Fermion-monopole system reexamined. II

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Following a preceding paper, we reexamine the interaction of magnetic monopoles (dyons) with Dirac particles. For an infinitely heavy SU(2) dyon with one isodoublet fermion, we show that the vacuum angle in the original theory may be incorporated as a boundary condition in an effective one-particle Hamiltonian, which allows us to determine the charge distribution and the fermionic structure of the stable dyons. The resulting boundary conditions turn out to be Abelian and charge conserving, in contrast to that of previous authors. In particular, the dyon degree of freedom is essentially frozen, and there is no charge-exchange scattering off the dyon. With a Dirac mass term for the fermion, we also find that the ground state is twofold degenerate, owing to a spontaneous breakdown of fermion-number conjugation. For a Higgs mass, we find that the conjugation symmetry is unbroken, as in the original analysis of Jackiw and Rebbi. However, because of the changed boundary condition, the characteristic zero mode ceases to exist, and the ground state is nondegenerate with zero fermion number.

I. INTRODUCTION

In a previous paper[1] (referred to as I), we have reexamined the problem of a Dirac particle in a fixed Abelian monopole field. It was found that the system admitted the existence of θ vacua: For massless fermions chiral symmetry was spontaneously broken, whereas for massive fermions the monopole exhibited the Witten effect,[3–5] and became a dyon. In particular, the resulting charge was essentially given by the η invariant of Atiyah, Patodi, and Singer. Furthermore, the chiral symmetry breaking for the massless case and the Witten effect for the massive case were related by an analog of Levinson's theorem.

However, the paper left open several problems, one of which was the relation between Abelian and non-Abelian monopoles[6–9] with respect to the vacuum angle. For the Abelian monopole[6] θ entered the problem as a boundary condition at the origin (the location of the monopole), whereas for the non-Abelian monopole[7,8] it enters as a coupling constant in the Lagrangian,[3–5] a manner which at first sight seems to be quite distinct.

In this paper, we will show that the actual situation is otherwise. In Sec. II, we take a simple example to illustrate how a coupling constant may wind up as a boundary condition. Section III then gives a brief review of the properties of the Dirac equation for a fixed non-Abelian monopole field.[10–17] We proceed to the coupled fermion-monopole system in Sec. IV, which comprises the main part of this paper. Following previous authors,[15,18,19] we consider a model Hamiltonian, which retains only the lowest partial wave for the fermions. The Hamiltonian is solved by a variational calculation in the case where the dyon mass is much greater than that of the fermion. In particular, we are able to determine the charge distribution and the fermionic structure of the stable dyons in that limit. We find that the result is quite different from what one would naively expect on the basis of the Dirac equation in Sec. III. In particular, the effective boundary condition at the dyon core turns out to be charge conserving, in contrast to that of previous authors.[11–19] The reason for the discrepancy turns out to be quite simple[5,19]: The large Coulomb energy associated with the dyon core invalidates the usual semiclassical approximation.[20] This point is further discussed in Sec. V, where we also discuss the relation between our results and the previous work on the monopole-fermion system.[21]

II. A SIMPLE EXAMPLE

As in I, let us consider a Dirac particle in a fixed Abelian monopole field. This time, however, we also include the Coulomb self-energy. A suitable Hamiltonian would be

$$\mathscr{H}=\frac{1}{2}\int d\vec{x}\,\Psi^\dagger(\vec{x},t)H\Psi(\vec{x},t)+\frac{1}{2}\int d\vec{x}\,(H\Psi)^\dagger(\vec{x},t)\Psi(\vec{x},t)+\frac{1}{2}\int d\vec{x}\,d\vec{y}\,j_0(\vec{x},t)\frac{1}{4\pi|x-y|}j_0(\vec{y},t)\;, \tag{2.1}$$

where

$$j_\mu(\vec{x},t)=(e/2)[\overline{\Psi}(\vec{x},t),\gamma_\mu\Psi(\vec{x},t)]\;, \tag{2.2}$$

$$H=\vec{\alpha}\cdot\vec{\pi}+M\beta e^{-i\theta\gamma_5}\;, \tag{2.3}$$

$$\vec{\pi}=-i\vec{\nabla}-e\vec{A}\;. \tag{2.4}$$

(We have suppressed spinorial indices.) At first sight, it is not clear how a boundary condition could enter the problem;

$\mathscr{H}$ as it stands is already Hermitian,[22] without any boundary condition on H. Furthermore, the angle θ also seems to be without effect, since it may be rotated away by a chiral rotation

$$\Psi(\vec{x},t)\rightarrow e^{i\theta\gamma_5/2}\Psi(\vec{x},t) \tag{2.5}$$

without affecting the canonical anticommutation relations

$$\{\Psi(\vec{x},t),\Psi^\dagger(\vec{y},t)\}=\delta(\vec{x}-\vec{y})\mathbb{1},\quad \{\Psi(\vec{x},t),\Psi(\vec{y},t)\}=\{\Psi^\dagger(\vec{x},t),\Psi^\dagger(\vec{y},t)\}=0\ . \tag{2.6}$$

However, the results of I indicate that transformations such as (2.5) actually may take us from one Hilbert space to another. In other words, the physical content of $\mathscr{H}$ is not completely determined unless we also specify which representation of the canonical anticommutation relation to use.[23]

To do that on a rigorous basis of course would be very difficult; however, if we restrict ourselves to Fock representations[24] and formulate the question as which is the best basis to expand the field operator Ψ in, an approximation procedure immediately suggests itself: We simply apply the (Hartree-Fock) variational principle.

A straightforward way to proceed would be to take the Hartree-Fock Hamiltonian as

$$\mathscr{H}_{\rm HF}[A_0]=\tfrac{1}{2}\int d\vec{x}\,\Psi^\dagger(\vec{x},t)H\Psi(\vec{x},t)+\tfrac{1}{2}\int d\vec{x}(H\Psi)^\dagger(\vec{x},t)\Psi(\vec{x},t)+\int d\vec{x}\,A_0(\vec{x})j_0(\vec{x},t)+\tfrac{1}{2}\int d\vec{x}\,A_0(\vec{x})\Delta A_0(\vec{x}) \tag{2.7}$$

and to determine the potential A_0 through

$$\delta\langle 0_{\rm HF}|\mathscr{H}|0_{\rm HF}\rangle/\delta A_0=0\ , \tag{2.8}$$

where $|0_{\rm HF}\rangle$ is the ground state of $\mathscr{H}_{\rm HF}[A_0]$.

Although the solution of (2.8) is intractable analytically, it is not hard to guess its qualitative features. We may assume that A_0 is spherically symmetric, otherwise rotational symmetry would suffer spontaneous breakdown. Therefore the one-particle wave function may be decomposed into partial waves. We may also assume that A_0 is regular at the origin, corresponding to an extended charge distribution. According to I however, that means that a boundary condition at the origin must be imposed for the lowest partial wave so that $\mathscr{H}_{\rm HF}[A_0]$ possesses a decent vacuum. Therefore, we must also vary with respect to the boundary conditions; that is how they may enter the problem.

