Asymptotic behavior of the fixed-angle on-shell quark scattering amplitudes in non-Abelian gauge theories

Ashoke Sen
Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510*
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In this paper we find the asymptotic behavior of the fixed-angle on-shell quark-quark scattering amplitude in non-Abelian gauge theories in the limit of very large center-of-mass energy $\sqrt{s}$. We sum the perturbation series to all orders in the coupling constant and all powers of $\ln s$, but ignore terms which are suppressed by a power of s order by order in perturbation theory. In the $s\to\infty$ limit the amplitude vanishes as $\exp(-\alpha \ln s \ln\ln s)$, where α is a constant. The phase of the amplitude is shown to be free from infrared divergences. Hence the phase is a perturbatively calculable function and may provide important tests of QCD.

I. INTRODUCTION

In a previous paper[1] we showed how to systematically sum up all the logarithms that appear in the calculation of the asymptotic behavior of the Sudakov form factor in perturbation theory. In this paper, we generalize the approach to calculate the asymptotic behavior of the elastic quark-quark scattering amplitude in the $s\to\infty$, s/t-fixed limit. This method can also be applied to calculate the asymptotic behavior of a process with more than four external quarks, in the limit $p_i\cdot p_j\to\infty$ for all external momenta p_i, p_j (except for $i=j$), $p_i^2=m^2$, and the ratio $p_i\cdot p_j/p_{i'}\cdot p_{j'}$ fixed. In these calculations, we include all powers of the coupling constant g and all powers of logarithms of the external energy variable Q, but neglect terms which are suppressed by a power of Q, order by order in perturbation theory.

The asymptotic behavior of the $qq\to qq$ (and $q\bar{q}\to q\bar{q}$) amplitudes has been of interest in the recent past. They appear as subdiagrams in hadron-hadron elastic scattering amplitudes in the Landshoff diagrams.[2] Order by order in perturbation theory, Landshoff diagrams give contributions which are asymptotically larger by some power of s than the quark-counting result of Ref. 3. It was, however, argued[4] that such contributions involve near on-shell $qq\to qq$ and $q\bar{q}\to q\bar{q}$ scattering amplitudes as subdiagrams and these are suppressed due to the exponentiation of the Sudakov double logarithms[5] in the form $\exp(-A\ln^2 s)$, A being a constant and s the square of the total center-of-mass energy of the qq or the $q\bar{q}$ pair. Duncan and Mueller,[6] on the other hand, have argued that the actual hadronic elastic scattering amplitude is neither fully determined by the quark-counting rule, nor the power law given by the Landshoff pinch singular point, but by some function intermediate between the two. In order to find out the correct asymptotic behavior, we must sum up the Sudakov double logarithms in a systematic fashion. Calculation of the asymptotic behavior of the $qq\to qq$ or the $q\bar{q}\to q\bar{q}$ amplitude may be considered as a first step towards this process. In fact, the result of this paper supports Mueller's conjecture about the asymptotic form of the wide-angle hadron-hadron elastic scattering amplitudes. Recently, it has also been pointed out by Pire and Ralston[7] that the oscillation of the experimental value of the elastic hadron-hadron scattering cross sections about the quark-counting-rule prediction, as observed by Brodsky and Lepage,[8] may be due to the s dependence of the phase of the $q\bar{q}\to q\bar{q}$ and $qq\to qq$ amplitudes. In this paper, we also find out the s dependence of the phase in $s\to\infty$ limit. We show that the phase is free from infrared divergences. Hence, they are perturbatively calculable and may provide important tests of QCD.

The asymptotic behavior of the scattering amplitudes, in the limit considered in this paper, was calculated by several authors in QED and QCD in various approximations.[9] On the basis of their calculations up to a few orders in perturbation theory, Cornwall and Tiktopoulos, conjectured that in the leading-logarithmic order the amplitude for such a process goes as

$$\exp\left[-\frac{g^2}{32\pi^2}\left[\sum_\nu C_\nu\right]\ln^2(s/m^2)\right], \tag{1.1}$$

where C_ν is the value of the quadratic Casimir operator for the representation to which the νth external particle belongs. Note that the amplitude goes rapidly to zero as $s\to\infty$.

The amplitude under consideration is infrared divergent. In actual hadron-hadron scattering amplitudes, the color-singlet property, and the finite size of the hadron provide the necessary infrared cutoff. In our calculation, we must somehow simulate this cutoff. In the case of Abelian gauge theories, we can regulate the infrared divergence by giving the gluon a finite mass, since this is a gauge-invariant regularization procedure. This does not work in non-Abelian gauge theories, since the theory with massive gluons is not gauge invariant. Another way of regulating the infrared divergence is by keeping the external fermions off-shell by a fixed amount. This procedure, however, is not gauge invariant either in Abelian or in non-Abelian gauge theories.

There exists, however, a gauge-invariant way of regulating the infrared divergences in non-Abelian gauge theories, e.g., by dimensional regularization. We keep the external particles on-shell and work in $4+\epsilon$ dimensions. The reader may wonder whether the result in $4+\epsilon$ dimensions has any physical relevance. It will become clear later that our result (6.5) for the scattering amplitude is not sensitive to the way we regulate the infrared divergences in our theory, provided this is a gauge-invariant regularization procedure, and we keep the external particles on-shell. Thus, had there existed another gauge-invariant regularization procedure for non-Abelian gauge theories, we would have gotten the same final form (6.5), where the regulator R now stands for the new regulator. (The infrared-divergent functions f_2, C, and A_i will now have different forms, but these functions are independent of s. The functions γ_1, f_1, and λ, which depend on s, but are free from infrared divergence, will have the same functional form.) For hadron-hadron scattering there exists an infrared regulator, which is the off-shell behavior of the quarks inside the hadron. This is roughly determined by the inverse of the transverse size of the hadron. It was, however, found by Mueller[6] by explicit one-loop calculation that when we add all the relevant diagrams, the relevant regions of integration which contribute to the amplitude is $|l^2| \gtrsim Xs$, where X is an integration variable which runs from m^2/s to 1, and l is the momentum of the internal gluon. Thus, in one-loop order, the effective regularization may be obtained by giving the gluons a mass $\sqrt{Xs}$ (or by cutting off the gluon momentum at $l^2 \sim Xs$) and setting the external particles on-shell. This is precisely the type of regularization for which we expect our result to be valid. We expect that when we consider all the higher-order diagrams in the hadron-hadron scattering amplitude, this general feature will remain valid, i.e., in the first approximation the full hadron-hadron scattering amplitude may be expressed in terms of four-quark amplitudes, where the infrared divergences in these amplitudes are regulated by cutting off the internal gluon momenta at $l^2 \sim Xs$ in some complicated way, so as to preserve the gauge invariance of the amplitude. The result (6.5) may then be used to analyze the contribution from this part. There will, of course, be nontrivial corrections to this result, and we hope, with the method developed in this paper, we shall be able to systematically compute those corrections in the future.

In this paper we shall show that the suppression of the amplitude in the $s\to\infty$, s/t-fixed limit, due to the exponentiation of the double logarithms, persists even when we include the effect of all the nonleading logarithms, but the $\ln^2 s$ term in the exponential is replaced by a term proportional to $\ln s \ln\ln s$, due to the asymptotic-freedom effect. We also give an algorithm to make systematic corrections to the above result. The paper is organized as follows. In Sec. II, we describe the kinematics of the problem. We work in the c.m. frame and in the axial gauge. In Sec. III, we analyze the amplitude using a power-counting method developed by Sterman[10] and express it as a sum of four independent amplitudes, each of which is a convolution of the eight-quark Green's function and a hard core (with four external quarks), all of the internal lines of the core being constrained to carry momenta of order Q. In Sec. IV, we show that each of these amplitudes may be expressed as a product of wave-function renormalization constants on external lines and an amplitude Γ_i, which is free from collinear divergences. Each of these Γ_i's may be expressed as a convolution of a regularized eight-quark Green's function and a hard core with four external quarks. In Sec. V we derive a set of differential equations involving Γ_i's, and show that the coefficients of these equations may be analyzed by using renormalization-group equations. The differential equations for the Γ_i's may then be solved and the solution gives us the asymptotic behavior of Γ_i. In Sec. VI, we find the asymptotic behavior of the wave-function renormalization constants, using the method of Ref. 1. Combining this with the asymptotic behavior of the Γ_i's, we find the asymptotic behavior of the full amplitude. We summarize our result and its possible applications in Sec. VII.

For skeptic readers, who may object to the use of the axial gauge in the analysis of the problem, because of the extra singularities in the axial-gauge propagator, we mention here that the analysis may also be carried out in the Coulomb gauge in a similar way.

II. KINEMATICS, GAUGE, RENORMALIZATION

In order to find the asymptotic behavior of the $qq\to qq$ amplitude in the $s\to\infty$, $t\to\infty$, s/t-fixed limit, we choose a frame in which the incoming quark momenta p_a, p_b and the outgoing quark momenta p_c, p_d are given by

$$\begin{aligned} p_a &= ((Q^2+m^2)^{1/2},0,0,Q)\ , \\ p_b &= ((Q^2+m^2)^{1/2},0,0,-Q)\ , \\ p_c &= ((Q^2+m^2)^{1/2},0,Q\sin\theta,Q\cos\theta)\ , \\ p_d &= ((Q^2+m^2)^{1/2},0,-Q\sin\theta,-Q\cos\theta)\ . \end{aligned} \tag{2.1}$$

We denote the color indices and helicities carried by the external particles by a,b,c,d and s_a,s_b,s_c,s_d, respectively. We define

$$s=(p_a+p_b)^2=4(Q^2+m^2)\ , \tag{2.2}$$

$$t=(p_a-p_c)^2=-2Q^2(1-\cos\theta)\ . \tag{2.3}$$

Thus, we can take the $s\to\infty$, t/s-fixed limit by taking the $Q\to\infty$ limit at fixed θ. We shall be interested in the dependence of the amplitude on Q.

We work in the axial gauge, where the gluon propagator takes the form

$$-iN^{\mu\nu}_{\alpha\beta}(k)/(k^2+i\epsilon)=\delta_{\alpha\beta}[-i/(k^2+i\epsilon)][g^{\mu\nu}-(k^\mu n^\nu+k^\nu n^\mu)P(1/n\cdot k)+n^2k^\mu k^\nu P(1/n\cdot k)^2]\ , \tag{2.4}$$

where

$$P(1/n\cdot k)^r=\lim_{\epsilon\to 0}[1/(n\cdot k+i\epsilon)^r+1/(n\cdot k-i\epsilon)^r]/2\ . \tag{2.5}$$

Here μ,ν are the Lorentz indices and α,β are the color indices in the adjoint representation. n is any spacelike vector. For reasons which will become clear later, we shall keep n in the plane of p_a, p_b, p_c, and p_d.

