# Quantum chromodynamics for static sources in the $A^0=0$ gauge and Schrödinger representation—Quantum fluctuations about the classical limit

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Quantum chromodynamics for static $c$-number sources is studied nonperturbatively using the canonical formalism with $A^0=0$. Solutions to the functional differential homogeneous Gauss-law equation are found and the issue of Gauss-law constraints on the state vector is approached by transforming to a set of curvilinear gauge-invariant coordinates. When $\hbar$ is small, but finite, collective oscillations of the gluon field about a mean field are found. An illustrative example of a single color charge source is considered.

In recent years, many nonperturbative studies of solutions to the classical Yang-Mills equations with static $c$-number sources have been carried out (see, for example, Refs. 1–8). While the non-Abelian nature of these equations makes their solution difficult and interesting from a mathematical viewpoint, the real motivation behind these efforts is the hope that certain features of the solutions will be preserved by the true quantum solutions. Nevertheless, the quantal significance of the classical solutions has remained quite obscure.

With a view towards understanding the relation between QCD and its classical counterpart, we shall consider the quantum theory with fixed $c$-number sources in the Schrödinger representation and $A^0=0$ gauge. In this representation the "coordinates" of the time-independent Schrödinger equation can be chosen to be the gauge field $\vec{A}(x)$, and $-\vec{E}(x)$ as the conjugate momentum. Explicit covariance is sacrificed in favor of an equal-time surface. While such a Lorentz-frame choice would not be suitable for scattering problems, it is entirely appropriate for bound states.

The gauge choice $A^0=0$ is a highly desirable one for at least two reasons. First, one completely avoids the issue of the Gribov singularity[9] for strong fields. This singularity is a feature of the Coulomb gauge, and arises from the noninvertibility of a certain operator when the field strength exceeds a critical value. No unambiguous way of formulating the strong-field problem beyond the critical value is known. The second advantage is the existence of a straightforward canonical quantization procedure. The price paid for these gains is well known—one must require all physical states to obey the Gauss law. This means that the state vector must obey an infinite number of constraints, in addition to being a solution of the equation of motion. For the gauge group SU(2), Goldstone and Jackiw[10] (see also Baluni and Grossman[11]) were able to construct an effective Hamiltonian which operates on unconstrained states. Unfortunately, the complex structure of this Hamiltonian makes practical calculations unfeasible.

To confront the fundamental problem of gauge invariance of solutions to the Schrödinger equation, we shall not try to construct an effective Hamiltonian as in Refs. 10 and 11. Rather, an infinite set of gauge-invariant coordinates $\{Q\}$ will be constructed. By requiring that the amplitude of the wave functional $\Psi[A]$ be expressed only in terms of $Q_i$, this amplitude automatically satisfies gauge invariance. One is still left with a condition for the phase and a functional differential equation whose exact solution is, of course, impossible. Nevertheless, by posing the problem in gauge-invariant coordinates, some gains can be made. First, the classical wave functional is seen to be proportional to a $\delta$ function in an appropriate gauge-invariant coordinate. Second, by assuming $\hbar$ infinitesimal the classical solution can be corrected for minimal quantum fluctuations of the gluon field.

The organization of this paper is as follows.

In Sec. I, the necessary notation and formalism will be established by briefly reviewing the classical Yang-Mills theory, and then the issues of gauge invariance and canonical quantization in the $A^0=0$ gauge.

In Sec. II, the wave functional $\Psi[A]$ will be expressed as the product of an amplitude and a phase, and the corresponding Schrödinger and Gauss equations for these quantities written down. Next, the case $g=0$ will be solved and a functional introduced which generates a complete set of states obeying the Gauss law for $g=0$. This functional will then be generalized to the case of arbitrary $g$, and a set of generalized curvilinear gauge-invariant coordinates $\{Q\}$ defined. After displaying the transformed Schrödinger equation, the relation to the classical Yang-Mills theory will be made explicit. Then, by assuming $\hbar$ to be small but finite, a quantum wave functional will be postulated which reduces to the classical form as $\hbar\to 0$. Collective oscillations of the gluon field centered around the classical field value are found. Finally, an application to a single static charge source is made.

Section III contains a brief summary and closing comments.

## I. NOTATION AND THEORY

The canonical formalism for the Yang-Mills theory has been extensively reviewed elsewhere (see, for example, Refs. 12 and 13). Therefore, only aspects of direct relevance will be covered here.

The Yang-Mills Hamiltonian in the limit of infinite quark masses and the $A^0=0$ gauge is given by

$$H=\tfrac{1}{2}\int d^3x(\vec{E}^l\cdot\vec{E}^l+\vec{B}^l\cdot\vec{B}^l)\ , \tag{1.1}$$

 

where $\vec{E}_i^l(x)$ and $\vec{B}_i^l(x)$ are the color electric and magnetic fields, respectively. (Color indices will always be denoted by $l$, $m$, and $n$ and space indices by $i$, $j$, and $k$.) The color matrices $\lambda^l$ obey the relation

$$[\lambda^l,\lambda^m]=if^{lmn}\lambda^n . \tag{1.2}$$

In terms of the fundamental gauge fields $\vec{A}_i^l(\vec{x})$, the electric and magnetic fields are defined by

$$\vec{E}^l(x)=-\dot{\vec{A}}^l(x) , \tag{1.3}$$

$$\vec{B}^l(x)=\vec{\nabla}\times\vec{A}^l-\tfrac{1}{2}gf^{lmn}\vec{A}^m\times\vec{A}^n . \tag{1.4}$$

Considering the fields to be classical at this stage, the following equal-time Poisson bracket relations hold:

$$\{\vec{A}_i^l(\vec{x}),\vec{A}_j^m(\vec{x}')\}=0 , \tag{1.5}$$

$$\{\vec{E}_i^l(\vec{x}),\vec{E}_j^m(\vec{x}')\}=0 , \tag{1.6}$$

$$\{\vec{E}_i^l(\vec{x}),\vec{A}_j^m(\vec{x}')\}=-\delta^{lm}\delta_{ij}\delta(\vec{x}-\vec{x}') . \tag{1.7}$$

Equation (1.7) is a statement that $\vec{E}^l$ is the momentum canonically conjugate to the dynamical variable $\vec{A}^l$.