In fact, for the purpose of illustration, let us take the trial Hamiltonian to be simply

$$\mathscr{H}_0=\tfrac{1}{2}\int d\vec{x}\,\Psi^\dagger(\vec{x},t)\vec{\alpha}\cdot\vec{\pi}\Psi(\vec{x},t)+\tfrac{1}{2}\int d\vec{x}(\vec{\alpha}\cdot\vec{\pi}\Psi)^\dagger(\vec{x},t)\Psi(\vec{x},t)\ . \tag{2.9}$$

Again, if $\mathscr{H}_0$ is to have a decent vacuum, the wave equation for the lowest partial wave

$$\vec{\alpha}\cdot\vec{\pi}\frac{\chi(r)}{r}\eta_{jm}(\Omega)=E\frac{\chi(r)}{r}\eta_{jm}(\Omega) \tag{2.10}$$

must be supplemented by the boundary condition

$$\chi(0)\propto e^{i\tilde{\omega}\gamma_5/2}\begin{bmatrix} i\\ 1\end{bmatrix},\quad \gamma_5=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\ . \tag{2.11}$$

The solutions of (2.10) and (2.11) are given by

$$u_{k\tilde{\omega}}(r)=e^{i\tilde{\omega}\gamma_5/2}\begin{bmatrix} i\sin\left[kr+\frac{\pi}{4}\right]\\ \cos\left[kr+\frac{\pi}{4}\right]\end{bmatrix}\quad (E=k>0)\ ,$$

$$v_{k\tilde{\omega}}(r)=e^{i\tilde{\omega}\gamma_5/2}\begin{bmatrix} -i\sin\left[kr+\frac{\pi}{4}\right]\\ \cos\left[kr+\frac{\pi}{4}\right]\end{bmatrix}\quad (E=-k<0)\ , \tag{2.12}$$

and we may expand the field operator Ψ as

$$\Psi(\vec{x},t)=\frac{1}{\pi r}\sum_m\int_0^\infty dk[b_{k\tilde{\omega}m}e^{-ikt}u_{k\tilde{\omega}}(r)\eta_{jm}(\Omega)+d^\dagger_{k\tilde{\omega}m}e^{ikt}v_{k\tilde{\omega}}(r)\eta_{jm}(\Omega)]+(\text{higher } j)\ . \tag{2.13}$$

By virtue of the boundary condition (2.11), $b_{k\tilde{\omega}m}$ and $d_{k\tilde{\omega}m}$ also obey the canonical anticommutation relations, and determine the trial vacuum through

$$b_{k\tilde{\omega}m}|\tilde{\omega}\rangle=d_{k\tilde{\omega}m}|\tilde{\omega}\rangle=0\ . \tag{2.14}$$

Using (2.12), it is not hard to show that

$$\langle\tilde{\omega}|\mathscr{H}|\tilde{\omega}\rangle=-M\cos(\tilde{\omega}-\theta)\frac{2j+1}{2\pi}\int_0^\infty\frac{dk}{k}+(\tilde{\omega}\text{- independent terms})\ . \tag{2.15}$$

(The only thing we need to know about the higher partial waves is that they are chirality conserving.) Hence, the vacuum energy is minimized when $\tilde{\omega}=\theta$. Needless to say, $\mathscr{H}_0$ is a crude approximation to $\mathscr{H}$; nevertheless, we hope we have made our point. Different θ's lead to different Hilbert spaces; so do different boundary conditions.

III. THE DIRAC EQUATION IN A FIXED EXTERNAL FIELD

Let us now consider a Dirac particle in a non-Abelian monopole field, where the field is taken to be external and fixed. The solution of the corresponding Dirac equation has been discussed by many authors[10–17]; hence we shall review only the salient features. (Our conventions are summarized in the Appendix.)

We first take the case of an isospinor fermion in a point SU(2) monopole field.[25] Then the system is essentially Abelian[26] and we may immediately carry over our results from the Abelian case. In particular, the radial equation for the lowest partial wave reads[27]

$$H_0\chi(r)=E\chi(r) , \tag{3.1}$$

$$H_0=-i\gamma_5\tau_3\frac{d}{dr}+\beta M . \tag{3.2}$$

Again, it is necessary to impose a boundary condition at $r=0$; Hermiticity requires

$$\begin{aligned}0&=(\chi^{(1)},H_0\chi^{(2)})-(H_0\chi^{(1)},\chi^{(2)})\\&=i\chi^{(1)\dagger}(0)\gamma_5\tau_3\chi^{(2)}(0) .\end{aligned} \tag{3.3}$$

(After second quantization, this would translate into the conservation of fermion-number current $j_\mu=\overline{\Psi}\gamma_\mu\Psi$.)

It is convenient to work in a representation in which both γ_5 and τ_3 are diagonal:

$$\chi(r)=\begin{bmatrix}\chi_+(r)\\ \chi_-(r)\end{bmatrix},\quad \tau_3=\begin{bmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{bmatrix} , \tag{3.4}$$

$$\chi_\pm(r)=\begin{bmatrix}R_\pm(r)\\ L_\pm(r)\end{bmatrix},\quad \gamma_5=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \beta=-\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} . \tag{3.5}$$

We may then introduce the eigenvectors of $\gamma_5\tau_3$,

$$X_+=\begin{bmatrix}R_+\\ L_-\end{bmatrix},\quad X_-=\begin{bmatrix}L_+\\ R_-\end{bmatrix} , \tag{3.6}$$

so that (3.3) reads

$$X_+^{(1)\dagger}(0)X_+^{(2)}(0)=X_-^{(1)\dagger}(0)X_-^{(2)}(0) . \tag{3.7}$$

The desired boundary condition is therefore

$$X_+(0)=iUX_-(0) , \tag{3.8}$$

where U is an arbitrary 2×2 unitary matrix. (The i is conventional.) We also define the associated Pauli matrices t_1,t_2,t_3 corresponding to the action of $\beta\tau_1,\beta\tau_2,\tau_3$ on X; in particular, t_3 corresponds to the charge.

In terms of the solutions of the Abelian equation

$$\left[-i\gamma_5\frac{d}{dr}+\beta M\right]\hat{\chi}(r)=E\hat{\chi}(r) , \tag{3.9}$$

$$\hat{\chi}_E^{(R)}(r)=\begin{bmatrix}\cos kr+i(E/k)\sin kr\\ -i(M/k)\sin kr\end{bmatrix} , \tag{3.10}$$

$$\hat{\chi}_E^{(L)}(r)=\begin{bmatrix}i(M/k)\sin kr\\ \cos kr-i(E/k)\sin kr\end{bmatrix} , \tag{3.11}$$

the solutions of (3.1) may be expressed as

$$u_{kU}(r)=\begin{bmatrix}R_+(0)\hat{\chi}_E^{(R)}(r)+L_+(0)\hat{\chi}_E^{(L)}(r)\\ R_-(0)\hat{\chi}_E^{(R)*}(r)+L_-(0)\hat{\chi}_E^{(L)*}(r)\end{bmatrix} ,\quad E=(k^2+M^2)^{1/2} \tag{3.12}$$

$$v_{kU}(r)=\begin{bmatrix}R_+(0)\hat{\chi}_E^{(R)}(r)+L_+(0)\hat{\chi}_E^{(L)}(r)\\ R_-(0)\hat{\chi}_E^{(R)*}(r)+L_-(0)\hat{\chi}_E^{(L)*}(r)\end{bmatrix} ,\quad E=-(k^2+M^2)^{1/2} . \tag{3.13}$$

Using the formula

$$\begin{aligned}(\hat{\chi}_E,\hat{\chi}_{E'})=&\,|\hat{\chi}_E^\dagger(0)(E-\beta M)\hat{\chi}_E(0)|\frac{\pi}{k}\delta(E-E')\\&-i\hat{\chi}_E^\dagger(0)\gamma_5\hat{\chi}_{E'}(0)\frac{P}{E-E'}\end{aligned} \tag{3.14}$$

we find that the solutions will be orthonormal,

$$\begin{aligned}&(u_{kU},u_{k'U})=(v_{kU},v_{k'U})=\pi\delta(k-k') ,\\&(u_{kU},v_{k'U})=0 ,\end{aligned} \tag{3.15}$$

if we take $X_\pm(0)$ to be the eigenvectors of U

$$UX_\pm^{(i)}(0)=e^{i\theta_i}X_\pm^{(i)}(0)\ (i=1,2) \tag{3.16}$$

with length $k/[2E(E-M\sin\theta_i)]^{1/2}$.

If $\cos\theta_i<0$, there are also bound states

$$\begin{aligned}&B_U^{(i)}(r)=B_U^{(i)}(0)e^{-\kappa_i r} ,\\&E=M\sin\theta_i,\quad \kappa_i=M\,|\cos\theta_i| ,\\&UX_\pm^{(i)}(0)=e^{i\theta_i}X_\pm^{(i)}(0),\quad X_\pm^{(i)\dagger}(0)X_\pm^{(i)}(0)=\kappa_i .\end{aligned} \tag{3.17}$$

Together, u, v, and B form a complete set.

