main results and point out some special features arising in this limit.

The self-duality constraint is (Appendix B)

$$r^2(G_rG^{-1})_r+(1+y\bar{y})^2(G_yG^{-1})_{\bar{y}}=0 \ . \tag{4.1}$$

Let

$$rD_1\psi \equiv [r\partial_r-\Lambda(1+y\bar{y})\partial_{\bar{y}}-(1+y\Lambda)\Lambda\partial_\Lambda]\psi$$
$$= r(G_rG^{-1})\psi \ , \tag{4.2}$$
$$(1+y\bar{y})D_2\psi \equiv [\Lambda r\partial_r+(1+y\bar{y})\partial_y+(\Lambda-\bar{y})\Lambda\partial_\Lambda]\psi$$
$$=(1+y\bar{y})(G_yG^{-1})\psi \ , \tag{4.3}$$

when D_1 and D_2 commute and

$$[D_1,D_2]\psi=0 \tag{4.4}$$

yields (4.1). The pole equations are now

$$r\mu_r-\mu(1+y\bar{y})\mu_{\bar{y}}+(1+y\mu)\mu=0 \ , \tag{4.5}$$
$$\mu r\mu_r+(1+y\bar{y})\mu_y-(\mu-\bar{y})\mu=0 \ . \tag{4.6}$$

Let

$$B(\mu) \equiv \left[\frac{\mu-\bar{y}}{1+y\mu}\right] \tag{4.7}$$

and

$$\Gamma(\mu)=[r(1+y\bar{y})]^{-1}[(y\mu-\bar{y}\mu^{-1})+(1-y\bar{y})] \tag{4.8}$$
$$=[r(1+y\bar{y})]^{-1}(\mu-\bar{y})(\mu^{-1}+y) \tag{4.9}$$
$$=r^{-1}[\sin\theta\tfrac{1}{2}(\mu e^{i\phi}-\mu^{-1}e^{-i\phi})+\cos\theta] \ . \tag{4.10}$$

It can be shown that

$$H(\Gamma(\mu),B(\mu))=0 \ , \tag{4.11}$$

where H is an arbitrary (nice) function, solves the pole equations. To zeroth order in ρ, $B_\pm(\mu)$ of (2.9) coincide and give the r-independent $B(\mu)$ of (4.7). The other argument of H is provided by the limiting form (rescaled) of the combination (2.16). Corresponding to (4.2) and (4.3) one has

$$D_1B(\Lambda)=0=D_2B(\Lambda) \ , \tag{4.12}$$
$$D_1\Gamma(\Lambda)=0=D_2\Gamma(\Lambda) \ . \tag{4.13}$$

Setting

$$G_{(0)}=\mathrm{diag}(e^r,e^{-r}) \ , \tag{4.14}$$

one has

$$\psi_{(0)}=\mathrm{diag}(e^h,e^{-h}) \ , \tag{4.15}$$

with

$$h=-r\bar{y}(\Lambda-\bar{y})^{-1}+F(\Gamma(\Lambda),B(\Lambda)) \ , \tag{4.16}$$

such that $F(\Lambda=0)=0$. The limiting form of (2.25), namely,

$$h=r(1+y\Lambda)^{-1} \tag{4.17}$$

corresponds to

$$F=\Gamma^{-1}(\Lambda) \ , \tag{4.18}$$

whereas $F=0$ corresponds to (2.26). The *restriction to axial symmetry* gives the following situation. The self-duality constraint is

$$r^2(G_rG^{-1})_r+(\sin\theta)^{-1}(\sin\theta\, G_\theta G^{-1})_\theta=0 \ . \tag{4.19}$$

For the Lax pair,

$$rD_1\psi \equiv [r\partial_r - \Lambda\partial_\theta - (1-\Lambda\cot\theta)\Lambda\partial_\Lambda]\psi$$
$$= r(G_r G^{-1})\psi \ , \tag{4.20}$$
$$D_2\psi \equiv [\Lambda r\partial_r + \partial_\theta + (\Lambda+\cot\theta)\Lambda\partial_\Lambda]\psi$$
$$= (G_\theta G^{-1})\Psi \ , \tag{4.21}$$

the consistency condition

$$[D_1,(\sin\theta D_2)]\psi = 0 \tag{4.22}$$

gives back (4.19). The pole equations

$$r\mu_r - \mu\mu_\theta + (1-\mu\cot\theta)\mu = 0 \ , \tag{4.23}$$
$$\mu r\mu_r + \mu_\theta - (\mu+\cot\theta)\mu = 0 \tag{4.24}$$

now have the following one-parameter family of solutions, with a constant c, namely,

$$\mu = (c\sin\theta)^{-1}[(r - c\cos\theta) \mp (r^2 - 2rc\cos\theta + c^2)^{1/2}] \ . \tag{4.25}$$