We regularize our theory by dimensional regularization. We use the physical mass of the quark as the renormalized mass parameter. For other counterterms, we use the minimal-subtraction scheme. If $G(p_a,p_b,p_c,p_d)$ is the sum of all Feynman diagrams contributing to the amplitude, including the self-energy insertions on the external lines, the amplitude is given by

$$\bar{u}(p_c)(\not{p}_c-m)[Z_2(p_c)]^{-1/2}\bar{u}(p_d)(\not{p}_d-m)[Z_2(p_d)]^{-1/2}G(p_a,p_b,p_c,p_d) \times[Z_2(p_a)]^{-1/2}(\not{p}_a-m)u(p_a)[Z_2(p_b)]^{-1/2}(\not{p}_b-m)u(p_b)\,, \quad (2.6)$$

where the Z_2's are the external wave-function renormalization factors. In Eq. (2.6), we have left out all the Dirac indices. In the axial gauge, $Z_2(p)$ may have nontrivial dependence on p through the combination $n\cdot p$. In fact, we shall see that the double-logarithmic contribution comes solely from the Z_2's in the axial gauge.

As mentioned in the Introduction, we regularize the infrared divergences by dimensional regularization, although our result is valid for any gauge-invariant regularization procedure. We shall denote the infrared regulator by R. Thus the $R\to 0$ limit corresponds to the infrared-divergent limit. In the dimensional-regularization scheme we work in $4+\epsilon$ dimensions. We first compute the ultraviolet counterterms by working in $4-\epsilon$ dimensions, and then analytically continue the results to $4+\epsilon$ dimensions. This removes all the ultraviolet divergences from the Green's functions, in particular, all the off-shell Green's functions are finite in the $\epsilon\to 0$ limit. As long as we keep ϵ finite and positive, all the on-shell Green's functions are also finite, since in $4+\epsilon$ dimensions there are no infrared divergences in the on-shell amplitudes. The infrared divergences now appear as poles in ϵ in the on-shell amplitudes in the $\epsilon\to 0$ limit. Hence, in this scheme the infrared regulator R may be taken to be ϵ. This method of regularizing the infrared divergences is both Lorentz and gauge invariant.

The diagrams that contribute to the process considered may be divided into two classes, one in which the line carrying momentum p_c is the continuation of the line carrying momentum p_a and the line carrying momentum p_d is the continuation of the line carrying momentum p_b, and the other where the situation is reversed. The sum of all diagrams in each class is separately gauge and Lorentz invariant, thus we may analyze each of them separately. For definiteness, we shall carry out the analysis for the sum of diagrams belonging to the first class. The second class of diagrams may be analyzed in an exactly similar way.

III. A CONVOLUTION FORM FOR THE AMPLITUDE

In this section we shall analyze the important regions of integration in the loop-momentum space which contribute to the amplitude in the leading power of s and express the amplitude as a convolution of a central hard core and an eight-quark Green's function. To do this, we make use of a power-counting method developed by Sterman.[10] If p is any of the momenta p_a, p_b, p_c, or p_d, we say momentum k is parallel or collinear to p if

$$k^0\sim p^0,\;\; p\cdot k\sim k^2\sim\lambda Q^2\,, \quad (3.1)$$

where λ is a scaling parameter which scales to zero. For example, we say k is parallel to p_a if

$$k^0\sim p^0,\;\; k^0-k^3\sim\lambda Q,\;\; k^1,\;\; k^2\sim\lambda^{1/2}Q\,. \quad (3.2)$$

We shall say a momentum is soft if all its components are small compared to Q, whereas a momentum k is said to be hard if all its components are of order Q.

It can be seen from the power-counting argument of Sterman[10] that the regions in loop-momentum space that contribute to the amplitude in the leading power in Q must have the structure shown in Fig. 1. Here J_a, J_b, J_c, and J_d are blobs containing lines parallel to p_a, p_b, p_c, and p_d, respectively. The blob marked H contains hard lines only, whereas the blob marked S contains soft lines only. All the gluon lines connecting the blob S to the jets are also soft. These soft-gluon lines may attach to the jet lines through elementary or composite three-point vertices only. Here, by a composite three-point vertex we mean a subdiagram with three external lines, all of whose internal lines are hard. The soft blob S contains connected as well as disconnected diagrams.

With the knowledge that we gain from Fig. 1, we shall make a topological decomposition of a general graph contributing to the amplitude. First, we shall give a few definitions. A subgraph of any graph is called a four-quark subdiagram if it satisfies the following two properties: (1) it has two incoming quarks, two outgoing quarks, and no gluons as its external lines, and (2) the full graph may be topologically decomposed in the form of Fig. 1, with the subgraph as its central hard core. We also define a gluon subdiagram to be a connected subdiagram, with only gluons as external lines, the external gluons being attached directly to the quark lines ac or bd. Figure 2 shows examples of four-quark subdiagrams and gluon subdiagrams. If there are n such gluon subdiagrams in a given Feynman diagram, we denote by $k_1,\ldots,k_n$ the momenta transferred from the ac quark line to the bd quark line by these n subdiagrams. These momenta must satisfy the constraint

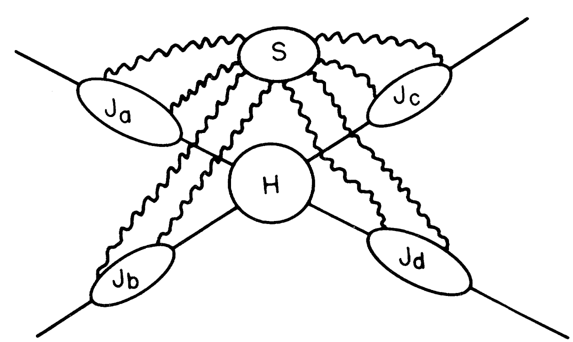

FIG. 1. A diagrammatic representation of the regions of integration in the loop momentum space, which contribute to the $qq\to qq$ amplitude in the leading power in Q.

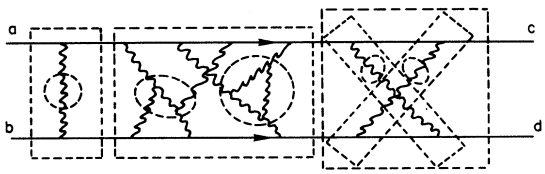

FIG. 2. Examples of four-quark subdiagrams (square boxes) and gluon subdiagrams (circular boxes).

$$\sum_{i=1}^{n} k_i = p_a - p_c \ . \tag{3.3}$$

Let f be the integrand of the Feynman integral corresponding to the graph. We may multiply it by $(\sum k_i)^2/t\,(\equiv 1)$ without changing the value of the integral and write the integral as

$$\sum_{i=1}^{n} \int (k_i{}^2/t) f + 2\sum_{i<j} \int (k_i \cdot k_j/t) f \ . \tag{3.4}$$

In the integral $\int (k_i{}^2/t)f$, the contribution from the region of integration, where the momentum k_i is soft, is suppressed by a power of Q, due to the presence of the extra factor $k_i{}^2/t \sim k_i{}^2/Q^2$; hence, k_i must be hard. Then, in order to get an integration region consistent with the picture shown in Fig. 1, all the internal lines of the minimal four-quark subdiagram, containing the ith-gluon subdiagram, must also carry hard momenta. Similarly, in the integral $2\int (k_i \cdot k_j/t)f$, all the internal lines of the minimal four-quark subdiagram, containing the ith- and the jth-gluon subdiagram, must carry hard momenta. Let us denote by ϕ the sum of all such possible four-quark subdiagrams, all of whose internal lines are constrained to be hard due to the presence of the extra factors of $k_i{}^2/t$ or $k_i \cdot k_j/t$ in the internal lines. Typical contributions to ϕ have been shown in Fig. 3. We may represent ϕ as

$$\phi^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}(p_a + l_1, p_b + l_2, p_c + l_3) \ ,$$

where a',b',c',d'; $\alpha',\beta',\gamma',\delta'$, and $p_a + l_1$, $p_b + l_2$, $p_c + l_3$,

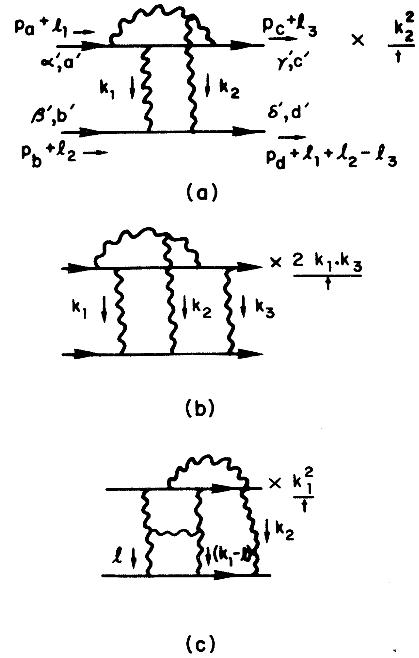

FIG. 3. Some typical contributions to ϕ.

and $p_d + l_1 + l_2 - l_3$ are, respectively, the color and Dirac indices and momenta carried by the quark lines external to ϕ. In later discussions, we shall often drop the color and the Dirac indices from ϕ.

Let F denote the Green's function shown in Fig. 4. In F, we sum all the diagrams, connected and disconnected, and self-energy insertions on external lines, and then multiply the sum by $[Z_2(p_a)]^{-1}(\not p_a - m)u(p_a)$, $[Z_2(p_b)]^{-1} \times (\not p_b - m)u(p_b)$, $\bar u(p_c)(\not p_c - m)[Z_2(p_c)]^{-1}$, and $\bar u(p_d)(\not p_d - m)[Z_2(p_d)]^{-1}$ for the external lines carrying momenta p_a, p_b, p_c, and p_d, respectively, thus truncating the propagators corresponding to these lines. The total contribution to the amplitude under consideration is then given by

$$\mathscr{A} = [Z_2(p_a)Z_2(p_b)Z_2(p_c)Z_2(p_d)]^{1/2} \int \prod_{j=1}^{3} \frac{d^4 l_j}{(2\pi)^4} F^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}(p_a + l_1, p_b + l_2, p_c + l_3)\phi^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}(p_a + l_1, p_b + l_2, p_c + l_3) \ , \tag{3.5}$$

where $\alpha',\beta',\gamma',\delta'$ and a',b',c',d' are, respectively, the Dirac and the color indices of the external quark lines of F, as shown in Fig. 4. For convenience of notation, we have dropped the dependence of F on the color, helicities, and momenta of the external on-shell quarks in the above equation. If we take the convolution of a particular diagram contributing to F with a particular diagram contributing to ϕ according to (3.5), the l integral may have some spurious ultraviolet divergences, due to the presence of the extra factor $k_i{}^2/Q^2$ or $k_i \cdot k_j/Q^2$ in the internal lines of ϕ, but these divergences must cancel when we sum over all the diagrams in F and ϕ.

The integral of (3.5) is diagrammatically represented as in Fig. 5. We write it as

$$\mathscr{B} = \int \prod_{j=1}^{3} \frac{d^4 l_j}{(2\pi)^4} F^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}(p_a + l_1, p_b + l_2, p_c + l_3)\phi^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}(p_a + l_1, p_b + l_2, p_c + l_3) \equiv F\phi \ . \tag{3.6}$$

We shall stick to the convention that whenever we draw a graph contributing to F, we shall draw the external on-shell quark lines, carrying momenta p_a, p_b, p_c, and p_d to the left, and the off-shell quark lines, carrying momenta $p_a + l_1$, $p_b + l_2$, $p_c + l_3$, and $p_d + l_1 + l_2 - l_3$ to the right. Thus, as we move from the left to the right in a graph contributing to F, we move towards the core ϕ in the corresponding amplitude shown in Fig. 5. Let us consider a

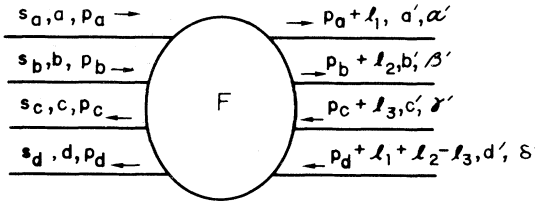

FIG. 4. The eight-quark Green's function F.