As in I, we may calculate the properties of the Dirac sea. The (one-dimensional) charge density is given by

$$\rho_U(r)=-\sum_{i=1}^{2}\frac{M\langle t_3\rangle_i\sin\theta_i}{2\pi}\int_M^\infty d\frac{\kappa}{(\kappa^2-M^2)^{1/2}}\frac{\kappa}{\kappa+M\cos\theta_i}e^{-2\kappa r} , \tag{3.18}$$

where

$$\langle t_3\rangle_i=X_\pm^{(i)\dagger}(0)t_3X_\pm^{(i)}(0)/X_\pm^{(i)\dagger}(0)X_\pm^{(i)}(0) . \tag{3.19}$$

Hence the total charge[28] is

$$Q_V(U)=-\sum_{i=1}^{2}\langle t_3\rangle_i\theta_i/4\pi \tag{3.20}$$

with the convention

$$|\theta_i| < \pi\ . \tag{3.21}$$

Similarly, the energy density is given by

$$T_{00}(r) = \sum_{i=1}^{2} \frac{M\sin\theta_i}{2}\delta(r) - \sum_{i=1}^{2} \frac{M\cos\theta_i}{\pi}\int_M^\infty d\kappa \frac{(\kappa^2-M^2)^{1/2}}{\kappa+M\cos\theta_i} e^{-2\kappa r}\ . \tag{3.22}$$

We note that the total energy may be finite if $\cos\theta_1+\cos\theta_2=0$,[15] in contrast with the Abelian case.

We now turn to the Dirac equation in the field of an extended non-Abelian monopole (dyon). Again, we restrict ourselves to the case of an SU(2) gauge field with one isospinor fermion.

In the static gauge, the Julia-Zee solution[8] takes the form

$$A_i^a(\vec{r}) = \epsilon_{aij}\hat{x}_j \frac{K(r)-1}{er}\ , \tag{3.23}$$

$$A_0^a(\vec{x}) = \hat{x}_a \frac{J(r)}{er}\ , \tag{3.24}$$

where $r=|\vec{x}|$, $\hat{x}_i=x_i/r$, and the space index i and the SU(2) index a both run from 1 to 3. (Warning: Sign errors and inconsistent positioning of spacetime indices are frequent in the literature.) There is also an isovector Higgs field of the form

$$\phi^a(\vec{x}) = \hat{x}_a \frac{H(r)}{er}\ . \tag{3.25}$$

The functions J,K,H satisfy the coupled differential equations

$$\begin{aligned} r^2J'' &= 2JK^2\ ,\\ r^2H'' &= 2HK^2 + \frac{\lambda}{e^2}(H^3 - m_W{}^2r^2H)\ ,\\ r^2K'' &= K(K^2-1) + K(H^2-J^2)\ , \end{aligned} \tag{3.26}$$

where m_W is the mass acquired by the vector bosons through the Higgs-Kibble mechanism, and λ is the self-coupling of the scalar ϕ.

The solution of (3.26) is not known analytically except in the Prasad-Sommerfield limit[29] ($\lambda\to 0$, e, m_W fixed):

$$\begin{aligned} J &= \sinh\gamma(m_W r \coth m_W r - 1)\ ,\\ K &= m_W r/\sinh m_W r\ ,\\ H &= \cosh\gamma(m_W r\coth m_W r - 1)\ . \end{aligned} \tag{3.27}$$

The qualitative features however are similar for the general case except for H in particular, the length scale is set by $r_0 = m_W{}^{-1}$, and the behavior as $r_0\to 0$ is required to be

$$J(r) = O(r^2),\quad K(r) = 1 + O(r^2)\ ,\qquad H(r) = O(r^2)\ , \tag{3.28}$$

so that the solution has finite energy. On the other hand, for $r\to\infty$,

$$\frac{K(r)}{er} \sim 0\ , \tag{3.29}$$

$$\frac{J(r)}{er} \sim eQ_D\left[-\frac{1}{Ie^2} + \frac{1}{4\pi r}\right]\ , \tag{3.30}$$

$$\frac{H(r)}{er} \sim \frac{m_W}{e}\ , \tag{3.31}$$

where the approach to the asymptotic form is exponential. The constant Q_D has an obvious physical meaning; it is the electric charge of the dyon.[28] The meaning of the other constant I is not so obvious; here we shall only remark that it is positive and of order r_0/e^2.

The radial equation for the lowest partial wave ($J=0$) now reads

$$\left[H + \frac{\tau_3}{2}\frac{J(r)}{r}\right]\chi(r) = E\chi(r)\ , \tag{3.32}$$

$$H = -i\gamma_5\tau_3\frac{d}{dr} + \beta M - \gamma_5\tau_2\frac{K(r)}{r}\ . \tag{3.33}$$

Unlike the case of the point monopole, the boundary condition at the origin is automatically determined, owing to the $1/r$ singularity of $K(r)/r$. The regular solution vanishes at the origin, whereas the irregular solutions fail to be square integrable there.

The behavior of the regular solution is such that

$$\begin{bmatrix} R_+(r_0) \\ L_-(r_0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} L_+(r_0) \\ R_-(r_0) \end{bmatrix} + O(kr_0)\ . \tag{3.34}$$

Therefore, it is quite tempting to conclude that

$$U = -it_1 \tag{3.35}$$

is the correct boundary condition to use in the limit $r_0\to 0$. Since (3.35) is charge violating, one may also expect such processes such as charge emission and charge-exchange scattering.[16,17]

The conclusion also seems to be justified on the basis of the semiclassical approximation[11,20]: The back reaction of the fermions is of higher order in the loop expansion.

Unfortunately, the argument turns out to have a loophole. As we shall see in the next section, effects which are formally of order e^2 are actually of magnitude e^2m_W, and in a sense dominate when $m_W\to\infty$. As a result, the effective boundary condition turns out to be charge conserving, in contrast with (3.35)

IV. THE COUPLED FERMION-MONOPOLE SYSTEM

We now treat the monopole (dyon) as a dynamical object, instead of as a fixed field configuration. To this end, it is convenient at first to go to the $A_0=0$ gauge, where in matrix notation

$$A_0^G(\vec{x},t) = GA_0^a(\vec{x})\frac{\tau^a}{2}G^\dagger + \frac{i}{e}G\dot{G}^\dagger = 0\ , \tag{4.1}$$

$$A_i^G(\vec{x},t) = GA_i^a(\vec{x})\frac{\tau^a}{2}G^\dagger - \frac{i}{e}G\nabla_i G^\dagger\ , \tag{4.2}$$

$$G = \exp\left[\frac{i}{2}\vec{\tau}\cdot\hat{x}\mathscr{J}(r)\varphi(t)\right]\ , \tag{4.3}$$

with

$$\mathscr{J}(r)=-\frac{I}{Q_D}\frac{J(r)}{r}\to 1\ (r\to\infty)\ . \tag{4.4}$$

(The overdot indicates differentiation with respect to time.) Comparison with (3.30) gives

$$Q_D=I\dot{\varphi}=\text{constant} \tag{4.5}$$

so the gauge transformation G is periodic in time at $r=\infty$.

Quantization[20] of the dyon degree of freedom consists in taking φ as a dynamical variable with period 2π.[30] The procedure should be valid to all orders in the semiclassical expansion if $M\neq 0$; we shall take it to be valid even beyond the semiclassical scheme.

If we ignore all other degrees of freedom, the gauge field part of the action

$$\tfrac{1}{2}\int dt\int d\vec{x}[E_i^a(\vec{x},t)E_i^a(\vec{x},t)-B_i^a(\vec{x},t)B_i^a(\vec{x},t)] + \frac{e^2\theta}{8\pi^2}\int dt\int d\vec{x}\,E_i^a(\vec{x},t)B_i^a(\vec{x},t) \tag{4.6}$$

reduces to that of a free rotator

$$S_\varphi=\frac{I}{2}\int dt\,\dot{\varphi}^2+\frac{\theta}{2\pi}\int dt\,\dot{\varphi} \tag{4.7}$$

identifying the constant I as the moment of inertia. It is easy to check that (4.6) correctly reproduces the Witten formula for the dyon charge in the absence of fermions

$$Q_D=n-\frac{\theta}{2\pi} \tag{4.8}$$

since the conjugate momentum

$$p_\varphi=I\dot{\varphi}+\frac{\theta}{2\pi} \tag{4.9}$$

can only take integer values.

