Rotation generators in two-dimensional space and particles obeying unusual statistics

Gerald A. Goldin
*Department of Physics, Princeton University, Princeton, New Jersey 08544**

David H. Sharp
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
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We describe systems of particles obeying unusual statistics in two-dimensional space, as well as solenoid—charged-particle composites, in terms of a complete set of local gauge-invariant currents.

I. INTRODUCTION

The unusual statistics which could be obeyed by particles moving in two-dimensional space, or by idealized solenoid—charged-particle composites, have been discussed recently by Wilczek,[1] Goldhaber,[2] Kobe,[3] Lipkin and Peshkin,[4] and Jackiw and Redlich,[5] raising questions concerning the correct choice of rotation generator.[6] The statistics of identical particles in two dimensions were discussed earlier by Leinaas and Myrheim.[7] Here we show how the description of these systems previously obtained using local currents[8] allows a rigorous, clear, and systematic treatment of the questions raised. We conclude in agreement with Refs. 1, 2, and 7 that in the strictly two-dimensional case the eigenvalues of the rotation generator can be shifted from integer multiples of $\hbar$ by an arbitrary fixed constant, and a corresponding one-parameter family of possible statistics interpolates between Bose and Fermi systems. We describe a simple model in which the exchange of particles in two dimensions is implemented exclusively by means of observable rotations in bounded regions, constructed from local currents. For solenoid—charged-particle composites, the unusual statistics occur only in the idealization of actually infinite solenoids with no return flux, and cannot be obtained in the limit of long, finite solenoids, as has been remarked in Refs. 2, 4, and 5. In contrast to Ref. 5, our description is in terms of a complete set of manifestly gauge-invariant quantities.

II. THE LIE ALGEBRA OF LOCAL CURRENTS AND THE PHYSICAL ANGULAR MOMENTUM OPERATOR

The fundamental objects in our approach are the operators

$$\rho(f)=\int \rho(\vec{x})f(\vec{x})d\vec{x}$$

and

$$J(\vec{g})=\int \vec{J}(\vec{x})\cdot\vec{g}(\vec{x})d\vec{x}\ ,$$

where f and the components of $\vec{g}$ are smooth test functions which vanish at infinity. The mass density $\rho(\vec{x})$ and the kinetic momentum density $\vec{J}(\vec{x})$ can be written in terms of a nonrelativistic second-quantized field $\psi(\vec{x})$ and its adjoint $\psi^*(\vec{x})$ as

$$\rho(\vec{x})=m\psi^*(\vec{x})\psi(\vec{x})\ , \tag{1}$$

$$\vec{J}(\vec{x})=\tfrac{1}{2}\psi^*(\vec{x})\{[(\hbar/i)\nabla-(e/c)\vec{A}(\vec{x})]\psi(\vec{x})\} + \text{Hermitian conjugate}\ , \tag{2}$$

where $\vec{A}(\vec{x})$ is the vector potential. Thus $\rho(f)$ is the mass density operator averaged in space by $f(\vec{x})$, and $J(\vec{g})$ is the momentum density averaged by $\vec{g}(\vec{x})$. The operators $\rho(f)$ and $J(\vec{g})$ are gauge invariant and satisfy the commutation relations

$$[\rho(f_1),\rho(f_2)]=0\ , \tag{3}$$

$$[\rho(f),J(\vec{g})]=i\hbar\rho(\vec{g}\cdot\nabla f)\ , \tag{4}$$

$$[J(\vec{g}_1),J(\vec{g}_2)]=i\hbar J([\vec{g}_1,\vec{g}_2]) + i(\hbar e/mc)\rho(\vec{B}\cdot(\vec{g}_1\times\vec{g}_2))\ , \tag{5}$$

where $[\vec{g}_1,\vec{g}_2]=\vec{g}_2\cdot\nabla\vec{g}_1-\vec{g}_1\cdot\nabla\vec{g}_2$, and the external magnetic field $\vec{B}=\nabla\times\vec{A}$ has not been quantized.[9] Here $[\vec{g}_1,\vec{g}_2]$ is the Lie bracket of the vector fields, allowing the geometrically meaningful interpretation of the operators $J(\vec{g})$ as infinitesimal generators of diffeomorphisms. For test functions restricted to be nonzero only in the field-free region, the second term on the right-hand side of Eq. (5) vanishes, leaving