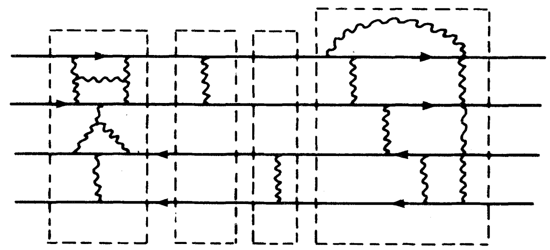

FIG. 6. Examples of four-particle irreducible eight-quark subdiagrams of F (square boxes).

subgraph of any graph contributing to F, with the following properties: (1) the subgraph has eight external quark lines and no external gluon lines, (2) the eight external quark lines of the subgraph are the continuations of the eight external quark lines of F into the graph. Such a subgraph is called four-particle irreducible (4PI) if it is not possible to divide the subgraph into two parts by drawing a vertical line through it, which cuts only four fermion lines. Examples of such 4PI subgraphs of F have been shown in Fig. 6 by enclosing them in square boxes. Let $K_{(a)}$ be the sum of all eight-quark graphs satisfying the following properties. In any graph contributing to $K_{(a)}$, one and only one of its 4PI subgraphs has nontrivial interaction with the a line and this 4PI subdiagram lies unambiguously to the left of all other 4PI subgraphs of that graph. Typical contributions to $K_{(a)}$ have been shown in Fig. 7. Note that a diagram of the type shown in Fig. 8 is not included in $K_{(a)}$, since the gluon line marked 2 may be taken to lie to the right or to the left of the gluon line marked 1. We also define $F_{(bcd)}$ to be the total contribution to F from those diagrams where the line a does not take part in any interaction. Typical contributions to $F_{(bcd)}$ have been shown in Fig. 9. If in $K_{(a)}$ we include the propagators of the external quark lines to the right, but truncate the propagators of the external quark lines to the left, F satisfies the equation

$$F=F_{(bcd)}+FK_{(a)} \tag{3.7a}$$

in the shorthand notation used in writing (3.6).

In an exactly similar way we may define $K_{(b)}$, $K_{(c)}$, $K_{(d)}$, $F_{(acd)}$, $F_{(abd)}$, and $F_{(abc)}$. Equations analogous to (3.7a) are

$$F=F_{(acd)}+FK_{(b)} \ , \tag{3.7b}$$

$$F=F_{(abd)}+FK_{(c)} \ , \tag{3.7c}$$

$$F=F_{(abc)}+FK_{(d)} \ . \tag{3.7d}$$

We shall find it more convenient to redefine F by dividing it by a factor of Q^2 and redefine ϕ by multiplying it by a factor of Q^2. As a result, the dependence of F on Q due to the presence of the factors $(p_a^0)^{1/2}$, $(p_b^0)^{1/2}$, $(p_c^0)^{1/2}$, and $(p_d^0)^{1/2}$ from the external spinors, goes away. On the other hand, multiplication by Q^2 makes ϕ dimensionless, thus ensuring that $\phi(p_a+l_1,p_b+l_2,p_c+l_3)$ is independent of Q at the tree level, in the limit $|l_i^\mu|/Q \to 0$, $i=1,2,3$. These redefinitions leave (3.5) unchanged. If we redefine $F_{(bcd)}$, $F_{(acd)}$, $F_{(abd)}$, and $F_{(abc)}$ by dividing each of them by Q^2, then these redefinitions also leave Eqs. (3.7) unchanged.

With the help of Eqs. (3.7), we shall bring (3.6) to a different form. Let us first analyze the tensor structure of

$$\phi^{\alpha\beta\gamma\delta}_{a'b'c'd'}(p_a+l_1,p_b+l_2,p_c+l_3) \ .$$

In color space, it can have two independent tensor structures, which may be taken as $\delta_{a'c'}\delta_{b'd'}$ and $\delta_{a'd'}\delta_{b'c'}$, respectively. In Dirac space, it may have many different tensor structures in general, but we shall be interested in only

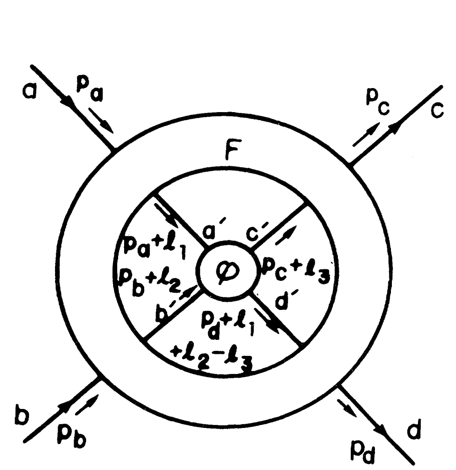

FIG. 5. Graphical representation of $\mathscr{B}$, defined in (3.6).

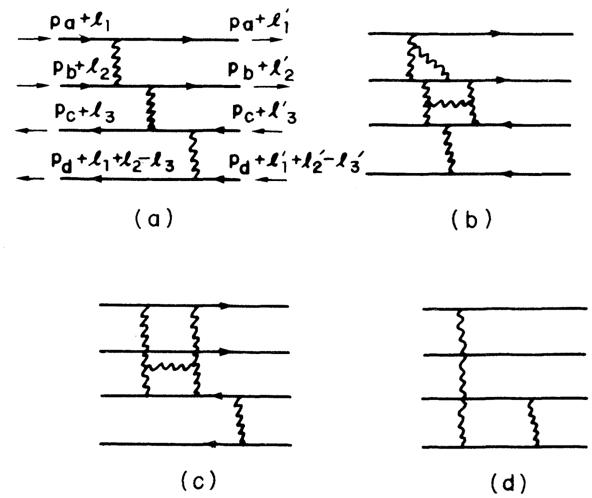

FIG. 7. Some typical contributions to $K_{(a)}$.

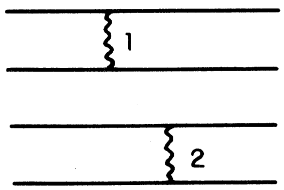

FIG. 8. A typical contribution to the eight-quark Green's function, suffering from ordering ambiguity.

those tensors, which contribute to (3.6) from the region $|l_i^\mu| \ll Q$, in leading power in Q. To identify such tensors, let us note that in the $l_i \to 0$ limit, the numerators of the propagators of the four-quark lines, entering and leaving ϕ, may be replaced by $p_a \cdot \gamma$, $p_b \cdot \gamma$, $p_c \cdot \gamma$, and $p_d \cdot \gamma$, respectively. In the $Q \to \infty$ limit,

$$(p_i \cdot \gamma)^{\alpha\beta} \propto \sum_{s_{i'}=\pm\frac{1}{2}} u^{\alpha}_{s_{i'}}(p_i)\bar{u}^{\beta}_{s_{i'}}(p_i) , \tag{3.8}$$

$s_{i'}$ referring to the helicity. For the subset of diagrams considered here, $s_{a'}=s_{c'}$, $s_{b'}=s_{d'}$ in leading power in Q, since there are an odd number of γ matrices on the $a'c'$ and the $b'd'$ lines. Using reflection symmetry in the plane of p_a, p_b, p_c, and p_d (n does not change under this reflection), we conclude that there are only two independent Dirac structures of ϕ, corresponding to the amplitudes $s_{a'}=\frac{1}{2}$, $s_{b'}=\frac{1}{2} \to s_{c'}=\frac{1}{2}$, $s_{d'}=\frac{1}{2}$, and $s_{a'}=\frac{1}{2}$, $s_{b'}=-\frac{1}{2} \to s_{c'}=\frac{1}{2}, s_{d'}=-\frac{1}{2}$. Thus, in the Dirac and the color space there are altogether four different tensor structures of ϕ that contribute to (3.6) from the $|l_i^\mu| \ll Q$ region, in leading power in Q. Let us choose a basis of linearly independent tensors $(\Lambda_i)^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}$ such that if we substitute Λ_1, Λ_2, Λ_3, or Λ_4 in place of ϕ in (3.6), we receive a nonsuppressed contribution to the integral from the $|l_j^\mu| \ll Q$ region, whereas if we substitute any of the other Λ_i's in place of ϕ in (3.6), the contribution from the region $|l_j^\mu| \ll Q$ is suppressed by a power of Q. Let ϕ_i be the component of ϕ along the direction of the tensor Λ_i. We may write ϕ as

$$\begin{aligned}\phi &= \sum_i \phi_i \Lambda_i \\ &\equiv \sum_{i=1}^{4} \phi_i \Lambda_i Q^4/[Q^4+(l_1{}^2)^2+(l_2{}^2)^2+(l_3{}^2)^2]+\phi_{\text{res}} .\end{aligned} \tag{3.9}$$

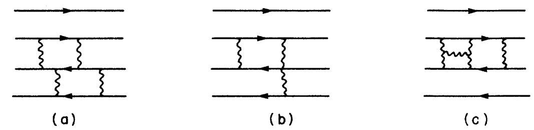

FIG. 9. Some typical contributions to $F_{(bcd)}$.

In the region $|l_j^\mu| \ll Q$, $j=1,2,3$, the contribution to (3.6) comes entirely from the $\sum_{i=1}^{4} \cdots$ term on the right-hand side of (3.9), thus the contribution from the ϕ_{res} term in this region is suppressed. The contribution to (3.6) from the $\sum_{i=1}^{4} \cdots$ term on the right-hand side of (3.9) is given by

$$\sum_{i=1}^{4} \int \prod_{j=1}^{3} \frac{d^4 l_j}{(2\pi)^4} F_i(p_a+l_1,p_b+l_2,p_c+l_3)\phi_i(p_a+l_1,p_b+l_2,p_c+l_3)Q^4/[Q^4+(l_1{}^2)^2+(l_2{}^2)^2+(l_3{}^2)^2] , \tag{3.10}$$

where

$$F_i = F^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}(\Lambda_i)^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'} . \tag{3.11}$$

The purpose of the term $Q^4/[Q^4+(l_1{}^2)^2+(l_2{}^2)^2+(l_3{}^2)^2]$ is to avoid ultraviolet divergences in the integral of (3.10) from the l integrals in graphs like Fig. 10(a). This also avoids spurious ultraviolet divergences in the l integrals in graphs like Fig. 10(b), due to the presence of the extra factors of $k_i{}^2/t$ or $k_i \cdot k_j/t$ in the internal lines of ϕ. All such divergences are dumped into the integral $\int F\phi_{\text{res}}$.