To incorporate fermions, we simply add the action integral

$$\int dt\int d\vec{x}\,\Psi^\dagger(\vec{x},t)\left[i\frac{\partial}{\partial t}-\alpha_i\pi_i^G-\beta M\right]\Psi(\vec{x},t)\ , \tag{4.10}$$

$$\pi_i^G=-i\nabla_i+eA_i^G\ . \tag{4.11}$$

Undoing the gauge transformation and keeping only the $J=0$ partial wave, we arrive at

$$S_\chi=\int dt\int dr\,\chi^\dagger(r,t)\left[i\frac{\partial}{\partial t}-H\right]\chi(r,t) + \frac{\dot{\varphi}}{2}\int dt\int dr[\chi^\dagger(r,t),\tfrac{1}{2}\tau_3\mathscr{J}(r)\chi(r,t)]\ , \tag{4.12}$$

where H is defined in (3.31).

The conjugate momentum is now given by

$$p_\varphi=I\dot{\varphi}+\frac{\theta}{2\pi}+\tfrac{1}{2}\int dt\int dr[\chi^\dagger(r,t),\tfrac{1}{2}\tau_3\mathscr{J}(r)\chi(r,t)] \tag{4.13}$$

and the Hamiltonian is[31]

$$\mathscr{H}=\mathscr{H}_F+\mathscr{H}_C\ , \tag{4.14}$$

$$\mathscr{H}_F=\tfrac{1}{2}\int dr\,\chi^\dagger(r,t)H\chi(r,t)+\tfrac{1}{2}\int dr\,(H\chi)^\dagger(r,t)\chi(r,t)\ , \tag{4.15}$$

$$\mathscr{H}_C=\frac{1}{2I}\left[p_\varphi-\frac{\theta}{2\pi}-\tfrac{1}{2}\int dr[\chi^\dagger(r,t),\mathscr{J}(r)\tfrac{1}{2}\tau_3\chi(r,t)]\right]^2\ . \tag{4.16}$$

Evidently, p_φ commutes with $\mathscr{H}$, and is a constant of motion. Since isodoublet fermions are present, p_φ can take half-integer values and θ is defined only modulo π. (One may equally say that p_φ takes integer values and θ is defined modulo 2π as before.)

Rewriting (4.13) as

$$\frac{n}{2}-\frac{\theta}{2\pi}=I\dot{\varphi}+\tfrac{1}{2}\int dr[\chi^\dagger(r,t),\mathscr{J}(r)\tfrac{1}{2}\tau_3\chi(r,t)] \tag{4.17}$$

we find that it is just a statement of charge conservation[32]: The first term is the charge of the dyon, whereas the second term is the charge of the fermion.

The physical meaning of the Hamiltonian is equally simple. $\mathscr{H}_C$ is equal to

$$\frac{1}{2I}(I\dot{\varphi})^2=\frac{Q_D{}^2}{2I}\simeq\frac{e^2Q_D{}^2}{2r_0} \tag{4.18}$$

and represents the Coulomb energy of the core. $\mathscr{H}_F$ is the energy of the fermions in the magnetic part of the dyon field. Equation (4.18) also confirms the remark in the previous section: $\mathscr{H}_C$ is formally of order e^2, however, in actual magnitude, it is of order e^2m_W.

To put these remarks on a more quantitative basis, we apply a Hartree-Fock procedure as in Sec. II. Since our system is one dimensional, we may even contemplate a search for the best Fock basis, i.e., we expand the field operator χ in terms of an arbitrary complete set

$$\chi(r)=\frac{1}{\pi}\int dk[b_ku_k(r)+d_k^\dagger v_k(r)]\ , \tag{4.19}$$

$$(u_k,u_{k'})=(v_k,v_{k'})=\pi\delta(k-k'),\quad (u_k,v_{k'})=0\ , \tag{4.20}$$

$$\frac{1}{\pi}\int dk[u_k(r)u_k^\dagger(r')+v_k(r)v_k^\dagger(r')]=\delta(r-r')\mathbb{1}\ , \tag{4.21}$$

and minimize $\langle\mathscr{H}\rangle$ with respect to variations $\delta u_k,\delta v_k$ under the constraints (4.20) and (4.21). (We have suppressed possible bound states for notational simplicity.)

Actually, it is not hard to see that the solution will not be uniquely determined; separate unitary transformations on $\{u_k\}$ and $\{v_k\}$ do not lead to different Fock representations. This may also be directly verified; in a condensed notation

$$\langle\mathscr{H}\rangle=\mathrm{Tr}HP^-+\frac{1}{2I}\left[\frac{n}{2}-\frac{\theta}{2\pi}-\tfrac{1}{2}\mathrm{Tr}\mathscr{J}\tau_3(P^--P^+)\right]^2 + \frac{1}{2I}\mathrm{Tr}\mathscr{J}\tau_3P^+\mathscr{J}\tau_3P^-\ , \tag{4.22}$$

where

$$P^+=\frac{1}{\pi}\int dk\,u_ku_k^\dagger,\quad P^-=\frac{1}{\pi}\int dk\,v_kv_k^\dagger\ . \tag{4.23}$$

The constraints

$$(P^+)^2=P^+,\quad (P^-)^2=P^-,\quad P^++P^-=1 \tag{4.24}$$

may be satisfied by taking the variations to be of the form

$$\delta P^{+}=-\delta P^{-}=P^{+}\delta Q P^{-}+P^{-}\delta Q^{\dagger}P^{+} . \quad (4.25)$$

Hence,

$$\delta\langle\mathscr{H}\rangle=-\mathrm{Tr}(P^{-}H_{\mathrm{sc}}P^{+}\delta Q+P^{+}H_{\mathrm{sc}}P^{-}\delta Q^{\dagger}) , \quad (4.26)$$

$$H_{\mathrm{sc}}=H-\frac{1}{I}\left[\frac{n}{2}-\frac{\theta}{2\pi}+\tfrac{1}{2}\mathrm{Tr}\mathscr{I}\tau_3(P^{+}-P^{-})\right]\mathscr{I}\tau_3 + \frac{1}{2I}\mathscr{I}\tau_3(P^{+}-P^{-})\mathscr{I}\tau_3 , \quad (4.27)$$

which gives

$$P^{-}H_{\mathrm{sc}}P^{+}=P^{+}H_{\mathrm{sc}}P^{-}=0 \quad (4.28)$$

or

$$[H_{\mathrm{sc}},P^{+}]=[H_{\mathrm{sc}},P^{-}]=0 \quad (4.29)$$

which may be regarded as a condition of self-consistency.

For actual computations, it is convenient to rewrite (4.24) and (4.29) as

$$[H_{\mathrm{sc}},P^{+}-P^{-}]=0,\quad (P^{+}-P^{-})^2=1 , \quad (4.30)$$

since the diagonal elements of $P^{+}-P^{-}$ are directly related to the charge density, and are expected to have no δ-function singularities.

However, the equations are still formidable, so we shall take a more simple-minded approach. Since we are interested in the limit $m_W\to\infty$, we may as well take our trial basis to be the eigenfunctions of the point Hamiltonian H_0 under the various boundary conditions (3.8). Unfortunately however, that will mean that we shall encounter ultraviolet divergences. In accordance with the fact that we have dropped all the massive excitations of the gauge fields and the Higgs fields, we shall impose a cutoff $\Lambda\lesssim m_W$ for the fermions as well. It is important that the cutoff may be implemented in a basis-independent manner; at least formally, that may be accomplished by the replacement

$$\chi(r,t)\to\mathscr{P}\chi(r,t)\mathscr{P} , \quad (4.31)$$

$$\mathscr{P}=\Theta\left[\tfrac{1}{2}\int dr\,\chi^{\dagger}(rt)H_0\chi(r,t) + \mathrm{H.c.}-E_0-(\Lambda^2+M^2)^{1/2}\right] , \quad (4.32)$$

where Θ is the step function and E_0 is a constant independent of U.

There is one more difference between this case and that of Sec. II.