This corresponds to

$$\Gamma(\mu) \equiv r^{-1}[\sin\theta \tfrac{1}{2}(\mu - \mu^{-1}) + \cos\theta] = c \ . \tag{4.26}$$

Defining ω and R through

$$\begin{aligned} R\cos(\omega-\theta) &= r - c\cos\theta \ , \\ R\sin(\omega-\theta) &= c\sin\theta \ , \end{aligned} \tag{4.27}$$

$$\mu = -\tan\left[\frac{\omega-\theta}{2}\right] \text{ and } \cot\left[\frac{\omega-\theta}{2}\right] \ , \tag{4.28}$$

for the upper and the lower signs, respectively. This should be compared to (3.10). In (I), $(\omega-\theta)$ played a crucial role in the construction of monopoles [Eqs. (2.19)–(2.23) of (I)]. Here, as was to be expected after (3.10), $(\omega-\theta)$ reappears in the poles. For

$$G_{(0)} = \mathrm{diag}(e^r, e^{-r}) \tag{4.29}$$

one has now

$$\psi_{(0)} = \mathrm{diag}(e^h, e^{-h}) \ , \tag{4.30}$$

where

$$h = r\left[1 - \Lambda\cot\frac{\theta}{2}\right]^{-1} + F(\Gamma(\Lambda)) \ , \tag{4.31}$$

with

$$F(\Gamma(0)) = 0 \ ,$$

corresponding to (4.17) one has

$$h = r\left[1 + \Lambda\tan\frac{\theta}{2}\right]^{-1} \ . \tag{4.32}$$

It will be noted that in the monopole limit, for example, many nontrivial factors in various parts of the results of Secs. II and III tend to unity. For such reasons no evident rules can be given for the inverse passage (monopoles→instantons).

V. τ-DEPENDENT FORMALISM

Although the τ-static case seems to be of particular interest, other aspects should not necessarily be ignored. For completeness, the main features of the τ-dependent case are presented briefly in this section. One has the self-duality constraint (1.9),

$$\sinh^2\rho(G_z G^{-1})_{\bar z} + (1+y\bar y)^2(G_y G^{-1})_{\bar y} = 0 \ , \tag{5.1}$$

where

$$\rho = z + \bar z, \quad \partial_z = \partial_\rho - i\partial_\tau, \quad \partial_{\bar z} = \partial_\rho + i\partial_\tau \ . \tag{5.2}$$

The pair (2.2) and (2.3) can be modified quite simply to

$$\sinh\rho\, D_1\psi \equiv [\sinh\rho\,\partial_z - \Lambda(1+y\bar y)\partial_{\bar y} - (\cosh\rho + y\Lambda)\Lambda\partial_\Lambda]\psi$$
$$= \sinh\rho(G_\rho G^{-1})\psi \ , \tag{5.3}$$
$$(1+y\bar y)D_2\psi = [\Lambda\sinh\rho\,\partial_{\bar z} + (1+y\bar y)\partial_y + (\Lambda\cosh\rho - \bar y)\Lambda\partial_\Lambda]\psi$$
$$= (1+y\bar y)(G_y G^{-1})\psi \ . \tag{5.4}$$

D_1, D_2 commute and

$$[D_1, D_2]\psi = 0 \tag{5.5}$$

leads back to (5.1). $B_\pm$, defined in Sec. II as

$$B_\pm(\Lambda) = (\Lambda e^{\pm\rho} - \bar y)(1 + \Lambda e^{\pm\rho}y)^{-1}$$
$$= (\Lambda e^{\pm z} - \bar y e^{\mp\bar z})(e^{\mp\bar z} + \Lambda e^{\pm z}y)^{-1} \ , \tag{5.6}$$

still satisfies

$$D_1 B_\pm(\Lambda) = 0 = D_2 B_\pm(\Lambda) \ . \tag{5.7}$$

But now one can also define the pair

$$\beta_1(\Lambda) \equiv (\Lambda e^z - \bar y e^{-\bar z})(\Lambda e^{-z} - \bar y e^{\bar z})^{-1}$$
$$= e^{-2\bar z}(\Lambda e^\rho - \bar y)(\Lambda e^{-\rho} - \bar y)^{-1} \ , \tag{5.8}$$
$$\beta_2(\Lambda) \equiv (y\Lambda e^z + e^{-\bar z})(y\Lambda e^{-z} + e^{\bar z})$$
$$= e^{-2\bar z}(\Lambda e^\rho y + 1)(\Lambda e^{-\rho}y + 1)^{-1} \ , \tag{5.9}$$