Application of the Hamiltonian equations of motion $\dot{O}=-\{H,O\}$, and exploitation of the definition Eq. (1.4), yield the following three generalized Maxwell equations:

$$\vec{E}^l=-\dot{\vec{A}}^l , \tag{1.8}$$

$$\dot{\vec{E}}^l=(D\times\vec{B})^l , \tag{1.9}$$

$$(D\cdot B)^l=0 , \tag{1.10}$$

where the $D$ operators are defined by

$$(D\times\vec{B})^l\equiv\vec{\nabla}\times\vec{B}^l-gf^{lmn}\vec{A}^m\times\vec{B}^n , \tag{1.11}$$

$$(D\cdot\vec{B})^l\equiv\vec{\nabla}\cdot\vec{B}^l-gf^{lmn}\vec{A}^m\cdot\vec{B}^n . \tag{1.12}$$

Gauss's law is conspicuously absent from Eqs. (1.8)—(1.10). It cannot be obtained as an equation of motion if $A_0^l=0$. Hence, one must arbitrarily impose as an extra constraint on the solutions of Eqs. (1.8)—(1.10) the condition

$$(D\cdot\vec{E})^l=\rho^l . \tag{1.13}$$

Some further insight into the issue of Gauss's law may be obtained from the following. If the static Maxwell equations were to be derived from an energy variational principle, then one would require the energy functional defined below to be stationary with respect to arbitrary variations of $\vec{E}_i^l(\vec{x})$, $\vec{A}_i^l(x)$ and $A_0^l(x)$,

$$\mathscr{E}[\vec{E}^l,\vec{A}^l,A_0^l]=\tfrac{1}{2}\int d^3x(\vec{E}^l\cdot\vec{E}^l+\vec{B}^l\cdot\vec{B}^l) - \int d^3x\,A_0^l(x)(D\cdot E^l-\rho^l) . \tag{1.14}$$

If $A_0^l(x)$ is restricted to zero, then clearly Eq. (1.13) cannot be recovered from Eq. (1.14).

Since the issue of gauge transformations will be of critical importance, let us briefly consider its relation with Gauss's law.

Under the following gauge transformation of fields $\vec{A}(x)$ ($\vec{A}\equiv\lambda^l\vec{A}^l$):

$$\vec{A}\to\vec{A}'=U^{-1}AU-g^{-1}U^{-1}\nabla U , \tag{1.15}$$

the Hamiltonian and equations of motion remain invariant, provided only that $U$ is a time-independent, but otherwise arbitrary, $N\times N$ matrix. To this symmetry there corresponds a Noether current, and hence a charge. Define now the Gauss's law operator $\mathscr{G}^l(x)$

$$\mathscr{G}^l(x)=(D\cdot\vec{E})^l-\rho^l . \tag{1.16}$$

Then, using Eqs. (1.5)—(1.7), the following Poisson relations can be proved:

$$\{\mathscr{G}^l(x),\mathscr{G}^m(x')\}=gf^{lmn}\mathscr{G}^n(x)\delta^3(x-x') . \tag{1.17}$$

It can be shown that the Gauss's law operator is the generator of infinitesimal gauge transformations. To see this, let

$$U(x)=1-ig\lambda^l\delta\Lambda^l(x) , \tag{1.18}$$

where $\delta\Lambda(x)$ is arbitrary. Then,

$$\delta\vec{A}^l(\vec{x})=\vec{A}'^l(\vec{x})-\vec{A}^l(\vec{x}) =\vec{\nabla}\delta\Lambda^l-gf^{lmn}\vec{A}^m\delta\Lambda^n . \tag{1.19}$$

If we express the unitary transformation $\mathscr{U}$ as

$$\mathscr{U}=1+i\int d^3x\,\delta\Lambda^l(x)\mathscr{G}^l(x) , \tag{1.20}$$

then it is readily verified that

$$\mathscr{U}\vec{A}^l\mathscr{U}^{-1}=\vec{A}^l+\vec{\nabla}\delta\Lambda^l-gf^{lmn}\vec{A}^m\delta\Lambda^n =\vec{A}'^l . \tag{1.21}$$

Finally, let us now consider quantization of the theory, which hitherto has dealt with the classical fields $\vec{A}^l$ and $\vec{E}^l$. By the usual canonical prescription, the Poisson brackets Eqs. (1.5)—(1.7) are now to be replaced by their corresponding commutators. We can choose $\vec{A}^l(x)$ to be a diagonal operator, and hence

$$\vec{E}^l(x)=i\frac{\delta}{\delta\vec{A}^l(x)} . \tag{1.22}$$

The system is described by a wave functional $\Psi[\vec{A}^l(\vec{x})]$ which has the interpretation of being the probability amplitude for a particular field configuration $\vec{A}^l(\vec{x})$ at time $t=0$. $\Psi$ is a solution of the functional differential Schrödinger equation,

$$\tfrac{1}{2}\int d^3x\left(-\frac{\delta}{\delta\vec{A}^l}\cdot\frac{\delta}{\delta\vec{A}^l}+\vec{B}^l\cdot\vec{B}^l\right)\Psi=E\Psi . \tag{1.23}$$

One might conceive of solving Eq. (1.23) approximately, subject to appropriate boundary conditions. This would be a mathematically consistent procedure. However, under the gauge transformation Eq. (1.15), the wave functional does not remain invariant but instead transforms as

$$\Psi\to\Psi'=\left(1+i\int d^3x\,\delta\Lambda^l(x)\mathscr{G}^l(x)\right)\Psi . \tag{1.24}$$

Thus, physical observables calculated from $\Psi$ will depend on the choice of gauge unless $\mathscr{G}^l\Psi=0$, i.e., unless Gauss's law is imposed.

An analogy with ordinary rotations is helpful in understanding the Gauss's law condition at an intuitive level.

First, let us observe from the commutator version of Eq. (1.17) that $\vec{\mathcal{G}}^{\,l}$ for the gauge group SU(2) behaves quite like the usual angular momentum operator for $(x=x')$,

$$[\mathcal{G}^l,\mathcal{G}^m]\sim i\epsilon^{lmn}\mathcal{G}^n\ . \tag{1.25}$$

Equation (1.24) then tells us that the wave function transforms under rotations into a different form unless it has "$l=0$," i.e., unless it satisfies Gauss's law. For ordinary space, any wave function expressed only in terms of $r$ will automatically be rotationally invariant. If one could find coordinates in function space analogous to $r$ and express $\Psi$ in terms of these coordinates, then $\Psi$ would be gauge invariant and "only" have to satisfy the Schrödinger equation, Eq. (1.23).