Owing to the $1/r$ singularity in H, substitution of

$$\chi(r)=\frac{1}{\pi}\sum_{i=1}^{2}\int dk[b_{kU}^{(i)}u_{kU}^{(i)}(r)+d_{kU}^{(i)\dagger}v_{kU}^{(i)}(r)] \quad (4.33)$$

into $\mathscr{H}$ will make sense only if

$$u_{kU}^{(i)\dagger}(0)\gamma_5\tau_2 u_{k'U}^{(j)}(0)=v_{kU}^{(i)\dagger}(0)\gamma_5\tau_2 v_{k'U}^{(j)}(0)=u_{kU}^{(i)\dagger}(0)\gamma_5\tau_2 v_{k'U}^{(j)}(0)=v_{kU}^{(i)\dagger}(0)\gamma_5\tau_2 u_{k'U}^{(j)}(0)=0 . \quad (4.34)$$

Otherwise, the coefficients of terms such as $b_k^{\dagger}b_{k'}$ will be divergent even with an ultraviolet cutoff.

In terms of the boundary conditions, (4.34) requires

$$U^{\dagger}t_1+t_1U=0 \quad (4.35)$$

which further implies that either

$$\theta_1=-\theta_2=\pm\pi/2,\quad U=\pm it_1 \quad (4.36)$$

or

$$\theta_1+\theta_2=\pm\pi,\quad U\neq\pm it_1 \quad (4.37)$$

must hold.

Having disposed of the preliminaries, we may now proceed to evaluate $\langle U\,|\,\mathscr{H}\,|\,U\rangle$, where $|\,U\rangle$ is our trial ground state:

$$b_{kU}^{(i)}\,|\,U\rangle=d_{kU}^{(i)}\,|\,U\rangle=0 . \quad (4.38)$$

The contribution of $\mathscr{H}_F$ is

$$\langle U\,|\,\mathscr{H}_F\,|\,U\rangle=-\frac{1}{\pi}\sum_{i=1}^{2}\int_0^{\infty}dr\int_0^{\Lambda}dk\,|\,E\,|\,v_{kU}^{(i)\dagger}(r)v_{kU}^{(i)}(r)-\frac{1}{\pi}\sum_{i=1}^{2}\int_0^{\infty}\frac{dr}{r}K(r)\int_0^{\Lambda}dk\,v_{kU}^{(i)\dagger}(r)\gamma_5\tau_2 v_{kU}^{(i)}(r) . \quad (4.39)$$

Apart from the U-independent constant

$$E_0=-\int_0^{\infty}dr\int_0^{\Lambda}dk\,|\,E\,| \quad (4.40)$$

the first term is finite and of order M; this is essentially a consequence of (3.22) and (4.35). On the other hand, from (3.12) and (3.13) we find

$$v_{kU}^{(i)\dagger}(r)\gamma_5\tau_2 v_{kU}^{(i)}(r)=-\frac{k}{|\,E\,|}\,\frac{|\,E\,|\sin\theta_i+M}{|\,E\,|+M\sin\theta_i}\langle t_1\rangle_i\sin 2kr . \quad (4.41)$$

Hence the second term is of order

$$\frac{m_W}{2\pi}\left[\ln\cosh\frac{\pi\Lambda}{m_W}\right]\sum_{i=1}^{2}\langle t_1\rangle_i\sin\theta_i , \quad (4.42)$$

where we have used the Prasad-Sommerfield solution (3.27) for explicit evaluation.

For the boundary condition (4.36)

$$\sum_{i=1}^{2}\langle t_1\rangle_i\sin\theta_i=\pm 2,\quad U=\pm it_1 \quad (4.43)$$

whereas for (4.37)

$$\sum_{i=1}^{2} \langle t_1 \rangle_i \sin\theta_i = 0 \; . \tag{4.44}$$

Therefore, in the absence of the Coulomb term $\mathcal{H}_C$, the energy is minimized for the charge-violating boundary (3.35) as expected.

Let us now consider the effect of $\mathcal{H}_C$. The decomposition

$$Q_F \equiv \tfrac{1}{2}\int dr[\chi^\dagger(r), \tfrac{1}{2}\tau_3 \mathcal{J}(r)\chi(r)] = :Q_F: + \langle U \mid Q_F \mid U \rangle \tag{4.45}$$

gives

$$\langle U \mid \mathcal{H}_C \mid U \rangle = \frac{1}{2I}\left[\frac{n}{2} - \frac{\theta}{2\pi} - \langle U \mid Q_F \mid U \rangle\right]^2 + \frac{1}{2I}\langle U \mid :Q_F:^2 \mid U \rangle \; , \tag{4.46}$$

where : : indicates normal ordering with respect to $\mid U \rangle$.

We find from (3.12) and (3.13) that

$$\langle U \mid Q_F \mid U \rangle = \int_0^\infty dr\, \mathcal{J}(r)\left[\rho_U(r) + O\left[\frac{M}{\Lambda r}\sin 2\Lambda r\right]\right] = Q_V(U) + O\left[\frac{M}{m_W}\ln\frac{m_W}{M}\right] \tag{4.47}$$

and the first term of (4.46) to be

$$\frac{1}{2I}\left[\frac{n}{2} - \frac{\theta}{2\pi} - Q_V(U)\right]^2 + O\left[e^2 M \ln\frac{m_W}{M}\right] . \tag{4.48}$$

On the other hand, the second term gives

$$\frac{1}{8\pi^2 I}\sum_{i=1}^{2}\sum_{j=1}^{2}\int_{0<k+k'<\Lambda} dk\, dk' \mid (u_{kU}^{(i)}, \mathcal{J}\tau_3 v_{k'U}^{(j)}) \mid^2 . \tag{4.49}$$

We find

$$u_{kU}^\dagger(r)\tau_3 v_{k'U}(r) = u_{kU}^\dagger(0)\tau_3 v_{k'U}(0)\cos(k+k')r - i u_{kU}^\dagger(0)\gamma_5 v_{k'U}(0)\sin(k+k')r + O\left[\frac{M}{k}, \frac{M}{k'}\right] , \tag{4.50}$$

$$(u_{kU}, \tau_3 v_{k'U}) = u_{kU}^\dagger(0)\gamma_5 v_{k'U}(0)\frac{-i}{E + \mid E' \mid} , \tag{4.51}$$

and therefore

$$(u_{kU}, \mathcal{J}\tau_3 v_{k'U}) \simeq u_{kU}^\dagger(0)\gamma_5 v_{k'U}(0)\frac{-i}{E + \mid E' \mid} + O\left[\frac{1}{m_W}\ln\frac{k+k'}{m_W}\right] . \tag{4.52}$$

The first term vanishes if and only if

$$\tau_3 v_{k'U}(0) = \pm v_{k'U}(0) \tag{4.53}$$

or in terms of U,

$$[U, t_3] = 0 \; . \tag{4.54}$$

In that case

$$\frac{1}{2I}\langle U \mid :Q_F:^2 \mid U \rangle \simeq \frac{1}{8\pi^2 I}\left[\frac{\Lambda}{m_W}\right]^2 , \tag{4.55}$$

otherwise

$$\frac{1}{2I}\langle U \mid :Q_F:^2 \mid U \rangle \simeq \frac{1}{32\pi^2 I}\mathrm{tr}[U, t_3][t_3, U^\dagger]\ln\frac{\Lambda}{M} \; . \tag{4.56}$$

In particular, as noted by Besson,[15] the semiclassical expansion leads to an infrared divergence as $M \to 0$.

From these results, we may rule out

$$\theta_1 = -\theta_2 = \pi/2, \quad U = it_1 \tag{4.57}$$

and

$$\theta_1 + \theta_2 = \pm\pi, \quad [U, t_3] \neq 0 \; , \tag{4.58}$$

since they have higher energies relative to

$$\theta_1 = -\theta_2 = -\pi/2, \quad U = -i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q_V(U) = 0 \tag{4.59}$$

and

$$\theta_1 + \theta_2 = \pm\pi, \quad U = \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix}, \quad Q_V = \frac{-\theta_1 + \theta_2}{4\pi} \; . \tag{4.60}$$

To decide between these cases, we also need to take into account the piece of the Coulomb energy given by (4.48).