$$[J(\vec{g}_1),J(\vec{g}_2)]=i\hbar J([\vec{g}_1,\vec{g}_2])\ . \tag{6}$$

Equations (3), (4), and (6) are taken as the basis for our discussion. It is remarkable that the same infinite-dimensional Lie algebra arises whether one starts with underlying fields satisfying canonical commutation relations or canonical anticommutation relations. A physical system corresponds to a *representation* of the Lie algebra (3), (4), and (6) by self-adjoint operators in Hilbert space; thus for any $\vec{g}$, $J(\vec{g})$ describes an observable. Alternatively, a physical system can be described by a unitary representation of the associated infinite-parameter Lie group.[10] Two such representations describe the same physics if they are unitarily equivalent. Distinguishable as well as indistinguishable particles can be described by means of the rich class of inequivalent representations existing for the Lie algebra of local currents.

In such a representation, the operator for physical angular momentum (also called orbital or kinetic angular momentum) about the z axis is obtained from $J(\vec{g})$. Let the vector field $\vec{q}$ be given by $q_1(\vec{x})=-x_2$, $q_2(\vec{x})=x_1$,

$q_3(\vec{x})=0$. Since $\vec{g}$ must vanish at infinity while $\vec{q}$ does not, let $\vec{g}(\vec{x})=\vec{q}(\vec{x})$ in a large compact region $|\vec{x}|\leq R$ and let $\vec{g}(\vec{x})$ fall smoothly to zero for $|\vec{x}|>R$. With this choice, $J(\vec{g})$ in units of angular momentum is approximately

$$\int [\vec{x}\times\vec{J}(\vec{x})]\cdot\hat{x}_3 d\vec{x} \ .$$

The possibility exists of a shift in the spectrum of the physical angular momentum operator from integer multiples of $\hbar$ by $\lambda\hbar/2\pi$, even for *distinguishable* particles moving in two-dimensional space or for distinguishable particle-solenoid composites moving in R^3, because in these cases the configuration spaces are not simply connected, and there are consequently unitarily inequivalent representations of the Lie algebra corresponding to distinct values of λ. It is not necessary to exclude the particles from occupying the same point by means of repulsive δ-function interactions in order for the unusual representations to occur. Still other inequivalent representations exist describing identical particles, in which the physical angular momentum spectrum is shifted from even-integer multiples of $\hbar$ by $\lambda\hbar/\pi$. The case $\lambda=0$ describes bosons, $\lambda=\pi$ describes fermions, and other values of λ in the interval $[0,2\pi)$ describe particles obeying unusual statistics.[8] In the case of particle-solenoid composites, λ is proportional to the magnetic flux.

The nontrivial connectedness of a configuration space can be characterized by means of its fundamental group. In three or more dimensions, the fundamental group for configurations of N identical particles is isomorphic to the symmetric group S_N, whose one-dimensional representations can lead only to Bose or Fermi representations of the local current algebra.[11] In two dimensions, however, the fundamental group for configurations of N identical particles is isomorphic to the braid group of order N,[12] whose one-dimensional representations include those leading to the unusual statistics discussed here.

It is especially noteworthy that *all* of these systems—N distinguishable particles (with or without a shift in the angular momentum spectrum), N indistinguishable fermions, bosons, or "anyons," for arbitrary N, as well as infinite gases as $N\rightarrow\infty$ in the thermodynamic limit—arise as inequivalent representations of the *same* manifestly gauge-invariant Lie algebra (3), (4), and (6) of local observables.[8,13]