## II. ANALYSIS OF SCHRÖDINGER AND GAUSS EQUATIONS

We begin the analysis of the functional Schrödinger and Gauss's equations by writing the wave functional $\Psi$ as

$$\Psi[\vec{\mathrm{A}}^{\,l}]=\chi[\vec{\mathrm{A}}^{\,l}]\exp(-i\Phi[\vec{\mathrm{A}}^{\,l}])\ , \tag{2.1}$$

where $\chi$ and $\Phi$ are purely real. Inserting Eq. (2.1) into Eq. (1.23) and equating real and imaginary parts yields

$$\tfrac{1}{2}\int d^3x\left[-\frac{\delta}{\delta\vec{\mathrm{A}}^{\,l}}\cdot\frac{\delta}{\delta\vec{\mathrm{A}}^{\,l}}+\vec{\mathrm{B}}^{\,l}\cdot\vec{\mathrm{B}}^{\,l}+\frac{\delta\Phi}{\delta\vec{\mathrm{A}}^{\,l}}\cdot\frac{\delta\Phi}{\delta\vec{\mathrm{A}}^{\,l}}\right]\chi=\epsilon\chi \tag{2.2a}$$

and

$$\int d^3x\frac{\delta}{\delta\vec{\mathrm{A}}^{\,l}}\cdot\left[\chi^2\frac{\delta\Phi}{\delta\vec{\mathrm{A}}^{\,l}}\right]=0\ . \tag{2.2b}$$

Similarly, Gauss's law becomes

$$\nabla\cdot\frac{\delta\chi}{\delta\vec{\mathrm{A}}^{\,l}}-gf^{lmn}\vec{\mathrm{A}}^{\,m}\cdot\frac{\delta\chi}{\delta\vec{\mathrm{a}}^{\,n}}=0 \tag{2.3a}$$

and

$$\nabla\cdot\frac{\delta\Phi}{\delta\vec{\mathrm{A}}^{\,l}}-gf^{lmn}\vec{\mathrm{A}}^{\,m}\cdot\frac{\delta\Phi}{\delta\vec{\mathrm{A}}^{\,n}}=\rho^l\ . \tag{2.3b}$$

The total charge-density operator $\hat{\rho}_T^l(\vec{\mathrm{x}})$ is the sum of quark and gluon color charges,

$$\hat{\rho}_T^l=\hat{\rho}^l+igf^{lmn}\vec{\mathrm{A}}^{\,m}\cdot\frac{\delta}{\delta A^n}\ . \tag{2.4}$$

The wave functional $\Psi$ is not an eigenstate of $\hat{\rho}_T^l$,

$$\begin{aligned}\hat{\rho}_T^l\Psi&=\left[\rho^l+gf^{lmn}\vec{\mathrm{A}}^{\,m}\cdot\frac{\delta\Phi}{\delta A^n}\right]\Psi+igf^{lmn}\vec{\mathrm{A}}^{\,m}\cdot\frac{\delta\chi}{\delta A^n}e^{-i\Phi}\\&=\rho_T^l\Psi+i\left[\nabla\cdot\frac{\delta\chi}{\delta A^l}\right]e^{-i\Phi}\ .\end{aligned} \tag{2.5}$$

This is as it should be—color is not conserved locally. However, integrating Eq. (2.5) and assuming that the surface integral originating from the divergence term vanishes, it is seen that $\Psi$ *is* an eigenstate of total charge.

It is interesting to note that gauge invariance causes the phase of the wave function to play a crucially important role even for time-independent states, whereas for the ordinary Schrödinger equation the phase is inconsequential. Further, Eqs. (2.2) are also obtainable from requiring stationarity of $\langle\Psi\,|\,H-E\,|\,\Psi\rangle$ under arbitrary variations of $\chi$ and $\Phi$.

To analyze Eqs. (2.2) and (2.3), we shall proceed by a number of steps.

### A. Electrodynamics

It is useful to first consider the $g=0$ case. This is equivalent, of course, to electrodynamics with $N$ independent fields. Hence we drop the label $l$. Equations (2.2) and (2.3) become

$$\tfrac{1}{2}\int d^3x\left[-\frac{\delta}{\delta\vec{\mathrm{A}}}\cdot\frac{\delta}{\delta\vec{\mathrm{A}}}+|\,\vec{\nabla}\times\vec{\mathrm{A}}\,|^2+|\,\nabla\phi\,|^2\right]\chi=\epsilon\chi\ , \tag{2.6}$$

$$\nabla^2\phi=\rho\ , \tag{2.7}$$

$$\nabla\cdot\frac{\delta\chi}{\delta\vec{\mathrm{A}}}=0\ . \tag{2.8}$$

Equation (2.7) is the replacement of Eq. (2.3b). We could, of course, have set $\delta\Phi/\delta A=\nabla\phi+$ curl of an arbitrary vector. However, we shall set this arbitrary vector to zero since its inclusion merely redefines $\epsilon$ in Eq. (2.6). Also, the relation Eq. (2.2b) is satisfied identically if the solutions of Eq. (2.7) are required to be finite at infinity.

Let us now define a generating functional $\Omega_0$ by

$$\Omega_0=\exp\left[-\tfrac{1}{2}\int d^3x(\vec{\nabla}\times\vec{\mathrm{A}})_i\,|\,\nabla\,|^{-1}(\vec{\nabla}\times\vec{\mathrm{A}})_i\right]\ , \tag{2.9}$$

where $|\,\nabla\,|^{-1}$ is defined in a Fourier transform sense, i.e., $|\,\nabla\,|^{-1}e^{i\mathrm{k}\cdot\mathrm{x}}=|\,k\,|^{-1}e^{i\mathrm{k}\cdot\mathrm{x}}$.

We now observe the following facts. First, $\Omega_0$ satisfies Gauss's equation

$$\vec{\nabla}\cdot\frac{\delta\Omega_0}{\delta\vec{\mathrm{A}}}=0\ . \tag{2.10}$$

Second, $\Omega_0$ is the lowest-energy solution of Eq. (2.6):

$$\tfrac{1}{2}\int d^3x\left[-\frac{\delta}{\delta\vec{\mathrm{A}}}\cdot\frac{\delta}{\delta\vec{\mathrm{A}}}+|\,\vec{\nabla}\times\vec{\mathrm{A}}\,|^2+|\,\nabla\phi\,|^2\right]\Omega_0=E_0\Omega_0 \tag{2.11}$$

with

$$\begin{aligned}E_0&=\int d^3x\,|\,\nabla\,|\,\delta(0)+\tfrac{1}{2}\int d^3x\,d^3x'\frac{\rho(x)\rho(x')}{4\pi\,|\,\vec{\mathrm{x}}-\vec{\mathrm{x}}'\,|}\\&=\sum_{\mathrm{k}}|\,\vec{\mathrm{k}}\,|+\tfrac{1}{2}\int d^3x\,d^3x'\frac{\rho(x)\rho(\vec{\mathrm{x}}')}{4\pi\,|\,\vec{\mathrm{x}}-\vec{\mathrm{x}}'\,|}\ .\end{aligned} \tag{2.12}$$

The first term in Eq. (2.12) represents the zero-point energy of photons in a finite box, whereas the second term is the usual Coulomb energy.