We first consider the case when the total charge lies within the range

$$-\tfrac{1}{4} < \frac{n}{2} - \frac{\theta}{2\pi} < \tfrac{1}{4} \; . \tag{4.61}$$

Then for (4.60), all the charge may be carried by the vacuum with

$$\theta_1 - \theta_2 = 2\theta, \quad \theta_1 + \theta_2 = \pm\pi, \quad \mid \theta_1 \mid , \mid \theta_2 \mid < \pi \; , \tag{4.62}$$

and (4.48) becomes of order

$$\frac{e^2 M^2}{m_W}\left[\ln\frac{m_W}{M}\right]^2 \tag{4.63}$$

which is negligible. On the other hand, (4.59) will lead to an extra Coulomb energy

$$\frac{1}{2I}\left[\frac{\theta}{2\pi}\right]^2 \tag{4.64}$$

since the dyon core must carry the charge.

Collecting all the pieces together and setting $\Lambda = m_W$, we find

$$U = -it_1 \ , \quad \langle \mathscr{H} \rangle = \frac{1}{4\pi^2 I} \ln \frac{m_W}{M} - \frac{m_W}{\pi} \ln \cosh \pi \ , \tag{4.65}$$

$$U = \pm i e^{i\theta t_3}, \quad \langle \mathscr{H} \rangle = 0 \tag{4.66}$$

up to terms of order $e^2 m_W$. Hence the charge-conserving boundary condition (4.66) is favored when

$$\frac{e^2}{4\pi} \ln \frac{m_W}{M} \gtrsim O(1) \ . \tag{4.67}$$

In particular, the condition is always met for $m_W \to \infty$ or $M \to 0$, and we conclude that (4.66) is the correct effective boundary condition in that limit.

Values of n and θ other than (4.61) may also be treated, provided we fill some positive-energy levels or empty some negative-energy levels with respect to $|U\rangle$. A few occupied or unoccupied levels will not make much of a difference in (4.65) and (4.66), and we again conclude that the correct boundary condition is charge conserving in the limit $m_W \to \infty$.

In this connection, an interesting point is whether the relevant level is a bound state or a scattering state, since the former corresponds to a stable dyon, whereas the lattice corresponds to an unstable one. (We assume that $M \neq 0$.) Owing to the constraints (3.21) and (4.37), there are no bound states if $\theta_1 = \theta_2 = \pm\pi/2$, whereas there is just one if otherwise; working out the quantum numbers of those states and taking into account the vacuum charge $Q_V(U)$, we find that stable dyons can exist only for

$$\left| \frac{n}{2} - \frac{\theta}{2\pi} \right| < \tfrac{1}{2} \ . \tag{4.68}$$

We may record their charge distribution and their fermion number

$$F = \int dr \, {:}\chi^\dagger(r)\chi(r){:} \tag{4.69}$$

as well as the relevant boundary conditions:

$$(1)\ n=0, \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad U = \pm i e^{i\theta t_3}, \quad F = 0 \ ,$$

$$\rho(r) \sim -(M/2)\sin\theta \quad (Mr_0 \ll Mr \ll 1)$$
$$\sim -(M/4\pi r)^{1/2} \tan\theta e^{-2Mr} \quad (Mr \gg 1) \ . \tag{4.70}$$

$$(2a)\ \theta - n\pi = \frac{\pi}{2}, \quad U = t_3 \ ,$$

$$\rho(r) = -(M/2)e^{-2Mr} \ . \tag{4.71a}$$

$$(2b)\ \theta - n\pi = -\frac{\pi}{2}, \quad U = -t_3 \ ,$$

$$\rho(r) = (M/2)e^{-2Mr} \ . \tag{4.71b}$$

$$(3)\ n = 1, \ 0 < \theta < \frac{\pi}{2} \text{ or } n = -1, \ -\frac{\pi}{2} < \theta < 0 \ ,$$

$$U = \pm i e^{i\theta t_3}, \quad F = \pm 1 \ ,$$

$$\rho(r) \sim (M/2)\sin\theta \quad (Mr_0 \ll Mr \ll 1)$$
$$\sim M \sin\theta e^{-2M|\sin\theta| r} \quad (Mr \gg 1) \ . \tag{4.72}$$

[Note that for (2a) and (2b) the zero-energy mode is completely occupied or unoccupied, because of the Coulomb term $\mathscr{H}_C$.]

The twofold degeneracy of the vacuum is a consequence of the fact that

$$\langle U | \mathscr{H} | U \rangle = \langle U' | \mathscr{H} | U' \rangle \ , \quad U' = t_2 U^* t_2 \ , \tag{4.73}$$

owing to the invariance of the Hamiltonian under fermion-number conjugation

$$\chi \to \rho_2 \tau_1 \chi^\dagger \ . \tag{4.74}$$

The two vacua belong to two different Hilbert spaces as $m_W \to \infty$; hence the symmetry is spontaneously broken. On the other hand, the charge-violating boundary condition (3.35) is invariant under the conjugation symmetry, so we may expect a phase transition as we vary the ratio m_W/M.

The existence of a broken symmetry poses some problems, one of which is the behavior of a system with a monopole and an antimonopole. Since the system is expected to lie in the same sector as the vacuum, the conjugation symmetry should be restored, and it is not clear how that may happen, although there seems to be nothing wrong with that either. (A related problem is whether there is a state which in some sense would interpolate between the two vacua for a single monopole.)

We may also note that the system has fractional charge but integer fermion number. The latter is partly a matter of definition; it is possible to define fermion number as

$$\tfrac{1}{2} \int dr [\chi^\dagger(r), \chi(r)] \tag{4.75}$$

which amounts to the addition of the c number

$$-\frac{\theta_1 + \theta_2}{2\pi} = \mp \frac{1}{2} \ . \tag{4.76}$$

The definition (4.75) is more suitable as an order parameter; however, we believe the assignment of (4.69)—(4.72) is more expressive of the fermionic structure.

Evidently, it is also of interest to see what happens for the case originally treated by Jackiw and Rebbi,[11] where the fermion gets its mass from the Higgs field.[33] As shown in the Appendix, that is equivalent to changing $\gamma_5 \tau_2 K(r)/r$ in H to $\tau_2 K(r)/r$, which changes (4.20) to

$$t_2 U - U^\dagger t_2 = 0 \tag{4.77}$$

and (4.21) and (4.22) to

$$\cos\theta_1 = -\cos\theta_2 = \pm 1, \quad U = \pm t_2 \ , \tag{4.78}$$

$$\theta_1 = -\theta_2, \quad U \neq \pm t_2 \ . \tag{4.79}$$

Again, we find that $\mathscr{H}_C$ favors a charge-conserving boundary condition over a nonconserving one, whereas $\mathscr{H}_F$ favors (4.78) over the others; in the limit $m_W \to \infty$, $\mathscr{H}_C$ dominates over $\mathscr{H}_F$ as before, leading to the boundary condition

$$U = e^{i\theta t_3} \tag{4.80}$$

for

$$-\frac{1}{4} < \frac{n}{2} - \frac{\theta}{2\pi} < \frac{1}{4} \ . \tag{4.81}$$

However, the difference between (4.37) and (4.79) turns out to be quite significant for larger values of the total charge. Comparing with (3.22), we find that with a Higgs mass, the vacuum energy is logarithmically divergent; in particular, $\cos\theta_i > 0$ is favored over $\cos\theta_i < 0$, by letting the fermions carry some of the charge if necessary. However, since there are no bound states for $\cos\theta_i > 0$, we find that stable dyons only exist within the range (4.81), in contrast with (4.54). Again, we may record their properties:

$$n=0,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2},\quad U=e^{i\theta t_3},\quad F=0\ ,$$
$$\rho(r)\sim-\frac{M\sin\theta}{\pi}\ln Mr\ \ (Mr_0 \ll Mr \ll 1)$$
$$\sim-\left[\frac{M}{4\pi r}\right]^{1/2}\tan\frac{\theta}{2}e^{-2Mr}\ (Mr \gg 1)\ . \tag{4.82}$$

We note that fermion-number conjugation for a Higgs mass

$$\chi \to \rho_2\tau_2\chi^\dagger \tag{4.83}$$

is unbroken throughout. In particular, for the charge-violating boundary condition (4.78), there is one level with exactly zero energy,[34] as in the original analysis of Jackiw and Rebbi. However, for the charge-conserving boundary condition (4.80), there is no such level within the range (4.81), and the ground state is nondegenerate with zero fermion number.