To analyze the integral $\int F\phi_{\text{res}}$, we break up ϕ_{res} as

$$\phi_{\text{res}} = \phi^a_{\text{res}} + \phi^b_{\text{res}} + \phi^c_{\text{res}} , \tag{3.12}$$

where

$$\phi^a_{\text{res}} = \left[(l_1{}^2)^2 \Big/ \sum_{j=1}^{3} (l_j{}^2)^2 \right] \phi_{\text{res}} , \tag{3.13a}$$

$$\phi^b_{\text{res}} = \left[(l_2{}^2)^2 \Big/ \sum_{j=1}^{3} (l_j{}^2)^2 \right] \phi_{\text{res}} , \tag{3.13b}$$

$$\phi^c_{\text{res}} = \left[(l_3{}^2)^2 \Big/ \sum_{j=1}^{3} (l_j{}^2)^2 \right] \phi_{\text{res}} . \tag{3.13c}$$

As we have already seen, when we substitute ϕ_{res} in place of ϕ in (3.5), the integral receives nonsuppressed contributions only from the regions where at least one of l_1, l_2, and l_3 is hard, thus $(l_1{}^2)^2+(l_2{}^2)^2+(l_3{}^2)^2$ must be of order Q^4 or more. The contribution from the ϕ^a_{res} term is then suppressed unless $l_1{}^2$ is of order Q^2 or more. Similarly ϕ^b_{res} and ϕ^c_{res} terms will give nonsuppressed contributions to (3.6) only from the $|l_2^\mu| \gtrsim Q$ and $|l_3^\mu| \gtrsim Q$ regions, respectively. If in the integral $\int F\phi^a_{\text{res}}$, we substitute the right-hand side of Eq. (3.7a), the $F_{(bcd)}$ term does not contribute since it has a $\delta(l_1)$ term. The contribution from the $FK_{(a)}$ term is

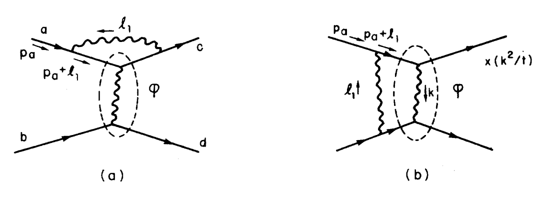

FIG. 10. Some typical contributions to the integral of (3.5), where the l integral suffers from ultraviolet divergences.

$$\int \prod_{j=1}^{3} \frac{d^4l_j}{(2\pi)^4}\frac{d^4l_j'}{(2\pi)^4} F(p_a+l_1,p_b+l_2,p_c+l_3)K_{(a)}(p_a+l_1,p_b+l_2,p_c+l_3,p_a+l_1',p_b+l_2',p_c+l_3')\phi^a_{\rm res}(p_a,p_b,p_c,l_1',l_2',l_3') \ , \tag{3.14}$$

where we have dropped all the color and Dirac indices of F, $K_{(a)}$, and $\phi^a_{\rm res}$ for convenience of writing. $\phi^a_{\rm res}$ is now separately a function of p_a, p_b, p_c, l_1, l_2, and l_3 instead of being a function of p_a+l_1, p_b+l_2, and p_c+l_3, because of the extra factors of

$$Q^4 \bigg/ \left[Q^4+\sum_{j=1}^{3}(l_j^2)^2\right]$$

and

$$(l_1^2)^2 \bigg/ \sum_{j=1}^{3}(l_j^2)^2$$

in (3.9) and (3.13a), respectively. In the integral in (3.14), l_1' is constrained to be hard. In order to get a momentum flow consistent with Fig. 1, all the internal lines of $K_{(a)}$ must also carry hard momenta. Similar analysis may be done for the $\int F\phi^a_{\rm res}$ and $\int F\phi^c_{\rm res}$ terms, by substituting for F the right-hand sides of Eqs. (3.7b) and (3.7c), respectively. The sum of all these terms may be written as

$$\int \prod_{j=1}^{3}\frac{d^4l_j}{(2\pi)^4} F^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}(p_a+l_1,p_b+l_2,p_c+l_3)\phi^{(1)\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}(p_a,p_b,p_c,l_1,l_2,l_3) \ , \tag{3.15}$$

where

$$\phi^{(1)}=K_{(a)}\phi^a_{\rm res}+K_{(b)}\phi^b_{\rm res}+K_{(c)}\phi^c_{\rm res} \ . \tag{3.16}$$

$\phi^{(1)}$ is calculated from diagrams, all of whose internal lines carry hard momenta. The integral (3.15) has the same structure as the integral of (3.6) and hence may be analyzed in the same way to give a sum of the term

$$\sum_{i=1}^{4}\int F_i(p_a+l_1,p_b+l_2,p_c+l_3)\phi_i^{(1)}(p_a,p_b,p_c,l_1,l_2,l_3)Q^4/[Q^4+(l_1^2)^2+(l_2^2)^2+(l_3^2)^2] \tag{3.17}$$

and the integral $\int F\phi^{(1)}_{\rm res}$. This may be analyzed in the same way as $\int F\phi_{\rm res}$. Continuing this process indefinitely, we may express $\mathscr{B}$ as

$$\sum_{i=1}^{4}\Gamma_i \ , \tag{3.18}$$

where

$$\Gamma_i=\int F_i(p_a+l_1,p_b+l_2,p_c+l_3)\Phi_i(p_a,p_b,p_c,l_1,l_2,l_3)\prod_{j=1}^{3}\frac{d^4l_j}{(2\pi)^4} \ , \tag{3.19}$$

$$\Phi_i(p_a,p_b,p_c,l_1,l_2,l_3)=\frac{Q^4}{Q^4+(l_1^2)^2+(l_2^2)^2+(l_3^2)^2}[\phi_i(p_a+l_1,p_b+l_2,p_c+l_3)+\phi_i^{(1)}(p_a,p_b,p_c,l_1,l_2,l_3)+\cdots] \ . \tag{3.20}$$

The right-hand side of Eq. (3.19) may be graphically represented as in Fig. 11. Note that, in (3.19), the l_j integrals do not have any ultraviolet divergence due to the presence of the

$$Q^4 \bigg/ \left[Q^4+\sum_{j=1}^{3}(l_j^2)^2\right]$$

factor in Φ_i. The spurious ultraviolet divergences which appear, due to the presence of the terms k_i^2/t or $k_i\cdot k_j/t$ in ϕ_i, are all included in the internal loop-momentum integrations in Φ_i and hence must cancel internally. The only ultraviolet divergences left are then due to the vertex and self-energy corrections which are canceled separately inside Φ_i and F_i by the usual counterterms. Hence, each of the terms

$$[Z_2(p_a)Z_2(p_b)Z_2(p_c)Z_2(p_d)]^{1/2}\Gamma_i \tag{3.21}$$

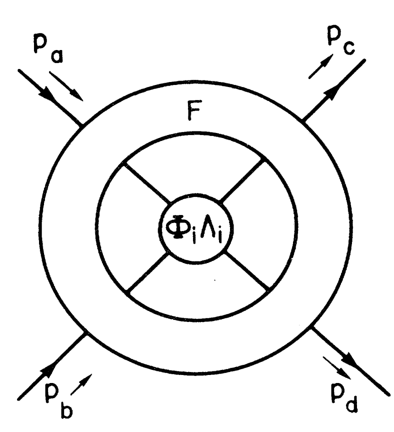

FIG. 11. Graphical representation of Γ_i.

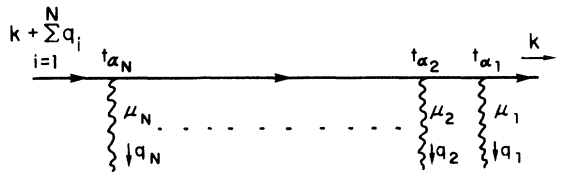

FIG. 12. A Green's function with N gluons attached to a fermion line.

is separately ultraviolet finite when expressed in terms of the renormalized parameters. Since the Z_2's are multiplicatively renormalized, so must be the Γ_i's. If $(Z_2^0)^{-1}$ is the multiplicative constant which renormalizes the $Z_2(p_i)$'s, Γ_i's are obtained from the corresponding unrenormalized quantities Γ_i^0's as

$$\Gamma_i=(Z_2^0)^2\Gamma_i^0 . \tag{3.22}$$

The renormalization-group equations for Γ_i's may be obtained by using (3.22) and the fact that Γ_i^0, when expressed as a function of the bare parameters of the theory, is independent of the renormalization mass μ.

IV. FACTORIZATION OF THE COLLINEAR DIVERGENCES

In this section we shall show that the contribution to the amplitudes Γ_i, defined in Sec. III, may be brought into a form, which receives contributions only from those regions of integration in momentum space, where none of the internal loop momenta are parallel to any of the external momenta p_a, p_b, p_c, or p_d. We shall use this result in the next section to derive a differential equation involving the amplitudes. We shall use a method developed by Collins and Soper,[11] rather than the method used in Ref. 1, to show the factorization of the collinear divergences. We shall explain the method briefly below.

For a set of gluons of momenta $q_1,\ldots,q_N$, polarizations $\mu_1,\ldots,\mu_N$, and color $\alpha_1,\ldots,\alpha_N$, attached to a quark line moving parallel to one of the external momenta p_i $(i=a,b,c,d)$, as shown in Fig. 12, we define the soft approximation as

$$\sum_{J=0}^{N} U_J^{-1}[i/(\not k+q_1+\cdots+q_J-m+i\epsilon)]U_{N,J} , \tag{4.1}$$

where

$$U_J^{-1}=\frac{i}{(q_1+\cdots+q_J)\cdot v_i\pm i\epsilon}(-igv_i^{\mu_1}t_{\alpha_1})\frac{i}{(q_2+\cdots+q_J)\cdot v_i\pm i\epsilon}(-igv_i^{\mu_2}t_{\alpha_2})\cdots\frac{i}{q_J\cdot v_i\pm i\epsilon}(-igv_i^{\mu_J}t_{\alpha_J}) \tag{4.2}$$

and

$$U_{N,J}=(igv_i^{\mu_{J+1}}t_{\alpha_{J+1}})\frac{i}{q_{J+1}\cdot v_i\pm i\epsilon}(igv_i^{\mu_{J+2}}t_{\alpha_{J+2}})\frac{i}{(q_{J+1}+q_{J+2})\cdot v_i\pm i\epsilon}\cdots\frac{i}{(q_{J+1}+\cdots q_N)\cdot v_i\pm i\epsilon} , \tag{4.3}$$

where t_α's are the representations of the group generators in the fermion representation, k and $k+\sum_{i=1}^{N}q_i$ are the momenta of the external fermion lines, and

$$v_i=\lim_{Q\to\infty} p_i/p_i^0 , \tag{4.4}$$

if the soft approximation is made for the momentum k being parallel to p_i $(i=a,b,c,d)$. In Eqs. (4.2) and (4.3), the negative sign in front of the $i\epsilon$ appears when k is parallel to p_a or p_b, the positive sign appears when k is parallel to p_c or p_d. This is to make sure that the poles in the $q\cdot v_i$ plane from the denominators of (4.2) and (4.3) are on the same side of the real axis as the poles from the original Feynman denominators of the graph. Equation (4.1) may be graphically represented by Fig. 13. The rules for the special vertices used in Fig. 13 are given in Fig. 14. Expression (4.1) approximates the graph shown in Fig. 12 in the region of integration where $q_1,\ldots,q_N$ are soft lines and k is collinear to the momentum p_i. Similar soft approximations may also be made for soft gluons attached to collinear gluons. Consider now a Green's function with a set A of external gluons and fermions, and a set B of gluons attached to it through the soft approximation given

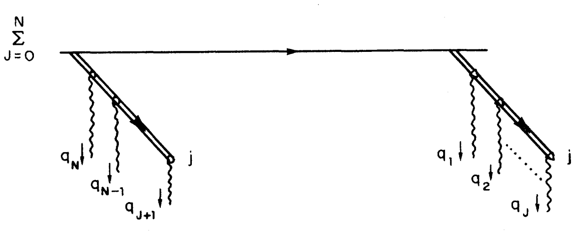

FIG. 13. Soft approximation for the Green's function shown in Fig. 12.