We shall now see that repeated functional differentiation of $\Omega_0$ generates a complete set of physical Hilbert-space vectors. Whereas one could equally well work in the coordinate representation, the momentum representation is a little more familiar.

With the definition

$$\vec{\mathrm{A}}(\vec{\mathrm{k}})=\int d^3x\,e^{i\vec{\mathrm{k}}\cdot\vec{\mathrm{x}}}\vec{\mathrm{A}}(\vec{\mathrm{x}})\ , \tag{2.13}$$

the generating functional $\Omega_0$ becomes

$$\Omega_0[A]=\exp\left[-\tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{|\vec{k}\times\vec{A}|^2}{k}\right]. \tag{2.14}$$

We now observe that

$$\frac{\delta\Omega_0}{\delta\vec{A}(-\vec{k})}=-k\vec{A}_T(\vec{k})\Omega_0\,, \tag{2.15}$$

where $A_T$ is the transverse part of the field

$$\vec{A}_T(\vec{k})=\vec{A}(\vec{k})-\hat{k}(\hat{k}\cdot\vec{A})\,. \tag{2.16}$$

Recalling now the form for the momentum operator

$$\vec{P}=-i\int d^3x\,\vec{B}\times\frac{\delta}{\delta\vec{A}}\,, \tag{2.17}$$

it may now be readily verified that

$$\vec{P}\Omega_0=0 \tag{2.18}$$

and

$$\vec{P}\psi(\vec{k}_1,\vec{k}_2,\dots)=(\vec{k}_1+\vec{k}_2+\dots)\times\psi(\vec{k}_1,\vec{k}_2,\dots)\,, \tag{2.19}$$

where

$$\psi(\vec{k}_1,\vec{k}_2,\dots)=\left[\frac{\delta}{\delta\vec{A}(-\vec{k}_1)}\frac{\delta}{\delta\vec{A}(-\vec{k}_2)}\cdots\right]\Omega_0\,. \tag{2.20}$$

Hence, an arbitrary Fock-space vector $|n_{k_1},n_{k_2},\dots\rangle$ can be generated by repeated functional differentiation. The functionals $\psi(\vec{k}_1,\vec{k}_2,\dots)$ or $\psi(\vec{x}_1,\vec{x}_2,\dots)$ can then be used as a basis for unconstrained diagonalization of the Hamiltonian.

### B. Generalization to $g\neq0$

Let us now consider the full theory, $g\neq0$. Define first the generating functional $\Omega$ by

$$\Omega[\vec{A}^{\,l}]=\exp\left[-\frac{1}{2\alpha}\int d^3x\,B_i^lB_i^l\right], \tag{2.21}$$

where the color-magnetic field is defined in Eq. (1.4) and $\alpha$ is any constant with dimensions of energy. The utility of $\Omega$ arises from its being an exact, nontrivial, solution of Eq. (2.3a),

$$\nabla\cdot\frac{\delta\Omega}{\delta\vec{A}^{\,l}}-gf^{lmn}\vec{A}^{\,m}\cdot\frac{\delta\Omega}{\delta\vec{A}^{\,n}}=0\,. \tag{2.22}$$

To see this we note that

$$\frac{\delta\Omega}{\delta A^l}\propto(D\times B)^l\Omega \tag{2.23}$$

and the easily proven identity

$$[D\cdot(D\times B)]^l=0\,. \tag{2.24}$$

Equation (2.22) then follows.

Consider now the generation of other solutions from $\Omega$, as was discussed earlier for the $g=0$ case. Differentiation with arbitrary numbers of $A$'s does not generate solutions of Eq. (2.3a). However, if $\hat{O}$ is an operator formed from products of $\delta/\delta A$, and if $[\mathscr{G}^l,\hat{O}]=0$, then $\hat{O}\Omega$ is a solution of Eq. (2.3a). It is easy to see that there are an infinite number of operators $\hat{O}$, and that they can be generated as sums and repeated products of a few basic units.[14]

### C. Coordinate transformation

Anticipating later use, let us now write the field $\vec{A}^{\,l}(x)$ as

$$\vec{A}^{\,l}(x)=\vec{A}^{\,l}_{\rm cl}(\vec{x})+\delta\vec{A}^{\,l}(\vec{x})\,, \tag{2.25}$$

where $\vec{A}^{\,l}_{\rm cl}(\vec{x})$ is arbitrary at this point. Further, define a functional $Z[\vec{A}^{\,l}]$ by

$$Z=\exp(-\tfrac{1}{2}u_0{}^2)\,, \tag{2.26}$$

where

$$u_0=\frac{\left[\int d^3x\,\vec{B}^{\,l}\cdot\vec{B}^{\,l}\right]^{1/2}-\left[\int d^3x\,\vec{B}^{\,l}_{\rm cl}\cdot\vec{B}^{\,l}_{\rm cl}\right]^{1/2}}{\left[\int d^3x\,\vec{B}^{\,l}_{\rm cl}\cdot\vec{B}^{\,l}_{\rm cl}\right]^{1/2}} \tag{2.27}$$

and

$$\vec{B}^{\,l}_{\rm cl}=\nabla\times\vec{A}^{\,l}_{\rm cl}-\tfrac{1}{2}gf^{lmn}\vec{A}^{\,m}_{\rm cl}\times\vec{A}^{\,n}_{\rm cl}\,. \tag{2.28}$$

Note that $u_0$ vanishes, and hence $Z$ peaks, when $\vec{B}^{\,l}=\vec{B}^{\,l}_{\rm cl}$. Further, it is assumed that $\vec{B}^{\,l}_{\rm cl}$ is not identically zero. We are not, however, identifying $\vec{B}^{\,l}_{\rm cl}$ with the classical value at the present time. With the help of the discussion in Sec. II B, it is readily shown that $Z$ satisfies the sourceless Gauss's law equation (2.3a). Also, one can once again generate all independent solutions of Eq. (2.3a) by operating with the operators $\hat{O}$. Let $\{Q\}$ be this set of functionals with $Q_0\equiv Z$.

We now make a transformation of coordinates from the original set $\{A_i^l(x)\}$ to a new set $\{Q,q\}$. The set $\{q\}$ is a set of functionals which have no common members with $\{Q\}$, and which are needed to make the transformation $\{\vec{A}^{\,l}_j(x)\}\rightarrow\{Q,q\}$ complete. The motivation for this transformation is that $\chi$ is automatically gauge invariant when expressed as a function of the $Q$'s, $\chi=\chi(Q_0,Q_1,\dots)$. The coordinate $Q_0\equiv Z$ plays the key role since it is the only one which explicitly occurs in the Hamiltonian. This fact will be crucially important later.