such that

$$D_1\beta_j(\Lambda) = 0 = D_2\beta_j(\Lambda) \quad (j = 1,2) \ . \tag{5.10}$$

One has however the relation

$$\beta_1(\Lambda)\beta_2^{-1}(\Lambda) = B_+(\Lambda)B_-^{-1}(\Lambda)$$
$$= [\Gamma(\Lambda)+1][\Gamma(\Lambda)-1]^{-1} \ , \tag{5.11}$$

where $\Gamma(\Lambda)$ is defined by (2.16). The pole equations are now

$$\sinh\rho\,\mu_z - \mu(1+y\bar{y})\mu_{\bar{y}} + (\cosh\rho + y\mu)\mu = 0 \ , \tag{5.12}$$

$$\mu \sinh\rho\,\mu_{\bar{z}} + (1+y\bar{y})\mu_y - (\mu\cosh\rho - \bar{y})\mu = 0 \ . \tag{5.13}$$

As compared to (2.13) H can now depend on *three* independent combinations of β_1, β_2, B_+, and B_-. In a symmetrical notation one can set

$$H(\beta_1(\mu),\beta_2(\mu),B_+(\mu),B_-(\mu)) = 0 \ , \tag{5.14}$$

where

$$\beta_1(\mu)\beta_2{}^{-1}(\mu) = B_+(\mu)B_-{}^{-1}(\mu) \ . \tag{5.15}$$

For the same $G_{(0)}$ as in (2.19), i.e.,

$$G_{(0)} = \mathrm{diag}(e^{\alpha\rho}, e^{-\alpha\rho}) \tag{5.16}$$

one now has

$$\psi_{(0)} = \mathrm{diag}(e^{h}, e^{-h}) \ , \tag{5.17}$$

where

$$e^{h} = e^{\alpha\rho}(\Lambda e^{\rho} - \bar{y})(\Lambda e^{-\rho} - \bar{y})^{-1}\mathscr{F}(\beta_1(\Lambda),\beta_2(\Lambda),B_+(\Lambda),B_-(\Lambda)) \tag{5.18}$$

along with (5.11) and $\mathscr{F}(\Lambda=0)=1$. The results in the scaling limit

$$(\tau,\rho) = (\alpha^{-1}t, \alpha^{-1}r), \quad \alpha \to \infty \tag{5.19}$$

can again be obtained very simply. This t-dependent generalization of the results of Sec. IV is left as an easy exercise.

VI. REMARKS

In this paper for all the cases considered the following goals have been attained:

(1) construction of Lax pairs,

(2) solutions of the pole equations,

(3) construction of a suitable seed solution $\psi_{(0)}$ suggested by the successful treatment of the axially symmetric case in (I).

Hence solutions are now in principle available (Appendix A) for an arbitrary number of poles. One crucial step remains to be taken however—to fix the criterion for a correct choice of poles, i.e., in Sec. II, for example, selecting the correct explicit form of H in (2.13). Only then [and fixing also the $M^{(K)}$'s in (2.28)] one can construct regular solutions with desired asymptotic behaviors. I will try to carry out this important part elsewhere.

Finally, I would like to collect together the reasons for my particular choice of techniques for constructing instanton chains both in (I) and, here, in (II). This will involve some repetitions, which I hope would be worthwhile. *Evidently I am trying to do much more than merely to reproduce monopole solutions in yet another way. The two-chain of (I) gives, for the first time so far as I know, explicitly (though in unfamiliar coordinates and gauge) a particular class of non-'t Hooft instantons (presumably a subclass of class 2 of Atiyah and Ward) for an infinite range of index.* That it is non-'t Hooft is assured by the fact that in the scaling limit it leads to a monopole of charge 2, which is in class 2 of Atiyah and Ward.[9,19] The general structure of the axially symmetric n-chain, leading to a monopole of charge n, can be written down as a straightforward generalization. As emphasized in Sec. I, such instanton chains should be studied for their many other interesting properties apart from the fact that they yield monopoles trivially as limits. (I, of course, do not imply that monopoles are trivial, but that there is practically no labor involved in extracting them at the end.) In Ref. 16, after a study of fluctuation determinants and explicit corrections to dilute-gas approximations for the one-chain, the desirability of exploring a hierarchy of chains related to limiting multimonopole configurations was pointed out. In (I) and (II) this program is being carried out. As explained before at length, the key technique that makes this feasible is the use of a static formalism for finite-action solutions through the use of a periodic time τ as an intermediate step. Now among all the approaches to multimonopole constructions, the FHP ones[2-5] turned out to be the most directly adaptable to my purpose. [It is important to note, however that, as explained in (I) and here in Appendix B, the spherical R gauge is necessary to start with. The relation with the standard R gauge is even difficult to make explicit.] This is the essential reason for selecting the FHP techniques as sources of "hindsight," as explained in Sec. I.