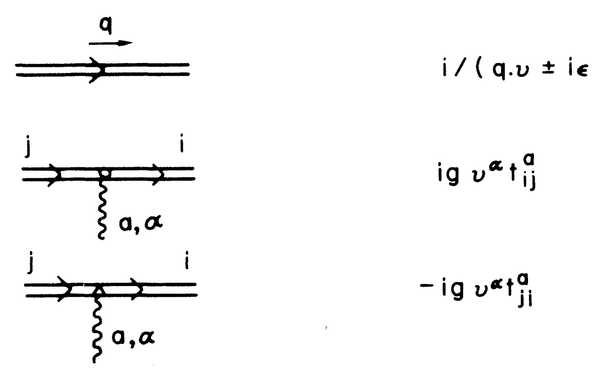

FIG. 14. Expressions for the special vertices and propagators shown in Fig. 13.

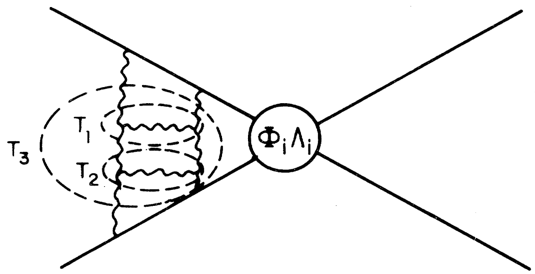

FIG. 15. Examples of tulips and gardens.

in (4.1). It was shown in Ref. 11 that if we sum over all insertions of the gluons of set B to the Green's function, using soft approximation every time, the final result is the sum of all possible graphs, where the gluons of set B are attached to the outer ends of the external fermions and the gluons in set A through the eikonal vertices given in Fig. 14 (as in Fig. 13). The graphs, where the external gluons of set B are attached to the ends of the internal lines of the Feynman graphs, cancel among themselves.

We now give some definitions. Let us consider any 4PI subgraph G of F. We may regard this as a subgraph of the amplitude Γ_i, after we plug F into Fig. 11. A subgraph T of G is called a tulip, if it satisfies the following property: The graph contributing to Γ_i, of which G is a subgraph, may be topologically decomposed in the form of Fig. 1, with T as a part of the central subgraph S, and all the lines in $G-T$ belonging to the various jets $J_a, \ldots, J_d$. The graph $G-T$ must be 1PI in the external gluon legs. A garden is a nested set of tulips $\{T_1, \ldots, T_n\}$ such that $T_j \subset T_{j+1}$ for $j=1,\ldots,n-1$. Examples of tulips and gardens are shown in Fig. 15. In this figure, T_1, T_2, T_3 are example of tulips and the sets $\{T_1\}$, $\{T_2\}$, $\{T_3\}$, $\{T_1,T_3\}$, and $\{T_2,T_3\}$ are examples of gardens.

For a given 4PI subgraph G, we define a regularized version G_R of G by

$$G_R = G + \sum_{\substack{\text{inequivalent}\\ \text{gardens}}} (-1)^N S(T_1)\cdots S(T_n)G \ . \tag{4.5}$$

We shall first explain the meaning of the symbol $S(T_1)\cdots S(T_n)G$. We start with the largest tulip T_n, belonging to a particular garden. We pretend that the gluons, coming out of the tulip, are soft gluons, attached to collinear lines in $G-T_n$, and replace these insertions by their soft approximation given in (4.1) or its analog for collinear gluons. This defines $S(T_n)G$. We now take the gluon lines coming out of T_{n-1}. If some of these gluons are identical to some of the gluons coming out of T_n, we leave them as they are. For the other gluons, we again pretend that they are soft gluons, attached to the collinear lines in $G-T_{n-1}$ and replace these insertions by their soft approximation. We proceed in this manner to calculate $S(T_1)\cdots S(T_n)G$. Two gardens are said to be equivalent if $S(T_1)\cdots S(T_n)G$ for the two gardens are the same, this happens if the two gardens have identical sets of boundaries. N is the maximum number of tulips in a garden.

It was shown in Ref. 11 that G_R, defined by Eq. (4.5), receives nonsuppressed contributions only from the integration region, where all its internal momenta are hard. Then according to Fig. 1, the subgraph of F, which lies unambiguously to the right of G_R, must also carry hard momenta.

We shall now show that the collinear divergences factorize into wave-function renormalization constants on external lines. We start with a given graph, contributing to the amplitude Γ_i, and number its 4PI graphs from outside to inside as $G_1, G_2, \ldots, G_n$. For graphs of the type shown in Fig. 8, it is not possible to say which 4PI graph is outside (or to the left side of) the other; let us for the time being ignore such ambiguities. We can then write the contribution to the amplitude Γ_i from the above graph as

$$G_1 G_2 \cdots G_n \Lambda_i \Phi_i \ . \tag{4.6}$$

We decompose G_i as

$$G_i = G_{iS} + G_{iR} \ , \tag{4.7}$$

where G_{iR} is the residue defined in (4.5) and G_{iS} is the term containing soft approximations. We write (4.6) as

$$G_{1R}G_2\cdots G_n + G_{1S}G_{2R}G_3\cdots G_n + G_{1S}G_{2S}G_{3R}G_4\cdots G_n + \cdots + G_{1S}\cdots G_{n-1S}G_{nR} + G_{1S}\cdots G_{nS})\Lambda_i\Phi_i \ . \tag{4.8}$$

In the first term, all the internal lines of the graph must carry hard momentum, since G_{1R} carries hard momentum and $G_2, \ldots, G_n$ are surrounded by G_{1R}. In the second term, G_{2R} carries hard momenta. This constrains $G_3, \ldots, G_n$ to carry hard momenta. And so on.

Let us now turn towards the case where we have the ordering ambiguity as shown in Fig. 8. The most general ambiguous subdiagram in F has the form shown in Fig. 16, except for possible permutations of the lines a,b,c,d. Here $G_1', G_2', \ldots, G_{n'}', G_1'', G_2'', \ldots, G_{n''}''$ are two-particle-irreducible subdiagrams. The product of $G_1' \cdots G_{n'}' G_1'' \cdots G_{n''}''$ is decomposed as

$$(G_{1R}'G_2'\cdots G_{n'}' + G_{1S}'G_{2R}'G_3'\cdots G_{n'}' + \cdots + G_{1S}'\cdots G_{n'-1S}'G_{n'R}' + G_{1S}'\cdots G_{n'S}') \times (G_{1R}''G_2''\cdots G_{n''}'' + G_{1S}''G_{2R}''G_3''\cdots G_{n''}'' + \cdots + G_{1S}''\cdots G_{n''-1S}''G_{n''R}'' + G_{1S}''\cdots G_{n''S}'') \ . \tag{4.9}$$

If we pick the R part from any of the G_i', all the G_j''s for $j \geq i$ and the part of F, which lies to the right of the subgraph of Fig. 16 in the full diagram, is constrained to carry hard momenta. Similarly, if we pick the R term from any of the G_i'''s, all the G_j'''s for $j \geq i$ are constrained to carry hard momenta, so is the part of F, lying to the right of the subgraph of Fig. 16 (remember that the right side of F refers to the part closer to the core $\Lambda_i\Phi_i$ in Fig. 11). For the term $G_{1S}'\cdots G_{n'S}'G_{1S}''\cdots G_{n''S}''$ we break up the part of F, lying to the right of the subdiagram of Fig.

16, into a product of 4PI parts and decompose them into R and S parts in the same way as we did in (4.8).

At the end of the decomposition procedure, we shall get a central hard core, which carries only hard momenta due to the presence of a G_R in its outermost shell, surrounded by shells of 4PI subdiagrams, each of which is replaced by its soft approximation

$$G_S = -\sum_{\substack{\text{inequivalent}\\ \text{gardens}}} (-1)^N S(T_1)\cdots S(T_n)G \ .$$

Let us call this central core to be $\Phi'_{(i)}$, the subscript (i) is to remind us that we started with the amplitude Γ_i. The contribution G_S from a given 4PI subdiagram may be written by grouping together the sum over all gardens with the largest tulip T. Thus, we may write

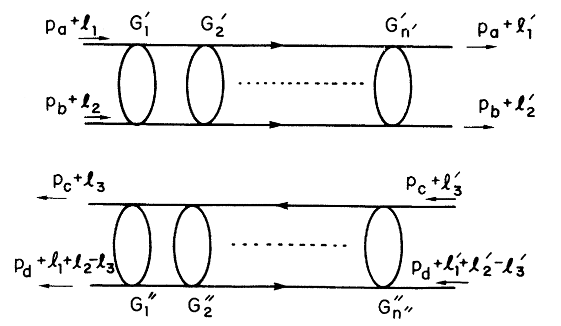

FIG. 16. The most general subdiagram of F, suffering from ordering ambiguity.

$$G_S = \sum_T \left[1 + \sum_{\substack{\text{inequivalent}\\ \text{gardens with } T_n = T}} (-1)^{N-1} S(T_1)\cdots S(T_{n-1}) \right] S(T)G \ . \tag{4.10}$$

Let

$$T_R = T + \sum_{\substack{\text{inequivalent}\\ \text{gardens with } T_n = T}} (-1)^{N-1} S(T_1)\cdots S(T_{n-1})T \ . \tag{4.11}$$

The right-hand side of (4.10) may then be interpreted as the insertion of T_R into $G-T$ using soft approximation for the lines coming out of T_R. Γ_i is then the sum of diagrams of the form shown in Fig. 17. Here M_R is the collection of disconnected regularized tulips T_R. The lines coming out of M_R are inserted into the blobs J_a, J_b, J_c, and J_d using soft approximation. The sum of all such insertions is given by Fig. 18. Let us now compare it with the graph shown in Fig. 19. Here $\chi_{(i)}$ is an unspecified hard core, we shall try to choose it in such a way that the graph of Fig. 18 becomes identical to the one in Fig. 19, except for the self-energy insertion on the external lines. To do this, let us note that the part of Fig. 19, involving M_R, may again be divided into 4PI subdiagrams, which are nothing but regularized tulips T_R, attached to the quark lines. Let $\tilde{G}$ be such a 4PI subgraph. We divide it into the soft part $\tilde{G}_S$, where the insertions of the gluon lines, coming out of T_R, on the quark lines in $\tilde{G}$ are replaced by their soft approximations, and $\tilde{G}_R \equiv \tilde{G} - \tilde{G}_S$, having the property that all its internal lines must be hard if $\tilde{G}$ is replaced by $\tilde{G}_R$ in the full graph. We then decompose the 4PI subgraphs of Fig. 19 using equations similar to (4.8) and (4.9), with G replaced by $\tilde{G}$. The result is a central hard core $\chi'_{(i)}$, surrounded by 4PI subgraphs T_R, inserted on the quark lines through soft approximation. The sum of all the soft insertions is the graph shown in Fig. 20. $\chi'_{(i)}$ is obtained from $\chi_{(i)}$ by using the equation

$$\chi'_{(i)} = \chi_{(i)} + \psi_1 \chi_{(i)} \ , \tag{4.12}$$

where ψ_1 is the sum of diagrams containing an arbitrary number of regularized 4PI subdiagrams $\tilde{G}$, the leftmost one of which is replaced by its R part, thus ensuring that all the lines in $\psi_1\chi_{(i)}$ are hard.