Continuing, the transformation of coordinates is characterized by a transfer matrix $M$,

$$M_{\alpha\beta}=\int d^3x\frac{\delta\mathsf{Q}_\alpha}{\delta A_i^l(\vec{x})}\frac{\delta\mathsf{Q}_\beta}{\delta\vec{A}^{\,l}_i(\vec{x})}\,, \tag{2.29}$$

where Q stands for either $Q$ or $q$. The Jacobian $\mathscr{J}$ is

$$\mathscr{J}=(\det M)^{1/2}\,. \tag{2.30}$$

In terms of these, the Hamiltonian [Eq. (2.2a)] is

$$H=K+V\,, \tag{2.31}$$

where

$$K=-\frac{\hbar^2}{2}\mathscr{J}\frac{\partial}{\partial\mathsf{Q}_\alpha}M_{\alpha\beta}\mathscr{J}^{-1}\frac{\partial}{\partial\mathsf{Q}_\beta}\,, \tag{2.32}$$

$$V=\tfrac{1}{2}(1+|2\ln Q_0|^{1/2})^2\int d^3x\,\vec{\mathbf{B}}^l_{\rm cl}\cdot\vec{\mathbf{B}}^l_{\rm cl} + \int d^3x\,\vec{\epsilon}^{\,l}\cdot\vec{\epsilon}^{\,l}\,. \tag{2.33}$$

In Eq. (2.32), $\hbar$ has been restored, whereas earlier it had been implicitly set equal to one. The quantity $\vec{\epsilon}^{\,l}$ in Eq. (2.33) is a solution of Eq. (2.3b),

$$\nabla\cdot\vec{\epsilon}^{\,l}-gf^{lmn}\vec{\mathbf{A}}^{\,m}\cdot\vec{\epsilon}^{\,n}=\rho^l\,. \tag{2.34}$$

The general solution of Eq. (2.34) is the sum of a particular solution and a homogeneous solution,

$$\vec{\epsilon}^{\,l}(\mathbf{Q})=\vec{\epsilon}^{\,l}_P(q)+\vec{\epsilon}^{\,l}_H(Q)\,. \tag{2.35}$$

We are at liberty to set $\epsilon_H=0$. [It is also known (Ref. 15) that there are no nontrivial finite-energy solutions to the classical homogeneous equations in three space dimensions.] Thus, $Q_0$ is the only element of $\{Q\}$ which enters into the potential Eq. (2.33).

### D. Classical theory

The relation to the classical theory is now easily seen from Eqs. (2.29)—(2.34). Setting $\hbar=0$, the total energy is now simply the potential $V$. Clearly, $V$ achieves its minimum as a function of the coordinate $\{Q\}$ when $Q_0=1$, or from Eqs. (2.26) and (2.27), when $u_0=0$. This condition is satisfied if $\vec{\mathbf{A}}^{\,l}=\vec{\mathbf{A}}^{\,l}_{\rm cl}$. Remembering $\vec{\mathbf{A}}_{\rm cl}$ had been hitherto arbitrary, let us define it now to be that field which minimizes the quantity

$$V(Q_0=1)=\tfrac{1}{2}\int d^3x(\vec{\epsilon}^{\,l}_{\rm cl}\cdot\vec{\epsilon}^{\,l}_{\rm cl}+\vec{\mathbf{B}}_{\rm cl}\cdot\vec{\mathbf{B}}_{\rm cl})\,, \tag{2.36}$$

subject to the condition

$$\nabla\cdot\vec{\epsilon}_{\rm cl}-gf^{lmn}\vec{\mathbf{A}}^{\,m}_{\rm cl}\cdot\vec{\epsilon}^{\,n}_{\rm cl}=\rho^l\,. \tag{2.37}$$

Then $\vec{\mathbf{A}}^{\,l}_{\rm cl}$ is indeed identical to the classical field which satisfies the classical Yang-Mills equations, and $V(Q_0=1)$ is the classical energy.

Finally, let us explicitly write down the total wave functional [Eq. (2.1)] which corresponds to the system being in its classical configuration,

$$\Psi_{\rm cl}=\delta(Q_0-1)\exp\left[-i\int d^3x\,\vec{\epsilon}^{\,l}_{\rm cl}\cdot\vec{\mathbf{A}}^{\,l}\right]\,. \tag{2.38}$$

The physical significance of the classical limit is clear. In the absence of kinetic energy ($\hbar=0$), the energy is minimized if the real wave functional $\chi$ is concentrated at a point (or points) in $Q$ space corresponding to a local minimum. However, since the potential is cyclic in $Q_1,Q_2,\ldots$, there is no advantage gained if the wave function concentrates itself around fixed values of these points. In other words, there can only be a single $\delta$ function multiplying the exponential in Eq. (2.38).

### E. Quantum correction

Having seen the relation to the classical ($\hbar=0$) limit, we can now attempt to include quantum corrections in a minimal way by assuming $\hbar$ to be small but finite.

From the form of the classical wave functional, Eq. (2.38), we are encouraged to seek a form for $\Psi$,

$$\Psi=\chi(Q_0)\exp\left[-i\int d^3x\,\vec{\epsilon}^{\,l}_{\rm cl}\cdot\vec{\mathbf{A}}^{\,l}_{\rm cl}\right]\,, \tag{2.39}$$

where $\chi(Q_0)$ is hypothesized to be very narrowly peaked around $Q_0=1$. (One could equally well formulate the problem here onwards in a variational language.)

The form postulated for $\chi$ can now be inserted into the Schrödinger equation $(K+V)\chi=E\chi$, where $K$ and $V$ are defined in Eqs. (2.32) and (2.33). However, it is convenient to revert to the quantity $u_0$ [Eq. (2.27)] and use $Q_0\equiv Z=\exp(-\frac{1}{2}u_0^2)$. With this change of variables, the Schrödinger equation becomes

$$\left[-\frac{\hbar^2}{2}M_{00}\frac{\partial^2}{\partial u_0{}^2}+\tfrac{1}{2}(1+u_0)^2\int d^3x\,\vec{\mathbf{B}}^l_{\rm cl}\cdot\vec{\mathbf{B}}^l_{\rm cl}\right]\chi = \left[E-\tfrac{1}{2}\int d^3x\,\vec{\epsilon}^{\,l}_{\rm cl}\cdot\vec{\epsilon}^{\,l}_{\rm cl}\right]\chi\,, \tag{2.40}$$

where

$$M_{00}=\int d^3x\frac{\delta u_0}{\delta\vec{\mathbf{A}}^{\,l}(x)}\cdot\frac{\delta u_0}{\delta\vec{\mathbf{A}}^{\,l}(x)}\bigg|_{A=A_{\rm cl}} =\frac{\int d^3x(D\times\vec{\mathbf{B}}_{\rm cl})^l\cdot(D\times\vec{\mathbf{B}}_{\rm cl})^l}{\left[\int d^3x\,\vec{\mathbf{B}}^l_{\rm cl}\cdot\vec{\mathbf{B}}^l_{\rm cl}\right]^2} \tag{2.41}$$

and

$$(D\times\vec{\mathbf{B}}_{\rm cl})^l=\vec{\nabla}\times\vec{\mathbf{A}}^{\,l}_{\rm cl}-gf^{lmn}\vec{\mathbf{A}}^{\,m}_{\rm cl}\times\vec{\mathbf{B}}^{\,n}_{\rm cl}\,. \tag{2.42}$$