In (I) the H-N-type transformations suitably adapted yielded efficiently and directly explicit, *real* gauge potentials. Here, I have chosen to work with Lax pairs. Apart from the sources already quoted, even in the context of instantons alone, there are interesting studies of Lax pairs.[20-23] Here I will briefly point out the relation to the Belavin-Zakharov approach.[20,21] With the definitions (1.8), setting

$D_z \equiv \partial z + iA_z$ and so on and defining

$$D_1 \equiv (\sinh\rho)^{-1}[\sinh\rho\, D_z - \Lambda(1+y\bar{y})D_{\bar{y}} - (\cosh\rho + y\Lambda)\Lambda\partial_\Lambda] \ , \qquad (6.1)$$

$$D_2 \equiv (1+y\bar{y})^{-1}[\Lambda \sinh\rho\, D_{\bar{z}} + (1+y\bar{y})D_y + (\Lambda\cosh\rho - \bar{y})\Lambda\partial_\Lambda] \quad (\rho = z+\bar{z}) \ . \qquad (6.2)$$

The condition

$$[D_1,D_2] = \Lambda^2[D_{\bar{z}},D_{\bar{y}}] + [D_z,D_y] + \Lambda[\sinh\rho(1+y\bar{y})]^{-1}\{\sinh^2\rho[D_z,D_{\bar{z}}] + (1+y\bar{y})^2[D_y,D_{\bar{y}}]\} = 0 \qquad (6.3)$$

yields the self-duality constraints. Hence setting

$$D_j\psi = 0 \quad (j=1,2) \qquad (6.4)$$

one has a Belavin-Zakharov-type formalism. Now gauge transforming $A_{\bar{z}}$ and $A_{\bar{y}}$ to zero (which is possible, consistently with $F_{\bar{y}\bar{z}}=0$), (6.4) can be shown to lead back to (5.3) and (5.4). (For the "Cartesian choice" this aspect is discussed in Refs. 22 and 23.) Since Belavin and Zakharov's multiplicative ansatz does not seem to be adapted to non-'t Hooft classes[21] and since the two-monopole case has already been studied[4,5] in a Belinski-Zakharov-type formalism, I directly generalized the latter, adapting it to our gauge. Algebraic topological methods can be powerful in demonstrating completeness[12] or in ensuring regularity.[13] But since in them really explicit forms for gauge potentials for higher monopole charges are not obtained, the prospect of constructing explicitly instanton chains, with infinite range of index, leading to monopoles (which is my aim) seems even more remote. As indicated before, I have not completed my program in this paper. But an eventual construction of the general two-chain [seven-parameter generalization of the two-chain of (I)] would already be quite interesting. It will not be merely two monopoles or two instantons, but an infinite sequence of non-'t Hooft instantons yielding the two-monopole solution as a by-product. *None of the other methods, since they concentrate on monopoles only, can achieve this.*

Even in the context of Lax pairs my formalism has special interesting features. In (2.21), choosing

$$\mathscr{F} = (B_+/B_-)^{\alpha/4} \qquad (6.5)$$

one has

$$e^h = e^{\alpha\rho/2}X(\Lambda) \ ,$$

where

$$X^{-1}(\Lambda) = \left[\frac{(\Lambda e^{-\rho} - \bar{y})(1+\Lambda e^{-\rho}y)}{(\Lambda e^{\rho} - \bar{y})(1+\Lambda e^{\rho}y)}\right]^{-\alpha/4} = \bar{X}(-\bar{\Lambda}^{-1}) \ . \qquad (6.6)$$

Thus from (A29) and (A25) in the limit $\mu_l \to -\bar{\mu}_k^{-1}$ one finally obtains simply

$$(1+\bar{\mu}_k\mu_l)N_{kl} \to \bar{M}_1^{(k)}M_1^{(l)} \pm \bar{M}_2^{(k)}M_2^{(l)} \ . \qquad (6.7)$$

This can simplify determination of the M's, which now must assure the vanishing of the right-hand sides for these critical points. Comparing with the FHP formalism,[4,5] where (in their notations)

$$h = z - \lambda y + F(\gamma,\lambda) \quad (\gamma \equiv y\lambda - \bar{y}/\lambda - z) \ , \qquad (6.8)$$

one gets a comparable feature by choosing $F = \frac{1}{2}\gamma$. But then

$$h = \frac{z}{2} - \tfrac{1}{2}(\lambda y + \bar{y}/\lambda) \qquad (6.9)$$

is *singular* for $\lambda = 0$.