If we choose $\chi_{(i)}$ in such a way that $\chi'_{(i)}$ equals $\Phi'_{(i)}$, then the graphs of Figs. 20 and 18 are identical, except for the

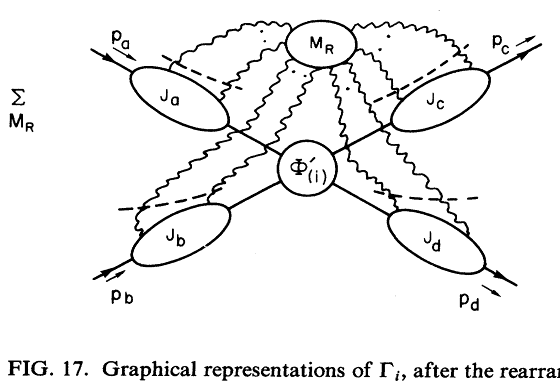

FIG. 17. Graphical representations of Γ_i, after the rearrangement given in Eqs. (4.8)–(4.10). The broken lines indicate that soft approximation is made for the gluon lines crossing the broken line.

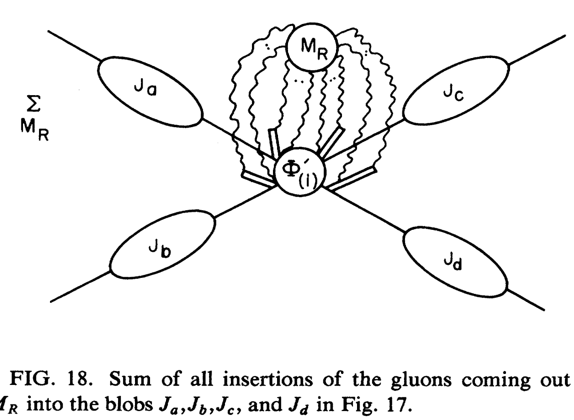

FIG. 18. Sum of all insertions of the gluons coming out of M_R into the blobs J_a,J_b,J_c, and J_d in Fig. 17.

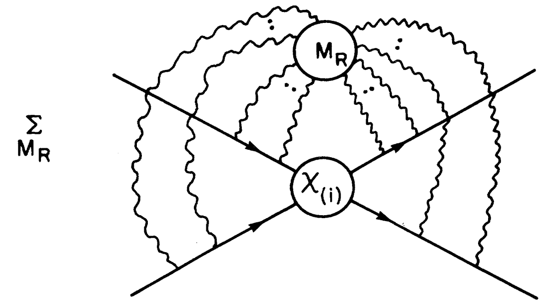

FIG. 19. A trial amplitude.

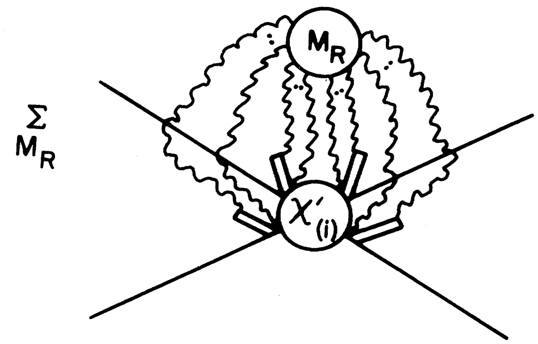

FIG. 20. The amplitude of Fig. 19, after the $S-R$ decomposition of its 4PI subgraphs and sum over all insertions of the gluons, coming out of the S part, on the quark lines.

self-energy insertion on external lines. The corresponding $\mathcal{X}_{(i)}$ is given by

$$\begin{aligned}\mathcal{X}_{(i)} &= (I+\psi_1)^{-1}\Phi'_{(i)}\\ &= (I-\psi_1+\psi_1{}^2-\psi_1{}^3+\cdots)\Phi'_{(i)} .\end{aligned} \tag{4.13}$$

In the definition of F, we have a $[Z_2(p_i)]^{-1}(\not p_i-m)$ factor for each external fermion, this removes the external self-energies from Fig. 18. Γ_i is then calculated by contracting Fig. 20, or equivalently Fig. 19, with the external spinors $u(p_i)$ and $\bar{u}(p_i)$ $(i=a,b,c,d)$. Figure 19 satisfies the property that none of the internal gluon lines in M_R, nor the gluon lines entering or leaving M_R, can be collinear. This is because if there is any such collinear gluon there will also be soft gluons attached to it, separating it from lines collinear to the other momenta. But the soft subtraction terms in M_R force the contribution from any such region to be suppressed by a power of Q. Thus, the internal lines of M_R may either be hard or soft. We express this contribution as

$$\int \prod_{j=1}^{3} \frac{d^4 l_j}{(2\pi)^4} F'(p_a+l_1,p_b+l_2,p_c+l_3) \times \mathcal{X}_{(i)}(p_a,p_b,p_c,l_1,l_2,l_3) , \tag{4.14}$$

where F' is the contribution from the part involving M_R. Again we have omitted all the Dirac and the color indices and the dependence of F' on the external momenta p_a,p_b,p_c for convenience of writing. F' has similar structure as F, defined in Sec. III, and we may write the equations

$$F'=F'_{(bcd)}+F'K'_{(a)} , \tag{4.15a}$$

$$F'=F'_{(acd)}+F'K'_{(b)} , \tag{4.15b}$$

$$F'=F'_{(abd)}+F'K'_{(c)} , \tag{4.15c}$$

$$F'=F'_{(abc)}+F'K'_{(d)} . \tag{4.15d}$$

$\mathcal{X}_{(i)}$ has a perturbation expansion, which may be obtained from Eq. (4.13). F' may be calculated using the subtraction scheme described in this section. Γ_i may then be calculated using (4.14), an expression which is free from collinear divergences.

V. ANALYSIS OF Γ_i

In this section we shall derive a set of differential equations involving the Γ_i's and show how the solutions of these equations give us the asymptotic behavior of the Γ_i's. The asymptotic behavior of the functions $Z_2(p_i)$'s $(i=a,b,c,d)$ will be derived in the next section. Combining these two results, we may find the asymptotic behavior of the full amplitude.

The color and the Dirac structure of $\mathcal{X}^{\alpha'\beta'\gamma'\delta'}_{(i)a'b'c'd'}$ may be analyzed in an exactly similar way as we did for ϕ. We express $\mathcal{X}_{(i)}$ as

$$\begin{aligned}(\mathcal{X}_i)^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}(p_a,p_b,p_c,l_1,l_2,l_3) &= \sum_{i'} \mathcal{X}_{(i)i'}(p_a,p_b,p_c,l_1,l_2,l_3)(\Lambda_{i'})^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}\\ &\equiv \sum_{i'=1}^{4} \mathcal{X}_{(i)i'}(p_a,p_b,p_c,0,0,0)(\Lambda_{i'})^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}+(\mathcal{X}_{i)\mathrm{res}})^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'} ,\end{aligned} \tag{5.1}$$

where the last line of the above equation defines $\mathcal{X}_{(i)\mathrm{res}}$. This definition is slightly different from the definition of ϕ_{res} given in Eq. (3.9). In the limit $|l_j^\mu| \ll Q$ $(j=1,2,3,)$, $\mathcal{X}_{(i)}$ becomes independent of these momenta, since all the internal lines of $\mathcal{X}_{(i)}$ are constrained to carry hard momenta. Hence, we may set l_1, l_2, l_3 to be zero in $\mathcal{X}_{(i)}$ in this region. The contribution to the integral of (4.14) from the region $|l_j^\mu| \ll Q$ then comes solely from the

$$\sum_{i'=1}^{4} \mathcal{X}_{(i)i'}\Big|_{l=0} \Lambda_{i'}$$

term, the $\mathcal{X}_{\mathrm{res}}$ term contributes when at least one of the l_j's is of order Q. Equation (4.14) may then be analyzed in a similar way as (3.5) and brought into a form analogous to (3.18):

$$\sum_{i'=1}^{4}\int\prod_{j}\frac{d^4l_j}{(2\pi)^4}F'_{i'}(p_a+l_1,p_b+l_2,p_c+l_3)[\chi_{(i)i'}(p_a,p_b,p_c,0,0,0)+\chi^{(1)}_{(i)i'}(p_a,p_b,p_c,0,0,0)+\cdots]\equiv\sum_{i'=1}^{4}\Delta_{i'}\tau_{ii'}\ , \tag{5.2}$$

where

$$\Delta_{i'}=\int\prod_{j=1}^{3}\frac{d^4l_j}{(2\pi)^4}F'_{i'}(p_a+l_1,p_b+l_2,p_c+l_3)\ , \tag{5.3}$$

$$\tau_{ii'}=(\chi_{(i)i'}+\chi^{(1)}_{(i)i'}+\cdots)_{l_j=0}\ . \tag{5.4}$$

Thus, $\tau_{ii'}$'s are calculated from Feynman graphs, all of whose internal lines carry hard momenta. We shall now try to evaluate $\partial\Gamma_i/\partial\ln Q$, keeping the angle θ defined in Eq. (2.1) fixed. Thus, the differentiation is a differentiation with respect to $\ln\sqrt{s}$, keeping the ratio s/t fixed. Taking the derivative operator inside the integral in (4.14) we may write

$$\frac{\partial\Gamma_i}{\partial\ln Q}=\int\prod_{j=1}^{3}\frac{d^4l_j}{(2\pi)^4}\left[\frac{\partial F'(p_a+l_1,p_b+l_2,p_c+l_3)}{\partial\ln Q}\chi_{(i)}(p_a,p_b,p_c,l_1,l_2,l_3)+F'(p_a+l_1,p_b+l_2,p_c+l_3)\frac{\partial\chi_{(i)}(p_a,p_b,p_c,l_1,l_2,l_3)}{\partial\ln Q}\right]\ . \tag{5.5}$$

F' may be written as a Feynman integral over its internal loop momenta. We may choose the independent loop momenta of F' in such a way that the only dependence of the integrand on p_a, p_b, p_c, or p_d comes from the propagators of the four fermion lines running through the graph. $\partial F'/\partial\ln Q$ may then be evaluated by acting with the derivative operation on each of these lines. A typical contribution to F' and the corresponding contribution to $\partial F'/\partial\ln Q$ have been shown in Fig. 21. The cross on a fermion line denotes the operation of $\partial/\partial\ln Q$ on that line. (Although we represent F' by a Feynman diagram, it should be understood that in these diagrams, the 4PI subgraphs have internal soft subtractions. Such subtraction terms do not affect our discussion.)