Finally, as a consistency check on our calculation, let us consider the induced vacuum current

$$j_\mu^a=-\frac{e}{2}\langle[\overline{\Psi},\gamma_\mu\tfrac{1}{2}\tau^a\Psi]\rangle\ . \tag{4.84}$$

We find the contribution of the lowest partial wave to be

$$j_0^a=\frac{e}{4\pi r^2}\hat{x}_a\rho(r)\ , \tag{4.85}$$
$$j_k^a=\frac{e}{8\pi r^2}\hat{x}_a\hat{x}_k\langle\tfrac{1}{2}[\chi^\dagger(r),\gamma_5\chi(r)]\rangle$$
$$+\frac{e}{8\pi r^2}(\delta_{ak}-\hat{x}_a\hat{x}_k)\langle\tfrac{1}{2}[\chi^\dagger(r),\gamma_5\tau_1\chi(r)]\rangle$$
$$+\frac{e}{8\pi r^2}\epsilon_{akl}\hat{x}_l\langle\tfrac{1}{2}[\chi^\dagger(r),\gamma_5\tau_2\chi(r)]\rangle\ , \tag{4.86}$$

for a Dirac mass, and

$$j_0^a=\frac{e}{4\pi r^2}\hat{x}_a\rho(r)\ , \tag{4.87}$$
$$j_k^a=\frac{e}{8\pi r^2}\hat{x}_a\hat{x}_k\langle\tfrac{1}{2}[\chi^\dagger(r),\gamma_5\chi(r)]\rangle$$
$$+\frac{e}{8\pi r^2}(\delta_{ak}-\hat{x}_a\hat{x}_k)\langle\tfrac{1}{2}[\chi^\dagger(r),\tau_1\chi(r)]\rangle$$
$$+\frac{e}{8\pi r^2}\epsilon_{akl}\hat{x}_l\langle\tfrac{1}{2}[\chi^\dagger(r),\tau_2\chi(r)]\rangle\ , \tag{4.88}$$

for a Higgs mass.

On the other hand, we find from (3.23)–(3.25) that

$$(D_\mu F_{\mu 0})^a=-\frac{1}{er^3}\hat{x}_a(r^2J''-2JK^2)\ , \tag{4.89}$$

$$(D_\mu F_{\mu k})^a=-\frac{1}{er^3}\epsilon_{akl}\hat{x}_l[r^2K''-K(K^2-1)]\ . \tag{4.90}$$

For the charge-conserving boundary conditions (4.66) and (4.80), j_k^a vanishes, and we find that the spin-isospin structure of $D_\mu F_{\mu\nu}$ and j_ν are the same.

This would not have been a cause for concern had the loop expansion been valid; as apparent from (4.89) and (4.90), $D_\mu F_{\mu\nu}$ is of order $1/e$, whereas j_ν is of order e. However, we have seen that the expansion is not valid in the limit $m_W \to \infty$, so a mismatch in the indices is potentially dangerous, and it is gratifying that our results have passed this check.

V. DISCUSSION

We have shown that the coupled fermion-monopole (dyon) system behaves quite differently from what one would naively expect from the solution of the Dirac equation for a fixed-field configuration. For a heavy dyon, the charge of the fermions and the charge of the dyon was essentially conserved separately, and the system was Abelian in character. Also, for a Dirac mass, fermion-number conjugation was spontaneously broken, whereas for a Higgs mass, there was no zero-mode degeneracy. The reason for the discrepancy was quite simple: There is a large Coulomb energy in the limit $m_W \to \infty$, if the charge resides on the dyon.

This aspect, as well as the existence of a fermionic structure has already been anticipated by Wilczek[5]; the difference is that our work is more specific and quantitative. In particular, within our approximation, the quantum nature of the dyon degree of freedom was seen to be quite important.[35] If the Coulomb energy is to be small

$$\langle\ |Q_D{}^2|\ \rangle \simeq 0\ , \tag{5.1}$$

we must have

$$Q_D\,|\ \rangle \simeq 0 \tag{5.2}$$

rather than just

$$\langle\,|Q_D|\,\rangle \simeq 0 \tag{5.3}$$

and the dyon degree of freedom is essentially frozen for the low-lying states.

Furthermore, (5.2) together with charge conservation gives

$$Q_F\,|\ \rangle \simeq \left[\frac{n}{2}-\frac{\theta}{2\pi}\right]|\ \rangle \tag{5.4}$$

which implies that the effective boundary condition must be charge conserving. Compatibility with a finite energy for the gauge field (3.28) then determines U to be (4.66) and (4.80).

Evidently, the argument is quite general and may be expected to hold even if other degrees of freedom are taken into account; also, there was a nontrivial consistency condition which was satisfied as well. Since previous authors have adopted the charge-violating boundary conditions (4.65) and (4.78) in conflict with these considerations,[35] we believe it is necessary to reassess their results.[11–19]

As for the work based on the one-particle equation with the charge-violating boundary condition, we have already argued that the results are not expected to be valid for the low-lying states in the case $m_W \gg M$. Actually, this point is already implicit in the remark of Blaer, Christ, and

Tang[16] that their results do not apply in the long-time limit, i.e., for stable dyons.

The work of Rubakov[18] and Callan[19] is more subtle, since they have gone beyond the one-particle approximation. While the authors use the charge-violating boundary condition, they also include the Coulomb energy, and the suggestion is that the original boundary condition is in effect wiped out. In particular, Callan has performed a classical analysis for a bosonized Hamiltonian in the limit $m_W \gg M$, and has found that there is strong coupling between the dyon degree of freedom and the Dirac sea; as a result, all the electric charge was found with the fermions, in agreement with our results.

Upon close inspection however, we find significant difference. For a Dirac mass, Callan's boundary conditions do not break fermion-number conjugation as ours do; for a Higgs mass, they give a degeneracy for $\theta=0$, which we do not find. The discrepancy apparently stems from the fact that the original boundary condition actually remains intact in Callan's case. It follows from the structure (in his notation):

$$\chi_u(r,t)=:\exp\left\{i\sqrt{\pi}\left[\phi(r,t)-\int_0^r ds\,\dot{\phi}(s,t)\right]\right\}: , \qquad (5.5)$$

$$\chi_l(r,t)=i:\exp\left\{i\sqrt{\pi}\left[\phi(r,t)+\int_0^r ds\,\dot{\phi}(s,t)\right]\right\}: , \qquad (5.6)$$

that whatever the boundary condition for the boson fields, one still has the charge-violating boundary condition

$$\overline{\gamma}_0\begin{bmatrix}\chi_u(0)\\ \chi_l(0)\end{bmatrix}=\begin{bmatrix}\chi_u(0)\\ \chi_l(0)\end{bmatrix} \qquad (5.7)$$

for the fermion fields.

To summarize, if we are to use a "microscopic" boundary condition, we should include the Coulomb energy from the beginning; on the other hand, if we are to use an effective boundary condition, we should not use the boson-fermion correspondence in the forms (5.5) and (5.6).

Evidently, an important question is what happens if there exist two or more isodoublets, since that may be considered as the prototype of monopole-induced proton decay,[5,18,19] if we identify the charge $\overline{Q}$ as the sum of ordinary electric charge and color hypercharge.[36] Unfortunately, the calculations of this paper do not furnish us with a clear clue as to what to expect. The process

$$u_{1L}+u_{2L}\underset{M}{\rightarrow}\overline{d}_{3R}+e_R^+ \qquad (5.8)$$

is certainly allowed as far as $\overline{Q}$ is concerned; on the other hand, there are no positive indications that it should occur either.

Also in our opinion, an equally important question is whether or not there exists a significant back reaction of the fermions on the magnetic part of the dyon field, since the bulk of the mass of the dyon comes from its magnetic energy. Although our results were negative, they certainly cannot be regarded as conclusive proof: As evident from (3.26), the electric field and the magnetic field are highly coupled for a non-Abelian dyon, and there is no reason *a priori* that a more sophisticated analysis would not yield a change in the magnetic structure and hence its mass as well.