Indeed, throughout my formalism (thanks to ratios such as $B_\pm$) $\Lambda = 0$ is not a danger point. So one does not get involved at this stage in complicated constraints to ensure regularity at $\Lambda = 0$, which is the final value to be retained to obtain G from ψ. Such constraints, even for two-monopoles, relate two parameters through an elliptic integral in the FHP formalism[4,5] which reproduces closely similar features in the Ward-Corrigan-Goddard formalism.[9,11] (The relation between the spectral parameter Λ in the Lax pairs and Ward's ζ is discussed, for example, in Ref. 22.) Full consequences of such features of our formalism and the way certain constraints arise in our ease, to ensure a finite action will not be discussed here.

I conclude with a piece of somewhat belated wisdom. Strictly speaking, it should not have been necessary at all to construct the multimonopoles (in the PS limit) separately, though the formalisms proposed to this end are admirable in many respects. Exploring special hierarchies, with various fascinating properties, in the space of instantons multimonopoles could have been obtained, among other interesting results, as by-products. I am trying to show, profiting from the hindsight provided by the monopole solutions, one of the ways in which this can yet be realized.

APPENDIX A

The essential steps of the method of Zakharov *et al.*[6–8] are given below. For the reader's convenience they are adapted to the specific context of Sec. II of this paper. The matrix equations to be solved are

$$\begin{aligned}\sinh\rho\, D_1\psi &\equiv [\sinh\rho\,\partial_\rho-\Lambda(1+y\bar{y})\partial_{\bar{y}}\\ &\quad-(\cosh\rho+y\Lambda)\Lambda\partial_\Lambda]\psi\\ &=\sinh\rho(G_\rho G^{-1})\psi\ ,\end{aligned} \tag{A1}$$

$$\begin{aligned}(1+y\bar{y})D_2\psi &\equiv [\Lambda\sinh\rho\,\partial_\rho+(1+y\bar{y})\partial_y\\ &\quad+(\Lambda\cosh\rho-\bar{y})\Lambda\partial_\Lambda]\psi\\ &=(1+y\bar{y})(G_yG^{-1})\psi\end{aligned} \tag{A2}$$

with the boundary condition

$$\psi(\Lambda=0)=G\ . \tag{A3}$$

Suppose that a particular solution $\psi_{(0)}$ for $G=G_{(0)}$ has been found (examples are given in Sec. II). Then for the "purely solitonic" case one assumes

$$\psi=\chi\psi_{(0)}\equiv\left[I+\sum_k R_k(\Lambda-\mu_k)^{-1}\right]\psi_{(0)}\ , \tag{A4}$$

where I and R_k are matrices like G and ψ (in our case 2×2, I being the unit matrix). The poles μ_k and the matrices R_k are free of Λ, as is evidently G. $\psi_{(0)}$ is supposed to be so constructed that it has no singularity as $\Lambda\to\mu_k$. The fact that the pole structure is explicit in the ansatz is exploited systematically and proves powerful enough a constraint to lead to a general solution. Substituting (A4) in (A1) and (A2) double poles appear on the left only. Hence their residues must vanish. This leads to the pole equations to be satisfied by the μ_k's for each k:

$$[\sinh\rho\,\partial_\rho-\mu(1+y\bar{y})\partial_{\bar{y}}+(\cosh\rho+y\mu)]\mu=0\ , \tag{A5}$$

$$[\mu\sinh\rho\,\partial_\rho+(1+y\bar{y})\partial_y-(\cosh\rho\,\mu-\bar{y})]\mu=0\ . \tag{A6}$$

These are solved in Sec. II and leads to (2.13). The consequences of these equations will be exploited repeatedly.

Now one proceeds to construct the R_k's as follows. From (A1), (A2), and (A4)

$$D_1\chi=(G_\rho G^{-1})\chi-\chi(G_{(0)\rho}G_{(0)}{}^{-1})\ , \tag{A7}$$

$$D_2\chi=(G_yG^{-1})\chi-\chi(G_{(0)y}G_{(0)}{}^{-1})\ . \tag{A8}$$

Using the Hermiticity of G and the relation

$$\sinh\rho\,\bar{D}_1(-\bar{\Lambda}^{-1})=(1+y\bar{y})\Lambda^{-1}D_2(\Lambda)\ , \tag{A9}$$

one obtains the important constraint

$$\chi(\Lambda)=G\chi^{\dagger-1}(-\bar{\Lambda}^{-1})G_{(0)}{}^{-1}\ . \tag{A10}$$

This may be seen as follows. From (A7)

$$\begin{aligned}\sinh\rho\,D_1(G^{-1}\chi G_{(0)})&=-\Lambda(1+y\bar{y})(G_{\bar{y}}{}^{-1}\chi G_{(0)}+G^{-1}\chi G_{(0)\bar{y}})\\ &=\Lambda(1+y\bar{y})[(G^{-1}G_{\bar{y}})(G^{-1}\chi G_{(0)})-(G^{-1}\chi G_{(0)})(G_{(0)}{}^{-1}G_{(0)\bar{y}})]\ .\end{aligned} \tag{A11}$$