Let us consider a quark propagator in a graph, contributing to F', which is a part of the continuation of the external quark line, carrying momentum p_a through the graph. The momentum carried by this line may be written as p_a+k, where k is some linear combination of the internal loop momenta of F' and the l_j's. The contribution to the Feynman integrand from this line is given by

$$(p_a+k)\cdot\gamma/[(p_a+k)^2-m^2+i\epsilon]\ . \tag{5.6}$$

If we consider the region of integration where k is soft, we may approximate the numerator by $p_a\cdot\gamma$ and the denominator by $(2p_a\cdot k+i\epsilon)$. In the $Q\to\infty$ limit, (5.6) may be written as

$$v_a\cdot\gamma/(2v_a\cdot k+i\epsilon)\ , \tag{5.7}$$

v_a being defined as in Eq. (4.4). Equation (5.7) is independent of Q, hence the $\partial/\partial\ln Q$ operator, acting on it, gives zero. So, in order to give a nonsuppressed contribution to $\partial\Gamma_i/\partial\ln Q$, a crossed line in a diagram must carry hard momentum (remember that there is no collinear loop momentum in F'). [This result is valid only for on-shell regularization. For off-shell regularization, the denominator is given by $(2p_a\cdot k+M_a{}^2+i\epsilon)$ in the k-soft region, where $M_a{}^2$ denotes the quantity $p_a{}^2-m^2$. $\partial/\partial\ln Q$ operator, acting on this term, will receive a contribution from the $k\sim M_a{}^2/Q$ region, besides the hard region.] Then, in order to get a momentum flow consistent with the result

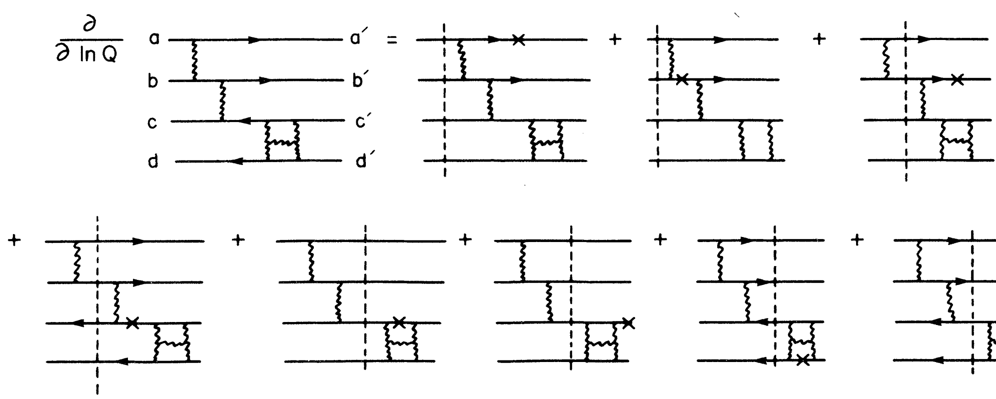

FIG. 21. A typical contribution to F' and the corresponding contributions to $\partial F'/\partial\ln Q$. In each graph, the subgraph to the right of the broken line is constrained to be hard due to the action of the derivative operations. (This excludes the quark lines cut by the broken lines.)

of Fig. 1, the part of the graph which lies unambiguously to the right of the crossed line, or which cannot be separated from the crossed line by drawing a vertical line through the diagram cutting four normal (not crossed) quark lines, must also carry hard momenta. Similar analysis may be carried out for crosses on the other fermion lines. In Fig. 21, in each graph, the part lying to the right of the broken line is constrained to be hard. As a result, the first term on the right-hand side of (5.5) may be expressed as a convolution of F' with a central hard core $\rho_{(i)}$. Typical contributions to $\rho_{(i)}$ have been shown in Fig. 22. Then (5.5) may be written as

$$\int \prod_{J=1}^{3} \frac{d^4 l_j}{(2\pi)^4} F'(p_a+l_1,p_b+l_2,p_c+l_3) \left[\rho_{(i)}(p_a,p_b,p_c,l_1,l_2,l_3) + \frac{\partial \chi_{(i)}(p_a,p_b,p_c,l_1,l_2,l_3)}{\partial \ln Q} \right] . \tag{5.8}$$

The above expression has the same structure as (4.14), with $\chi_{(i)}$ replaced by the hard core $\rho_{(i)}+\partial\chi_{(i)}/\partial \ln Q$. Thus, it may be analyzed in the same way and brought into a form analogous to (5.2):

$$\frac{\partial \Gamma_i}{\partial \ln Q} = \sum_{i'=1}^{4} \sigma_{ii'} \Delta_{i'} , \tag{5.9}$$

where σ is calculated from Feynman diagrams all of whose internal momenta carry hard momenta. Γ_i is given by (5.2). Treating Γ and Δ as four-dimensional vectors and σ and τ as 4×4 matrices, we may eliminate Δ between (5.2) and (5.9) and write

$$\partial \Gamma_i / \partial \ln Q = \sum_{i'=1}^{4} \lambda_{ii'} \Gamma_{i'} , \tag{5.10}$$

where

$$(\lambda)_{ii'} = (\sigma \tau^{-1})_{ii'} . \tag{5.11}$$

$\Delta_{i'}$, as defined in (5.3), suffers from ultraviolet divergences, since the l_j integrals diverge in (5.3). Consequently, σ and τ must also have ultraviolet divergences, so that the products $\sigma\Delta$ and $\tau\Delta$ are free from ultraviolet divergences. In Eq. (5.10), however, both $\Gamma_{i'}$'s and $\partial\Gamma_i/\partial \ln Q$'s are free from ultraviolet divergences. If we regard these as a set of linear equations in $\lambda_{ii'}$'s, we get 16 such independent equations (four i's and four different color and helicity structures of the external on-shell particles, on which the Γ_i's depend). By solving these, $\lambda_{ii'}$'s may be expressed in terms of Γ_i's and $\partial\Gamma_i/\partial \ln Q$'s. This shows that the $\lambda_{ii'}$'s are free from ultraviolet divergences. λ is also free from infrared divergences and independent of the quark mass m in the $Q\to\infty$ limit, since it is calculated from Feynman diagrams, all of whose internal lines are hard. Thus, λ may be expressed as a function of Q/μ and the coupling constant g (μ = renormalization mass).

If we multiply both sides of Eq. (5.10) by $(Z_2^0)^2$ then, using Eq. (3.22), we get

$$\partial \Gamma_i^0 / \partial \ln Q = \sum_{i'} \lambda_{ii'} \Gamma_{i'}^0 . \tag{5.12}$$

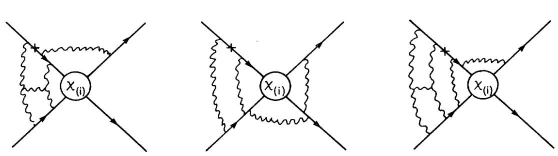

FIG. 22. Some typical contributions to $\rho_{(i)}$.

Solving these equations, we may express $\lambda_{ii'}$'s in terms of Γ_i^0's and $\partial\Gamma_i^0/\partial \ln Q$'s. Now, Γ_i^0's and also $\partial\Gamma_i^0/\partial \ln Q$'s are independent of μ, when expressed in terms of the bare parameters of the theory. Hence, $\lambda_{ii'}$'s must also be independent of μ if expressed in terms of the bare parameters of the theory. This leads to the renormalization-group equation for the λ's.

$$\left[\beta(g)\frac{\partial}{\partial g} + \mu \frac{\partial}{\partial \mu} \right] \lambda_{ii'}(Q/\mu,g) = 0 , \tag{5.13}$$

The solution of (5.13) is

$$\lambda_{ii'}(Q/\mu,g) = \lambda_{ii'}(1,\bar{g}(Q)) , \tag{5.14}$$

$\bar{g}$ being the running coupling constant. Thus, the solution of (5.10) is

$$\Gamma_i = \sum_{i'=1}^{4} \left[P \exp \left[\int_\mu^Q \lambda(1,\bar{g}(Q'))d \ln Q' \right] \right]_{ii'} \times A_{i'}(m,\mu,R,g) , \tag{5.15}$$

where P is the path ordering, which orders the terms in the expansion of the exponential, from right to left, in the order of increasing Q'. $A_{i'}$'s are constants, independent of Q.

In writing Eq. (5.13), we have set the quark mass to be zero in λ. A careful reader may object to this, since in (5.15) the Q' integral runs from μ to Q, and it may not be legitimate to set $m=0$ in λ near the lower limit of the integral ($Q'\sim\mu$). To see the effect of a finite quark mass m, let us note that $\lambda(Q',\mu,m)$ may be expressed as a sum of $\lambda(Q',\mu,0)$ and a function $(m/Q')f(Q',\mu,m)$, where the function f is at most logarithmically divergent in the $Q'\to\infty$ limit. The m/Q' factor in front of f reflects the fact that all the internal lines of λ carry a momentum of order Q', and hence any m dependence must be suppressed by a power of m/Q'. If we add the term $(m/Q')f(Q',\mu,m)$ to λ in (5.15), the effect of this term is to give a constant multiplicative factor that can be absorbed into $A_{i'}$. This shows that Eq. (5.15) is valid so long as we ignore terms of order m/Q.

λ has a perturbation expansion starting at g^2. Thus,

$$\lambda_{ii'}(1,\bar{g}(Q')) = \lambda_{ii'}^{(0)}[\bar{g}(Q')]^2 + O([\bar{g}(Q')]^4) . \tag{5.16}$$

In non-Abelian gauge theories,

$$\bar{g}^2(Q') = 16\pi^2/(\beta_0 \ln Q'^2/\Lambda^2) \tag{5.17}$$

in the $Q'\to\infty$ limit. Here β_0 is a constant related to the group structure. Thus,

$$\Gamma_i \simeq \exp\left[\frac{8\pi^2\lambda^{(0)}}{\beta_0}\left[\ln\ln\frac{Q}{\Lambda}-\ln\ln\frac{\mu}{\Lambda}\right]\right]_{ii'}$$

$$\times A_{i'}(m,\mu,R,g) \; . \tag{5.18}$$

Systematic corrections to (5.18) may be made by including higher-order terms in λ in Eq. (5.15) and higher-order corrections in the expression for $\bar{g}^2(Q)$. For the SU(3) group,

$$\beta_0 = 11 - 2n_f/3 \; , \tag{5.19}$$

where n_f is the number of flavors.

VI. ASYMPTOTIC BEHAVIOR OF THE FULL AMPLITUDE

In Sec. V, we found the asymptotic behavior of the Γ_i's. In this section, we shall show how to find out the asymptotic behavior of the Z_2's, using the results of Ref. 1. Combining these two results, we may find the asymptotic behavior of the full amplitude.