In deriving Eq. (2.40), consistent with the assumption of $\chi$ being very narrowly peaked, the derivatives of all quantities ($M$ and $\mathscr{V}$) were neglected, except that of $\partial\chi/\partial u_0$.

Equation (2.40) is, of course, the equation for a one-dimensional oscillator with the "collective coordinate" $v$,

$$v=1+u_0\,. \tag{2.43}$$

The solution is

$$\chi_n=\left[\frac{\alpha^2}{\pi 2^{2n}(n!)^2}\right]^{1/2}H_n(\alpha v)e^{-\alpha v^2/2}\,, \tag{2.44}$$

$$E_n=\tfrac{1}{2}\int d^3x(\vec{\epsilon}^{\,l}_{\rm cl}\cdot\vec{\epsilon}^{\,l}_{\rm cl}+\vec{\mathbf{B}}^l_{\rm cl}\cdot\vec{\mathbf{B}}^l_{\rm cl})+(n+\tfrac{1}{2})\hbar\omega\,, \tag{2.45a}$$

where the dimensionless parameter $\alpha$ frequency $\omega$ are given by

$$\alpha^2=\frac{1}{\hbar}\frac{\left[\int d^3x\,\vec{\mathbf{B}}^l_{\rm cl}\cdot\vec{\mathbf{B}}^l_{\rm cl}\right]^{3/2}}{\left[\int d^3x(D\times\vec{\mathbf{B}}_{\rm cl})^l\cdot(D\times\vec{\mathbf{B}}_{\rm cl})^l\right]^{1/2}}\,, \tag{2.45b}$$

$$\omega^2=\frac{\int d^3x(D\times\vec{\mathbf{B}}_{\rm cl})^l\cdot(D\times\vec{\mathbf{B}}_{\rm cl})^l}{\int d^3x\,\vec{\mathbf{B}}^l_{\rm cl}\cdot\vec{\mathbf{B}}^l_{\rm cl}}\,. \tag{2.45c}$$

The condition for validity of the energy expression Eq. (2.45) is that $\alpha\gg1$. It is also clear that in the limit $\hbar\to0$, $\chi_n\to\chi_{\rm classical}$. Equations (2.43)—(2.45) are the central results of this paper.

Although our derivation of the above results has consistently been in the $A_0=0$ gauge, we are now free to choose any gauge for solving the classical problem. Further, our identification of $\vec{\mathbf{A}}^{\,l}_{\rm cl}$ with the true classical field is really true only to $O(\hbar)$. To see this, consider minimization of the following functional (note all fields below are $c$-number fields),

$$E[\vec{\epsilon}^{\,l},\vec{A},\phi^l]=(n+\tfrac{1}{2})\hbar\left[\frac{\int d^3x(D\times\vec{B})^l\cdot(D\times\vec{B})^l}{\int d^3x\,\vec{B}^{\,l}\cdot\vec{B}^{\,l}}\right]^{1/2}+\tfrac{1}{2}\int d^3x(\vec{\epsilon}^{\,l}\cdot\vec{\epsilon}^{\,l}+\vec{B}^{\,l}\cdot\vec{B}^{\,l})-\int d^3x\,\phi^l(x)(\nabla\cdot\vec{\epsilon}^{\,l}-gf^{lmn}\vec{A}^{\,m}\cdot\vec{\epsilon}^{\,n}-\rho^l)\,. \tag{2.46}$$

Varying with respect to $\phi^l$, $\epsilon^l$, and $\vec{A}^{\,l}$ gives the three equations

$$\nabla\cdot\vec{\epsilon}^{\,l}-gf^{lmn}\vec{A}^{\,m}\cdot\vec{\epsilon}^{\,n}=\rho^l\,, \tag{2.47}$$

$$\vec{\epsilon}^{\,l}=-\nabla\phi^l+gf^{lmn}\vec{A}^{\,m}\phi^n\,, \tag{2.48}$$

and

$$\frac{(n+\frac{1}{2})\hbar(\{D\times[D\times(D\times\vec{B})]\}^l-gf^{lmn}\vec{B}^{\,m}\times(D\times\vec{B})^n-\omega^2(D\times B)^l)}{2\omega^{1/2}\int d^3x\,\vec{B}^{\,l}\cdot\vec{B}^{\,l}}+(D\times B)^l-gf^{lmn}\phi^m\vec{\epsilon}^{\,n}=0\,. \tag{2.49}$$

Equations (2.47) and (2.48), and Eq. (2.49) with $\hbar=0$, together with the definition of $\vec{B}^{\,l}$ in Eq. (1.4), are the usual static Maxwell equations. The term proportional to $\hbar$ in Eq. (2.49) makes the solution of Eqs. (2.47)–(2.49) different from the usual classical solution. However, the corrections enter as $O(\hbar^2)$ in the energy and can be discarded there.