Now after $\Lambda\to-\Lambda^{-1}$ and taking successively the inverse and adjoint of both sides one uses (A9). This shows that the right-hand side of (A10) satisfies (A8) and similarly also (A7). For solutions satisfying (A10) a pole μ_k of $\chi(\Lambda)$ is seen to correspond to a pole at $\Lambda=-\bar{\mu}_k{}^{-1}$ ($=\nu_k$ say) of $\chi^{-1}(\Lambda)$. These poles have to be different from those of χ for the inverse to exist. This is a constraint imposed by (A10). Finally it turns out that for

$$\chi(\Lambda)=I+\sum_k R_k(\Lambda-\mu_k)^{-1} \tag{A12}$$

one has the structure

$$\chi^{-1}(\Lambda)=I+\sum_k S_k(\Lambda-\nu_k)^{-1}\ . \tag{A13}$$

One need not construct explicitly the S_k's. It suffices to note that

$$\chi(\Lambda)\chi^{-1}(\Lambda)=I \tag{A14}$$

implies through the vanishing of the residue as $\Lambda\to\mu_k$, for each k,

$$R_k\chi^{-1}(\mu_k)=0\ . \tag{A15}$$

Hence denoting

$$(R_k)_{ab}=n_a^{(k)}m_b^{(k)} \tag{A16}$$

and

$$(\chi^{-1}(\mu_k))_{ab}=q_a^{(k)}p_b^{(k)}\ , \tag{A17}$$

$$\sum_a m_a^{(k)}q_a^{(k)}=0\ , \tag{A18}$$

and rewriting (A7) and (A8) as

$$(D_1\chi)\chi^{-1}+\chi(G_{(0)\rho}G_{(0)}{}^{-1})\chi^{-1}=G_\rho G^{-1} \ , \tag{A19}$$

$$(D_2\chi)\chi^{-1}+\chi(G_{(0)y}G_{(0)}{}^{-1})\chi^{-1}=G_y G^{-1} \ , \tag{A20}$$

the right-hand sides are independent of Λ. The residues of the double poles on the left vanish due to the pole equations. Equating to zero the residues of the simple poles one obtains

$$[\sinh\rho\,\partial_\rho-\mu_k(1+y\bar{y})\partial_{\bar{y}}]m_a^{(k)}+\sinh\rho\, m_b^{(k)} \times(G_{(0)\rho}G_{(0)}{}^{-1})_{ba}=0 \ , \tag{A21}$$

$$[\mu_k\sinh\rho\,\partial_\rho+(1+y\bar{y})\partial_y]m_a^{(k)}+(1+y\bar{y})m_b^{(k)} \times(G_{(0)y}G_{(0)}{}^{-1})_{ba}=0 \ , \tag{A22}$$

Again taking account of the pole equations one has

$$[\sinh\rho\,\partial_\rho-\mu_k(1+y\bar{y})\partial_{\bar{y}}]\psi(\mu_k) =\sinh\rho(G_\rho G^{-1})\psi(\mu_k) \ , \tag{A23}$$

$$[\mu_k\sinh\rho\,\partial_\rho+(1+y\bar{y})\partial_y]\psi(\mu_k) =(1+y\bar{y})(G_y G^{-1})\psi(\mu_k) \ . \tag{A24}$$

Hence

$$m_a^{(k)}=M_b^{(k)}(B_+(\mu_k),B_-(\mu_k))\psi_{(0)ba}{}^{-1}(\mu_k) \ , \tag{A25}$$

where the arbitrary functions $M^{(k)}$ [of $B_\pm$ defined in (2.9)] should eventually be so chosen as to be compatible with the sought for regularity and asymptotic properties.

To determine the $n^{(k)}$'s one starts from

$$G=\chi(\Lambda)G_{(0)}\chi^\dagger(-\bar{\Lambda}^{-1}) \ . \tag{A26}$$

Equating to zero the residues of the poles ($k=1,\ldots,J$) for J poles, say,

$$R_k G_{(0)}\left[I-\sum_{l=1}^{J}(\mu_k{}^{-1}+\bar{\mu}_l)^{-1}R_l^\dagger\right]=0 \ , \tag{A27}$$

or

$$n_a^{(k)}\left\{\mu_k{}^{-1}m_c^{(k)}G_{(0)cb}-\sum_{l=1}^{J}[(1+\mu_k\bar{\mu}_l)^{-1}m_c^{(k)}G_{(0)cd}\bar{m}_d^{(l)}]\bar{n}_b^{(l)}\right\}=0 \ . \tag{A28}$$