We hope to return to these issues in the near future, as well as to give a systematic comparison between our approach and that of Rubakov and Callan. Although our approach was successful in giving interesting information such as (4.69)–(4.71) and (4.81), further improvements are obviously necessary, even within the framework of the truncated Hamiltonian, not to mention all the other degrees of freedom we have ignored; in particular, the estimate (4.67) as it stands is quite unsatisfactory. Bosonization[37,38] should be useful in this regard, since much is known about the $(\sin\varphi)_2$ model both classically[39] and quantum mechanically.[40]

Note added. After the original version of this paper was submitted, we received a paper by D'Hoker and Farhi,[41] in which a different approach to the point limit is suggested. Their limiting procedure however ignores the dyon degree of freedom, and as a consequence requires charge symmetry breaking for the fermion mass term.

We have also received a paper by Kazama[42] which criticizes the use of charge-violating boundary conditions, apparently however for different reasons than we do. There have also appeared several papers[43] which question the validity of the charge superselection rule in the presence of a non-Abelian monopole, essentially because of the charge-violating boundary condition. In our opinion, these papers do not deal properly with the dyon degree of freedom φ and its associated Coulomb energy, and we cannot concur with their contents. As evident from Eq. (4.13), the canonical conjugate of φ is essentially the global U(1) charge, so a proper treatment is crucial in this respect. Our calculation also supports the superselection rule in the sense that there are no states with lower energy which violate the superselection rule. It should be mentioned however that so far there is no rigorous proof of the charge superselection rule in the presence of a monopole.[44] I would like to thank Dr. D'Hoker and Professor Yoneya for correspondence.

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APPENDIX

Our conventions are as follows: For spacetime indices, we adopt the Feynman notation

$$a_\mu b_\mu = a_0 b_0 - a_i b_i ,$$
$$x_\mu=(t,\vec{x}),\quad \nabla_\mu=(\partial/\partial t,-\vec{\nabla}),\quad A_\mu=(A_0,\vec{A}) , \qquad (A1)$$
$$D_\mu=\nabla_\mu - ieA_\mu,\quad F^a_{\mu\nu}=\nabla_\mu A^a_\nu-\nabla_\nu A^a_\mu+e\epsilon^{abc}A^b_\mu A^c_\nu .$$

(Note the sign change of eA relative to the Abelian convention of I.)

The Dirac equation reads as

$$\left[\vec{\alpha}\cdot\vec{\pi}+\beta M-\tfrac{1}{2}\vec{\tau}\cdot\hat{x}\frac{J(r)}{r}\right]\psi(\vec{x})=E\psi(\vec{x}) , \qquad (A2)$$

where

$$\vec{\pi}=-i\vec{\nabla}+\frac{e}{2}\tau^a A^a=-i\vec{\nabla}-\tfrac{1}{2}(\vec{\tau}\times\hat{x})\frac{K(r)-1}{r}\ . \tag{A3}$$

The Hamiltonian commutes with the angular momentum operator[10,11]

$$\vec{J}=-i\vec{x}\times\vec{\nabla}+\tfrac{1}{2}\vec{\sigma}+\tfrac{1}{2}\vec{\tau}\ . \tag{A4}$$

On the other hand, fermion-number conjugation

$$\psi\rightarrow\rho_2\sigma_2\tau_2\psi^* \tag{A5}$$

transforms a solution with energy E into a solution with $-E$.

We split ψ into its chiral components and introduce

$$(\mathcal{M}_R)_{ai}=(\tau_2)_{ij}(\psi_R)_{aj},\ \ (\mathcal{M}_L)_{ai}=(\tau_2)_{ij}(\psi_L)_{aj}\ . \tag{A6}$$

Following Ref. 11 and treating $\mathcal{M}$ as 2×2 matrices, we find

$$\begin{aligned}
&-i\vec{\sigma}\cdot\vec{\nabla}\mathcal{M}_R-\frac{K(r)-1}{2r}(\vec{\sigma}\times\hat{x})_i\mathcal{M}_R\sigma_i\\
&\qquad+\frac{1}{2}\frac{J(r)}{r}\mathcal{M}_R(\vec{\sigma}\cdot\hat{x})-M\mathcal{M}_L=E\mathcal{M}_R\ ,\\
&i\vec{\sigma}\cdot\vec{\nabla}\mathcal{M}_L+\frac{K(r)-1}{2r}(\vec{\sigma}\times\hat{x})_i\mathcal{M}_L\sigma_i\\
&\qquad+\frac{1}{2}\frac{J(r)}{r}\mathcal{M}_L(\vec{\sigma}\cdot\hat{x})-M\mathcal{M}_R=E\mathcal{M}_L\ .
\end{aligned} \tag{A7}$$

For the lowest partial wave, we may take $\mathcal{M}_R$ to be of the form

$$\mathcal{M}_R=\frac{1}{4\sqrt{\pi}r}[(1+\vec{\sigma}\cdot\hat{x})R_+(r)+(1-\vec{\sigma}\cdot\hat{x})R_-(r)] \tag{A8}$$

and similarly for $\mathcal{M}_L$. The result is

$$\begin{aligned}
&-i\left[\frac{d}{dr}\pm\frac{K(r)}{r}\right](R_+\mp R_-)-M(L_+\pm L_-)\\
&\qquad+\frac{1}{2}\frac{J(r)}{r}(R_+\mp R_-)=E(R_+\pm R_-)\ ,\\
&i\left[\frac{d}{dr}\pm\frac{K(r)}{r}\right](L_+\mp L_-)-M(R_+\pm R_-)\\
&\qquad+\frac{1}{2}\frac{J(r)}{r}(L_+\mp L_-)=E(L_+\pm L_-)
\end{aligned} \tag{A9}$$

for (A7) and

$$\begin{aligned}
&R_+\pm R_-\rightarrow\pm i(L_+^*\mp L_-^*)\ ,\\
&L_+\pm L_-\rightarrow\pm i(R_+^*\mp R_-^*)
\end{aligned} \tag{A10}$$

for (A5), which reduce to

$$\left[-i\gamma_5\tau_3\frac{d}{dr}-\gamma_5\tau_2\frac{K(r)}{r}+\beta M+\frac{\tau_3}{2}\frac{J(r)}{r}\right]\chi(r)=E\chi(r) \tag{A11}$$

and

$$\chi\rightarrow\rho_2\tau_1\chi^* \tag{A12}$$

after we introduce the new Pauli matrices (3.4).

For a Higgs mass, M in (A2) is replaced by $G(\vec{\tau}\cdot\hat{x})H(r)/2er$ where G is the Yukawa coupling. Since $H(r)/r$ is regular at both $r=0$ and $r=\infty$, in contrast with $J(r)/r$ and $K(r)/r$, we may replace it by a constant. This leads to the equation for the lowest partial wave,

$$\left[-i\gamma_5\tau_3\frac{d}{dr}-\gamma_5\tau_2\frac{K(r)}{r}-\beta\tau_3 M+\frac{\tau_3}{2}\frac{J(r)}{r}\right]\chi(r)=E\chi(r) \tag{A13}$$

with fermion-number conjugation

$$\chi\rightarrow\beta\tau_1\chi^*\ . \tag{A14}$$

To make use of the solutions for a Dirac mass, it is convenient to eliminate the τ_3 in the mass term by the transformation

$$\chi\rightarrow\left[\frac{1-\tau_3}{2}+\gamma_5\frac{1+\tau_3}{2}\right]\chi \tag{A15}$$

which leads to the final form of the equation

$$\left[-i\gamma_5\tau_3\frac{d}{dr}-\tau_2\frac{K(r)}{r}+\beta M+\frac{\tau_3}{2}\frac{J(r)}{r}\right]\chi(r)=E\chi(r) \tag{A16}$$

and fermion-number conjugation

$$\chi\rightarrow\rho_2\tau_2\chi^*\ . \tag{A17}$$

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[33]Note though that an abnormally small Yukawa coupling is required for $m_W \gg M$.

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