### F. Application

Many practical applications of the results derived above can be made. Consider, by way of illustration, one such example. Specifically, let us take a single external point source

$$\rho^l=q\delta^{l3}\delta^3(\vec{x})\,. \tag{2.50}$$

Following Sikivie and Weiss,[4] we take $\phi^l=0$ and the following form for $\vec{\epsilon}^{\,l}$ and $\vec{A}^{\,l}$:

$$\vec{A}^{\,l}=\delta^{l1}\hat{\phi}\frac{a}{g}\frac{(\mu r)^{p/2}}{r}\sin\theta\,, \tag{2.51}$$

$$\vec{\epsilon}^{\,l}=\frac{q}{4\pi r^2}[\delta^{l3}\hat{r}F(r,\theta)+\delta^{l2}\hat{\phi}G(r,\theta)] \tag{2.52}$$

with

$$F(r,\theta)=1-\beta\left[\frac{\mu r}{1+\mu r}\right]^p\sin^2\theta\,, \tag{2.53}$$

$$G(r,\theta)=\beta\frac{p}{a}\frac{(\mu r)^{p/2}}{(1+\mu r)^{p+1}}\,. \tag{2.54}$$

The parameters $\beta$, $p$ $(0<p<1)$, and $a$ are dimensionless while the parameter $\mu<0$ has dimensions of inverse length. The total charge is calculated to be

$$I^l=\delta^{l3}q(1-\tfrac{2}{3}\beta)\,. \tag{2.55}$$

The configuration described by Eqs. (2.51)–(2.54) satisfies Gauss's law, but is not a solution of either the static Maxwell equations or their modified counterparts, Eqs. (2.48) and (2.49). Nevertheless, the configuration has some interesting properties, as was shown in Ref. 4. First, by choice of parameters, the energy can be made to be arbitrarily smaller than the Coulomb energy. Second, screened solutions ($\beta>0$) have lower energy than unscreened ones. It is interesting to see if these properties differ when quantum corrections are included.

Using Eqs. (2.51)–(2.54), the energy is calculated to be

$$E=E_0+(n+\tfrac{1}{2})\hbar\omega,\quad n=0,1,2,\ldots, \tag{2.56}$$

where

$$E_0=\frac{q^2}{8\pi\delta}+\frac{q^2}{12\pi}\frac{1}{1-p}\frac{(\mu\delta)^p}{\delta}\times\left[-2\beta+\frac{p^2\beta^2}{a^2}+\frac{4\pi^2a^2}{g^2q^2}(8+p^2)\right], \tag{2.57}$$

$$\omega=(1-p)^{1/2}f_1(p)\frac{1}{\delta}\,. \tag{2.58}$$

Also, the value of $\alpha^2$ [Eq. (2.45b)] is

$$\alpha^2=\frac{1}{\hbar}\frac{a^2}{g^2}\frac{f_2(p)}{(1-p)^{3/2}}(\mu\delta)^p\,. \tag{2.59}$$

In Eqs. (2.58) and (2.59), $f_1(p)$ and $f_2(p)$ are functions of $p$ alone and have no poles or zeros in the range $0\le p\le 1$. The energy minimized with respect to $a$ and $n$ is

$$E=\tfrac{1}{2}\hbar(1-p)^{1/2}f_1(p)\frac{1}{\delta}+\frac{q^2}{8\pi\delta}+\frac{q^2}{12\pi}\frac{1}{1-p}\frac{(\mu\delta)^p}{\delta}\left[-2\beta+\frac{4\pi}{gq}|\beta|p(8+p^2)^{1/2}\right]+O\left[\frac{(\mu\delta)^{2p}}{\delta}\right]. \tag{2.60}$$

The cutoff parameter $\delta$ $(\delta\to 0)$ is made necessary[4] because of the pointlike nature of the source.

Let us now make the following observations for the case where the parameter[4,16] $\alpha\equiv gq/4\pi$ has values $\alpha>\frac{3}{2}$ and $\alpha<\frac{3}{2}$.

(a) If $\alpha>\frac{3}{2}$, and if $p$ is fixed $(0<p<1)$, then the screened configuration $(\beta>0)$ has lower energy than the unscreened one $(\beta>0)$. Further, the $\beta>0$ $(\beta<0)$ case has energy lesser (greater) than the Coulomb energy, which has $\beta=0$.

(b) If $\alpha>\frac{3}{2}$, and if $\beta\ge 0$, then $E$ [Eq. (2.60)] is minimized by letting $p\to 1$. This implies that $\omega\to 0$, and hence that there are no stable modes around the screened configurations.

(c) If $\beta<0$, then $E$ is once again minimized by letting $p\to 1$. However, it is now no longer true that $\omega\to 0$, even though the minimized energy is still greater than the Coulomb energy. Instead, $\omega\propto\delta^{-2/3}$ in this limit and one has the possibility of oscillations about a local minimum. These oscillations obviously cannot represent stable modes of the system since even the Coulomb solution has lower energy. However, our minimal quantum corrections do not allow for tunneling to a lower level.

## III. DISCUSSION

In this paper, we have considered QCD with static $c$-number sources and with arbitrary coupling constant $g$, but with $\hbar$ assumed small. In a sense, this is a weak-coupling approximation because $g^2\hbar$ is a dimensionless coupling constant and is small because $\hbar$ is small. Indeed, if $\hbar$ were exactly zero, then one would have only classical tree graphs.[17] In the language of perturbation theory, the present approach includes some subset of loop diagrams. However, it is difficult to be more precise since the physical basis of our approximation is not based on perturbation theory.

When QCD is expressed in the canonical formalism, the equation of motion becomes a functional differential equation for the system wave functional, which is the projection of the state vector onto the gauge fields. Worse, the wave functional must obey the infinite number of constraints imposed by Gauss's law. The key element that made progress possible here was the observation that a complete set of solutions could be constructed for the homogeneous Gauss's law equation, starting from the simple function

$$\exp\left[-\text{const}\times\int d^3x\,\vec{\mathrm{B}}^{\,l}\cdot\vec{\mathrm{B}}^{\,l}\right].$$

This is highly significant since the only dependence of the static Hamiltonian on the gauge field $\vec{\mathrm{A}}^{\,l}(\vec{\mathrm{x}})$ is through the term $\int d^3x\,\vec{\mathrm{B}}^{\,l}\cdot\vec{\mathrm{B}}^{\,l}$.

Seen in the context of the formalism described in this paper, the classical Yang-Mills theory results from simply ignoring a generalized kinetic energy term $K$, i.e., setting $\hbar=0$ in Eq. (2.32). The total gluon energy is then solely the "potential" energy $V$. Further, seen as a function of the gauge-invariant coordinates $\{Q\}$ defined in the text, $V$ depends on only *one* coordinate of this (infinite) set of coordinates. This dependence is of harmonic form.

Quantum corrections to the classical ($\hbar=0$) limit can be motivated by the following heuristic argument: Consider a particle in a harmonic-oscillator well $V=\frac{1}{2}ku^2$. If $\hbar=0$, the energy is minimized if the particle is localized at the well bottom, i.e., if its wave function is a $\delta$ function $\delta(u)$. If $\hbar$ is now "turned on," this is no longer the state of minimum energy—the uncertainty principle forces the wave function to spread. However, if $\hbar$ is small one may still expect the wave function to be narrowly peaked around $x=0$. Precisely this argument was applied to the QCD problem, where it was further assumed that all derivatives save that of the rapidly varying function can be neglected. One is then left with a simple one-dimensional oscillator problem and the total energy is simply $E=E_{\text{classical}}+(n+\frac{1}{2})h\omega$, where $\omega$ is defined in Eq. (2.45) in terms of the classical color magnetic field $\vec{\mathrm{B}}^{\,l}(\vec{\mathrm{x}})$. Since $\vec{\mathrm{B}}^{\,l}(\vec{\mathrm{x}})$ is caused by all gluons, it is fair to refer to $\omega$ as a collective-gluon frequency.