Defining the $J\times J$ matrix

$$N_{kl}=(1+\bar{\mu}_k\mu_l)^{-1}m_d^{(l)}G_{(0)dc}\bar{m}_c^{(k)} \tag{A29}$$

and using $\bar{G}_{(0)cd}=G_{(0)dc}$ one obtains in terms of the inverse matrix N^{-1}

$$n_b^{(l)}=(N^{-1})_{lk}(\bar{\mu}_k{}^{-1}G_{(0)bc}\bar{m}_c^{(k)}) \ . \tag{A30}$$

Now one has $(R_k)_{ab}=n_a^{(k)}m_b^{(k)}$ and hence $\psi(\Lambda)$ with a suitable choice of the μ_k's from the solutions (2.13). Finally

$$G=\psi(\Lambda=0) \ . \tag{A31}$$

More precisely one has to restore unimodularity by further multiplying[8] by a factor $(\det G)^{-1/2}$. To make $\det G$ positive for an odd number of poles one can start with $\det G_{(0)}=-1$ such as by setting

$$G_{(0)}=\mathrm{diag}(e^{\alpha\rho},-e^{-\alpha\rho}) \tag{A32}$$

instead of (2.19).

APPENDIX B

Certain relevant features of different formulations of the self-duality constraints are compared below (see also Refs. 17 and 18). Starting with the flat Euclidean metric

$$ds^2=dx_0{}^2+dx_1{}^2+dx_2{}^2+dx_3{}^2 \tag{B1}$$

let a change of coordinates be introduced such that in terms of the new coordinates $(z,\bar{z},y,\bar{y})$ one has

$$ds^2=g_{z\bar{z}}d_z d_{\bar{z}}+g_{y\bar{y}}d_y d_{\bar{y}} \ , \tag{B2}$$

all other $g_{\mu\nu}$'s vanishing. Two simple examples are provided by the "standard choice"

$$\begin{aligned} z&=\tfrac{1}{2}(x_3+ix_0), & \bar{z}&=\tfrac{1}{2}(x_3-ix_0) \ , \\ y&=\tfrac{1}{2}(x_1+ix_2), & \bar{y}&=\tfrac{1}{2}(x_1-ix_2) \ , \end{aligned} \tag{B3}$$

and the "spherical choice"

$$\begin{aligned} z&=\tfrac{1}{2}(r+it), & \bar{z}&=\tfrac{1}{2}(r-it) \ , \\ y&=\tan\frac{\theta}{2}e^{-i\phi}, & \bar{y}&=\tan\frac{\theta}{2}e^{-i\phi} \ . \end{aligned} \tag{B4}$$

Here $t=x_0$; r,θ,ϕ are the usual spherical coordinates and we will use in the following $\epsilon_{0123}=1=\epsilon_{tr\theta\phi}$. [When singularities (evidently removable) are introduced by a choice, the behavior of the gauge field solutions has to be checked in those domains in terms of coordinates well behaved there. This sometimes provides crucial constraints.] Through (1.2) one obtains (1.8) which is a choice particularly important for us. But the two familiar examples given here will suffice to illustrate the essential points. In terms of *any such choice,* the self-duality constraints can be shown to be

$$F_{yz}=F_{\bar{y}\bar{z}}=0 \ , \tag{B5}$$

$$g^{\mu\bar{\mu}}F_{\mu\bar{\mu}}=g^{z\bar{z}}F_{z\bar{z}}+g^{y\bar{y}}F_{y\bar{y}}=0 \quad (\mu=y,z) \ . \tag{B6}$$

Thus for (B3) and (B4), (B6) becomes, respectively,

$$F_{y\bar{y}}+F_{z\bar{z}}=0 \tag{B7}$$

and

$$r^2F_{z\bar{z}}+(1+y\bar{y})^2F_{y\bar{y}}=0 \tag{B8}$$

since in the latter case,

$$g_{z\bar{z}}=4, \ \ g_{y\bar{y}}=4r^2(1+y\bar{y})^{-2} \ . \tag{B9}$$

Whatever will be the choice for $(y,\bar{y},z,\bar{z})$ the gauge potentials A_μ are taken to be (with real λ and ζ, in general, complex)

$$\begin{aligned} -2i\lambda A_\mu &= \begin{bmatrix} \lambda_\mu & 0 \\ 2\zeta_\mu & -\lambda_\mu \end{bmatrix} \ , \\ -2i\lambda A_{\bar{\mu}} &= \begin{bmatrix} \lambda_{\bar{\mu}} & 2\bar{\zeta}_{\bar{\mu}} \\ 0 & -\lambda_{\bar{\mu}} \end{bmatrix} \ , \end{aligned} \tag{B10}$$