In Ref. 1, we showed that for an on-shell quark, moving along the $+Z$ direction with large momentum p, we have

$$[Z_2(p)]^{1/2} = B(m,\mu,R,g)\exp\left[\int_\mu^{2p^0}\frac{dx}{x}\left[-\int_\mu^x\frac{dy}{y}\gamma_1(\bar{g}(y))+f_1(\bar{g}(x))+f_2(m,\mu,R,g)\right]\right] . \tag{6.1}$$

In writing the above equation we have taken the gauge-fixing vector n to be a fixed vector and hence omitted the dependence of the functions B, γ_1, f_1, and f_2 on n. The above equation is not in a Lorentz-covariant form, it is valid only for particles moving along the $+Z$ direction. In order to find $Z_2(p')$ for any large on-shell momentum p', we note that it may be expressed as a function of m,μ,R,g and $|n\cdot p'|$, due to Lorentz covariance and the invariance of the theory under the transformation $n\to -n$. If p is an on-shell momentum lying along the $+ve$ Z axis, and satisfies the condition

$$|n\cdot p| = |n\cdot p'| \; , \tag{6.2}$$

then $Z_2(p')$ will be identical to $Z_2(p)$. Let us define

$$\eta = 2p^0/|n\cdot p| = 2/|n^0 - n^3| \; . \tag{6.3}$$

Then,

$$[Z_2(p')]^{1/2} = [Z_2(p)]^{1/2}$$

$$= B(m,\mu,R,g)\exp\left[\int_\mu^{\eta|n\cdot p'|}\frac{dx}{x}\left[-\int_\mu^x\frac{dy}{y}\gamma_1(\bar{g}(y))+f_1(\bar{g}(x))+f_2(m,\mu,R,g)\right]\right] . \tag{6.4}$$

The reader may wonder how $Z_2(p')$ can depend on the quantity $\eta = 2/|n^0-n^3|$. Since n and p' are the only two vectors available to us and the gluon propagator is invariant under the scaling of n, the final result for $Z_2(p')$ should depend only on the quantity $n\cdot p/(-n^2)^{1/2}$, and the various mass parameters of the theory, but not on (n^0-n^3). The solution to this puzzle lies in the fact that the way the functions γ_1, f_1, f_2 were defined in Ref. 1 is not Lorentz invariant. Hence, these functions may also have explicit dependence on various components of n. [This is the reason why, in the right-hand side of Eq. (6.1), the upper limit of the x integration is p^0, instead of $n\cdot p/(-n^2)^{1/2}$.] In the final answer however, all the spurious n dependence must cancel among themselves and $Z_2(p')$ will be a function of $n\cdot p'/(-n^2)^{1/2}$ only. Since we are interested only in the Q dependence of the amplitude, the spurious n dependence of the various components does not bother us.

Equation (6.4) gives the asymptotic expression for $Z_2(p')$ for a general p'. Note that, if we choose $n^0=0$, $Z_2^{1/2}(p_a)=Z_2^{1/2}(p_b)$, as was the case in Ref. 1 and $Z_2^{1/2}(p_c)=Z_2^{1/2}(p_d)$. The full asymptotic expression may be obtained by combining Eqs. (5.15) and (6.4):

$$\mathscr{A} = C(m,\mu,R,g)\left\{\prod_{j=a,b,c,d}\exp\left[\int_\mu^{\eta|n\cdot p_j|}\frac{dx}{x}\left[-\int_\mu^x\frac{dy}{y}\gamma_1(\bar{g}(y))+f_1(\bar{g}(x))+f_2(m,\mu,R,g\right]\right]\right\}$$

$$\times\sum_{i,i'=1}^{4}\left[P\exp\left[\int_\mu^Q\lambda(1,\bar{g}(Q'),\theta)d\ln Q'\right]\right]_{ii'}A_{i'}(m,\mu,R,g,a,b,c,d,s_a,s_b,s_c,s_d,\theta) \; . \tag{6.5}$$

In the above equation, we have explicitly shown the dependence on all the external variables, except n. The functions γ_1, f_1, and f_2 may be calculated using the prescription of Ref. 1. The 4×4 matrix λ may be calculated using the prescription of Sec. V. C and $A_{i'}$'s are unknown constants, independent of s. In the $Q\to\infty$ limit, the γ_1 term is the most dominant term in the exponential. If the $O(\bar{g}^2)$ term in the expansion of γ_1 is $C_1\bar{g}^2$, the term gives a contribution

$$\exp\left[-4C_1\int_\mu^Q\frac{dx}{x}\int_\mu^x\frac{dy}{y}\bar{g}^2(y)\right] . \tag{6.6}$$

From Ref. 1 we know that C_1 is $C_F/4\pi^2$, C_F being the eigenvalue of the quadratic Casimir operator in the fermion representation. Using the expression (5.17) for $\bar{g}^2(Q)$, we can write (6.6) as

$$\exp\left[-\frac{8C_F}{\beta_0}\left[\ln\frac{Q}{\Lambda}\ln\ln\frac{Q}{\Lambda}-\ln\frac{Q}{\Lambda}\ln\ln\frac{\mu}{\Lambda}-\ln\frac{Q}{\mu}\right]\right]. \tag{6.7}$$

Systematic corrections to (6.7) may be made by using the full Eq. (6.5) and including higher-order corrections in the expression for $\bar{g}^2(Q)$. The functions γ_1, f_1, and λ have perturbation expansions in $\bar{g}$, which can be calculated up to any order. The functions C, f_2, and A_i's are infrared divergent and hence cannot be calculated in perturbation theory. These functions are, however, independent of Q. We may take them as unknown constants in calculating the Q dependence of the amplitude from Eq. (6.7).

At the tree level, the amplitude for $qq\to qq$ is proportional to g^2. When we take asymptotic freedom into account, we may expect this factor of g^2 to be replaced by $\bar{g}^2(Q)$ and produce an explicit factor of $1/\ln(Q/\Lambda)$, multiplying the full amplitude. The reader may wonder what has happened to this factor in our expression (6.5). In our formalism, this factor is included in the matrix λ. If we look at expression (5.18), we see that an additive factor of -1 in the matrix $8\pi^2\lambda^{(0)}/\beta_0$ will produce a multiplicative factor of $1/\ln(Q/\Lambda)$ in the amplitude. This is how the effect of the $\bar{g}^2(Q)$ term is hidden in λ.

The phase of the amplitude comes solely from the $\sum_{i=1}^4\Gamma_i$ term. The $Z_2(p)$ factors cannot have any imaginary part, since they involve a single on-shell incoming and outgoing quark line, which cannot give rise to any intermediate state with on-shell particles. From (5.18), we see that the leading contribution to Γ_i comes from the eigenvalue of $\lambda^{(0)}$ with largest real part. If λ_I be the imaginary part of this eigenvalue of $\lambda^{(0)}$, then the phase goes as

$$(8\pi^2\lambda_I/\beta_0)\ln\ln Q/\Lambda\,. \tag{6.8}$$

Thus, the phase of the amplitude is determined by the 4×4 matrix λ, which is free from infrared singularities and hence may be calculated perturbatively in QCD. (This is true for the phase of the Sudakov form factor also, where λ is a number, rather than a matrix.) This is an important result, since this shows that the phase of the hard scattering processes may provide an important test of QCD.

We should remember that (6.5) represents the asymptotic behavior of the sum of only those graphs where the c line is the continuation of the a line and the d line is the continuation of the b line. The sum of the other set of graphs, where the c line is the continuation of the b line and the d line is the continuation of the a line, may be obtained from (6.5) by interchanging the color and the helicity quantum numbers of the lines c and d and the momenta p_c and p_d.

VII. CONCLUSION

In this paper we have found a systematic way of calculating the asymptotic behavior of the scattering amplitudes of on-shell quarks (antiquarks) in the $s\to\infty$, t/s-fixed limit. The method can also be applied to analyze amplitudes with more than four external on-shell quarks (antiquarks). The leading asymptotic behavior comes from the self-energy insertions on the external lines. This is given by the renormalization-group modified formula of Cornwall and Tiktopoulos[9]:

$$\exp\left[-\frac{1}{32\pi^2}\left[\sum C_i\right]8\int_\mu^Q\frac{dx}{x}\int_\mu^x\frac{dy}{y}[\bar{g}(y)]^2\right], \tag{7.1}$$

where C_i is the eigenvalue of the quadratic Casimir operator in the representation to which the ith external particle belongs and Q is some energy of order $(p_i\cdot p_j)^{1/2}$. Systematic corrections to (7.1) for the $qq\to qq$ amplitude may be made by using the full expression (6.5) and adding to it the term with s_c,s_d interchanged, c,d interchanged, and p_c,p_d interchanged. For $q\bar{q}\to q\bar{q}$ amplitude we get a similar form as (6.5), with different functions $\lambda_{ii'}$ and $A_{i'}$. For amplitudes involving more external quarks, we again get a similar form as (6.5), except that here the dimensionality of the matrix λ and the vector A is larger than four, being equal to the number of independent tensor structures in the amplitude.

The result derived in this paper supports Mueller's conjecture[6] on the asymptotic behavior of the wide-angle elastic scattering amplitudes of hadrons. For his result, Mueller used a form like (6.7) for the $q\bar{q}\to q\bar{q}$ amplitude. In his calculation, the color-singlet property of the external hadrons automatically provided an infrared cutoff $\sqrt{Xs}$, where $X\sim m^2/s$ corresponds to the Landshoff pinch point and $X\sim 1$ corresponds to the hard scattering region, where the quark-counting rule is valid. As mentioned in the Introduction, it is a plausible conjecture that the off-shell regularization effectively reduces to an on-shell one, when we sum over a set of graphs, and use the fact that the hadrons are color singlets. In our result (6.5), if we set the infrared regulator R to be $\sqrt{Xs}$ and also $\mu=\sqrt{Xs}$, so as to aviod logarithms of $\mu/\sqrt{Xs}$, the asymptotic expression (6.7) becomes

$$\exp\left[-\frac{8C_F}{\beta_0}\left[\ln\frac{Q}{\Lambda}\ln\ln\frac{Q}{\Lambda}-\ln\frac{Q}{\Lambda}\ln\ln\frac{\sqrt{Xs}}{\Lambda}-\ln\frac{1}{\sqrt{X}}\right]\right], \tag{7.2}$$

which is exactly the form assumed by Mueller.

On the basis of this equation, Mueller showed that the leading contribution to the wide-angle elastic $\pi\pi$ scattering amplitude comes from a region $Xs\sim s^{2C/(2C+1)}$ where $C=8C_F/\beta_0$, which gives a factor of $s^{1/2-C\ln[(2C+1)/2C]}$ multiplying the quark-counting-rule prediction for the amplitude. Thus, our result supports Mueller's conjecture. We hope that the technique used in this paper may be applied directly to the analysis of hadron-hadron elastic scattering amplitude and will enable us to make systematic corrections to Mueller's result.

Pire and Ralston[7] have suggested that the phase of the $q\bar{q}\to q\bar{q}$ and $qq\to qq$ amplitudes may be responsible for the small oscillation of the experimental data for hadron-hadron elastic scattering cross section about the quark-counting-rule prediction, as was noted by Brodsky and Lepage.[8] This is achieved by considering the interference between Mueller's result and the quark-counting result. We have seen that the phase of the amplitude is free from infrared divergences. Hence, it is calculable perturbatively and is proportional to $\ln\ln Q/\Lambda$ thus confirming Pire and Ralston's assumption that the scale of the Q dependence of the phase is set by the QCD scale parameter Λ. We hope that the analysis of the full hadron-hadron scattering amplitude, using the method used here, will also provide us with a quantitative result for the oscillation of the scattering cross section.

Note added in proof. After finishing this work, we learned about a paper by Cheng *et al.*,[12] which deals with the similar problem in Abelian gauge theories.

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