Could the collective-gluon modes predicted by the present analysis actually have a physical existence detectable, for instance, by resonantly exciting them with quark projectiles? It appears impossible to provide an answer within the present framework for two reasons. First, one simply does not know whether the assumption of small $\hbar$ has anything to do with physical reality. Second, and this is a more serious objection, we have considered only $c$-number sources. These are merely the components of an $(N^2-1)$-dimensional vector. Physically, on emitting a gluon, the source remains unchanged—"recoil" in color space is forbidden. All strengths of the charge source are permitted and $gq$ is the appropriate dimensionless parameter which characterizes a source. On the other hand, quantum sources are required to have a fixed strength $q=g\hbar$ because of gauge invariance. Also, these sources obey the commutation relation

$$[I^l,I^m]=if^{lmn}I^n. \tag{3.1}$$

There is no nontrivial classical limit if the quark sources are assumed to lie in the fundamental SU($N$) representation. It can be argued[17] that classical sources correspond to quarks occupying infinite-dimensional representations of the gauge group. Unfortunately, it is unclear how minimal corrections arising from large, but finite, dimensionality can be incorporated.

Finally, it is interesting to compare the semiclassical approach of this paper with the "one-mode" approximations of Giles and McLerran[16] and of Pottinger and Warner.[17] The two approaches are quite different, of course. Nevertheless the comparison is quite instructive.

The one-mode approximations[16,17] to QCD are formulated in the Coulomb gauge and take as a starting point the Hamiltonian

$$H=\tfrac{1}{2}\int d^3x(\vec{\mathrm{E}}^{\,l}_T\cdot\vec{\mathrm{E}}^{\,l}_T+\vec{\mathrm{B}}^{\,l}\cdot\vec{\mathrm{B}}^{\,l})$$
$$+\tfrac{1}{2}\int\int d^3x\,d^3x'\rho^l_T(\vec{\mathrm{x}})G^{ll'}(\vec{\mathrm{x}},\vec{\mathrm{x}}')\rho^{l'}_T(\vec{\mathrm{x}}'). \tag{3.2}$$

In Eq. (3.2), $\vec{\mathrm{E}}^{\,l}_T$ is the transverse electric field, $\rho^l_T$ is the total charge density, and $G^{ll'}(\vec{\mathrm{x}},\vec{\mathrm{x}}')$ is a matrix which is obtainable in principle as the solution of an integral equation. Despite the complexity of $H$, the fact that for small field strengths Gauss's law is automatically incorporated in it (i.e., the state vectors are unrestricted) makes the Coulomb gauge attractive for perturbative studies.

The essence of the one-mode approximation is

$$\vec{\mathrm{A}}^{\,l}(\vec{\mathrm{x}})=\sum_{n=0}\vec{\mathrm{V}}_n(\vec{\mathrm{x}})a^l_n$$
$$\approx\vec{\mathrm{V}}_0(\vec{\mathrm{x}})a^l_0\,. \tag{3.3}$$

In the above, $\{\vec{\mathrm{V}}_n\}$ are a complete set of basis functions which can all be chosen variationally in principle. In practice, only $\vec{\mathrm{V}}_0(\vec{\mathrm{x}})$ can be thus determined. Basically, therefore, the one-mode approximation restricts the possible configurations of the field $\vec{\mathrm{A}}^{\,l}(\vec{\mathrm{x}})$. Equivalently, the Hamiltonian Eq. (3.2) is approximated by a simpler and more tractable expression. It is now possible to solve (albeit approximately) a simplified Schrödinger equation.

It is not possible to make a one-mode approximation with the $A^0=0$ gauge Hamiltonian because one cannot satisfy the Gauss's law constraint with any finite set of modes. Nevertheless, there is a similarity in that we too approximate the QCD Hamiltonian (after making a transformation to a more appropriate set of coordinates). However, the approximation philosophy is different—the potential term is presumed to dominate over the kinetic term in determining the wave functional, i.e., $\hbar$ is taken as a small parameter. Thus, our approach essentially builds small quantum corrections into the classical Yang-Mills theory. Solutions corresponding to collective-gluon oscillations exist at this level of correction. That such oscillations may persist even at the physical value of $\hbar$, and with quantum sources, is an interesting possibility.

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[14]Taking SU(2) as an example, all solutions to the sourceless Gauss's equation (2.3a) are color scalars which are generated by taking all $p$th-rank color tensors $[\delta/\delta\vec{A}^{l_1}(\vec{x})]\cdots[\delta/\delta\vec{A}^{l_p}(x)]$ and contracting with any $p$th-rank tensor formed from the tensors $\delta^{lm}$ and $\epsilon^{lmn}$. No more than one $\epsilon^{lmn}$ can ever be needed because of the well-known relation

$$\epsilon^{l_1m_1n_1}\epsilon^{l_2m_2n_2}=\delta^{l_1l_2}\delta^{m_1m_2}\delta^{n_1n_2}+\delta^{l_1m_2}\delta^{m_1n_2}\delta^{n_1l_2}+\delta^{l_1n_2}\delta^{m_1l_2}\delta^{n_1m_2}-\delta^{l_1l_2}\delta^{m_1n_2}\delta^{n_1m_2}-\delta^{l_1n_2}\delta^{m_1m_2}\delta^{n_1l_2}-\delta^{l_1m_2}\delta^{m_1l_2}\delta^{n_1n_2}\,.$$

Thus, typical operators $\hat{O}$ would be formed from operators such as

$$\hat{O}_1(\vec{x})=\frac{\delta}{\delta\vec{A}^{l}_{i_1}(\vec{x})}\frac{\delta}{\delta\vec{A}^{l}_{i_2}(\vec{x})}\,,$$

$$\hat{O}_2(\vec{x})=\epsilon^{lmn}\frac{\delta}{\delta\vec{A}^{l}_{i_1}(\vec{x})}\frac{\delta}{\delta\vec{A}^{m}_{i_2}(\vec{x})}\frac{\delta}{\delta\vec{A}^{n}_{i_3}(\vec{x})}\,.$$

Because $H$ is rotationally invariant, one could further classify these solutions according to their behavior under rotations. However, this will not be necessary for our present purpose.

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