where $\mu=y,z,\bar{\mu}=\bar{y},\bar{z}$; $\lambda_\mu\equiv\partial_\mu\lambda$ and so on. Now (B5) and (B6) can be shown to lead to

$$\begin{aligned} L_1 &\equiv g^{\mu\bar{\mu}}(\lambda\lambda_{\mu\bar{\mu}}-\lambda_\mu\lambda_{\bar{\mu}}+\zeta_\mu\bar{\zeta}_{\bar{\mu}})=0 \ , \\ L_2 &\equiv g^{\mu\bar{\mu}}(\lambda\zeta_{\mu\bar{\mu}}-2\zeta_\mu\lambda_{\bar{\mu}})=0 \ , \\ L_{\bar{2}} &\equiv g^{\mu\bar{\mu}}(\lambda\bar{\zeta}_{\mu\bar{\mu}}-2\bar{\zeta}_{\bar{\mu}}\lambda_\mu)=0 \quad (\mu=y,z) \ . \end{aligned} \tag{B11}$$

Defining the matrix

$$G=\lambda^{-1}\begin{bmatrix} \lambda^2+\zeta\bar{\zeta} & -\bar{\zeta} \\ -\zeta & 1 \end{bmatrix} \tag{B12}$$

it can be shown that

$$g^{\mu\bar{\mu}}(G_\mu G^{-1})_{\bar{\mu}}=\lambda^{-3}\begin{bmatrix} (\lambda L_1+\bar{\zeta}L_2) & (2\lambda\bar{\zeta}L_1+\zeta^{-2}L_2-\lambda^2L_{\bar{2}}) \\ -L_2 & -(\lambda L_1+\bar{\zeta}L_2) \end{bmatrix} \ . \tag{B13}$$

Hence the system (B11) can be replaced by the single matrix equation

$$g^{\mu\bar{\mu}}(G_\mu G^{-1})_{\bar{\mu}}=0 \ . \tag{B14}$$

For different choices of the coordinates $(y,\bar{y},z,\bar{z})$ one has different equations for G. This evident fact is emphasized for its consequences. For example, (B3) leads to

$$(G_zG^{-1})_{\bar{z}}+(G_yG^{-1})_{\bar{y}}=0 \tag{B15}$$

and (B4) to

$$r^2(G_zG^{-1})_{\bar{z}}+(1+y\bar{y})^2(G_yG^{-1})_{\bar{y}}=0 \ . \tag{B16}$$

The fact that

$$\lambda=e^r, \ \ \zeta=0=\bar{\zeta} \tag{B17}$$

is a solution of (B16) [though not of (B15)] was used in (I) to construct a "seed solution" without breaking spherical symmetry. This choice replaced

$$\lambda=e^{x_3}, \ \ \zeta=0=\bar{\zeta} \tag{B18}$$

used for (B15) (Refs. 2—5). Certain consequences of the choice of (B16) which reduces to "Ernst-type" *(not* exactly Ernst) equations for axial symmetry were discussed in detail in (I). Here I recapitulate only the simplest result for comparison. It was shown in (I) that the solution

$$\begin{aligned} \lambda &= r^{-1}\sinh r\sin\theta \ , \\ \zeta &= \bar{\zeta}=\cos\theta \ , \end{aligned} \tag{B19}$$

of (B16) gives the unit-charge PS monopole. Corresponding to the "standard choice" the formalism of FHP[2,3] leads, in our notations, to

$$\lambda=(r\sin\theta)F^{-1} \ ,$$

$$\zeta=\bar{\zeta}=(x_3\cosh x_3-r\sinh x_3\coth x_3)F^{-1} \ ,$$

where

$$\begin{aligned} F &= r(\sinh r)^{-1}+r\cosh x_3\coth r \\ &\quad -x_3\sinh x_3 \quad (x_3=r\cos\theta) \ . \end{aligned} \tag{B20}$$

In the complex gauge formalism of Prasad[10] one obtains, in our notations,

$$\begin{aligned} \lambda &= e^{ix_0}r^{-1}\sinh r \ , \\ \zeta &= e^{ix_0}e^{i\phi}(r\sin\theta)^{-1}(\cos\theta\sinh r+\cosh r) \ , \\ \bar{\zeta} &= e^{ix_0}e^{i\phi}(r\sin\theta)^{-1}(\cos\theta\sinh r-\cosh r) \ , \end{aligned} \tag{B21}$$

which are then gauge transformed to give real solutions.[10]

Comparing (B19) with (B20) and (B21) it is evident that a choice different from the standard one can be of interest for special purposes. But the main interest of (B4), from my point of view, is that (B16) is the limit, in the sense (1.6), of (1.10). This is what permits the trivial extraction of self-dual monopoles from special classes of finite-action instantons. Other possible interesting choices and the precise connection between the G matrices for such different choices should be studied.

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