III. THE ROTATION GENERATOR IN TWO-DIMENSIONAL SPACE

Next we examine the correct choice of rotation generator in such a representation. For particles moving in a strictly two-dimensional region, the following model illustrates why it is not necessary to require a rotation generator (canonical angular momentum) distinct from the orbital angular momentum obtained from $J(\vec{g})$ as above. Consider a circular wire hoop of radius D ($D\rightarrow\infty$) with a thin liquid film suspended across the hoop, and let there be N first-quantized pointlike vortices in the liquid. At the boundary $|\vec{x}|=D$, require the film to be stationary. When the $\vec{x}$ coordinates of two vortices are exchanged by letting them "orbit" each other, we have *locally* generated a rotation using the two-dimensional orbital angular momentum operator $J(\vec{g})$, and we can have a phase shift in the probability amplitude by other than 0 or π. The boundary condition at the hoop restricts the test function $\vec{g}$ to become zero as $|\vec{x}|\rightarrow D$, effectively "tying down" the local rotation at the boundary. Thus, we can "keep track" of how many times the disturbances have circled each other, *and this information becomes part of the description of a "particle configuration."* When we let $D\rightarrow\infty$, set $\vec{g}(\vec{x})=\vec{q}(\vec{x})$ for $|\vec{x}|\leq R$, and permit $R\rightarrow\infty$, the operator $J(\vec{g})$ becomes the generator of a *projective* representation (i.e., a multivalued or ray representation) of the two-dimensional rotation group. Physically, a ray representation is all that is needed to ensure that the outcomes of measurements are rotationally invariant. It is a representation of the covering group of the two-dimensional rotation group.[14] When $N=2$, one might seek to argue that a "true" rotation exchanging $\vec{x}$ coordinates means physically turning the *whole* hoop, wire and all (or, equivalently, rotating the liquid while abandoning the boundary condition at $|\vec{x}|=D$), thus ruling out the unusual statistics. But, any quantum-mechanical measurement which we actually perform must be made in a *bounded* region of space, and cannot entail a truly global rotation such as that described. Furthermore, when $N\geq 3$, one can keep track of the number of times a vortex passes *between* two others when a *local* rotation exchanges $\vec{x}$ coordinates; then unusual (non-Bose and non-Fermi) representations still occur since particle configurations require parameters beyond the position coordinates to characterize them uniquely.

Thus particles such as point vortices in two-dimensional space are completely described by means of local observables, and there is no *a priori* necessity either mathematically or physically for introducing a canonical angular momentum operator distinct from the rotation generator discussed above.

It is interesting that for *fixed, arbitrary* λ, the two-dimensional orbital angular momentum operator can be represented by the differential operator $(\hbar/i)\partial/\partial\theta$ on a domain *(depending on* λ) in the space of square-integrable wave functions of $\vec{x}$, where $\vec{x}=(r,\theta)$ is the relative coordinate between two (distinguishable) particles.[15] Recall that an unbounded linear operator A on a Hilbert space $\mathscr{H}$ is in general defined only on a dense domain $D_A\subset\mathscr{H}$. For A to be self-adjoint, one needs both $D_A=D_{A^*}$ and $A=A^*$ on D_A. Choose an arbitrary axis along which $\theta=0$, and define L to be $(\hbar/i)\partial/\partial\theta$ on the largest possible domain of functions which vanish on this axis. Thus defined, L is *not* self-adjoint, but it has a *one-parameter family of self-adjoint extensions* L_λ to larger domains D_λ. A wave function in D_λ has a phase shift $e^{i\lambda}$ at $\theta=0$. Different self-adjoint extensions L_λ arise from the self-adjoint operators $J(\vec{g})$ in inequivalent representations. The apparent arbitrariness of the axis where $\theta=0$ can be eliminated by writing a unitarily *equivalent* representation on the Hilbert space of wave functions on the covering space of two-particle configurations ("multivalued" wave functions).[16] In all such representations the orbital angular momentum $(\hbar/i)\partial/\partial\theta$ has a discrete spectrum and *is* therefore quantized. Here we differ from Refs. 2 and 5, at least in terminology. To say orbital angular momentum is not quantized would be to say that as a self-adjoint operator it has a continuous spectrum, which is not true here. For particle-solenoid composites, a change in flux represents the orbital angular momentum by a *different* self-adjoint operator, changing not merely its eigenfunctions, but its whole domain of definition.

We also note that while the spectrum of L_λ is shifted by $(\hbar\lambda/2\pi)$ from the spectrum of $L_{\lambda=0}$, the operator L_λ does not equal the operator $(\hbar\lambda/2\pi)I+L_{\lambda=0}$ since they are defined on different domains. However, here the two operators *are* related by a unitary transformation of the representation of the (gauge-invariant) current algebra. This is the transformation Q in Eq. (3.7) of Ref. 8, also discussed by Kobe.[3]

Finally, we turn to the question of the correct rotation generator for two impenetrable solenoid—charged-particle composites. Wilczek exchanges such composites by means of a rotation, and unusual statistics occur when the orbital angular momentum operator L_λ is taken as the infinitesimal generator of the rotation.[1] At issue, then, is whether the rotation is achieved "orbitally" or "canonically." Demanding that the orbital angular momentum operator generate a single-valued representation of the two-dimensional rotation group would quantize the flux in the solenoids so as to eliminate the unusual statistics, but such a condition is not dictated by the physics. However, other authors have noted that when the infinite solenoids are considered as limits of long, finite solenoids, it is necessary to include the angular momentum in the crossed fields,

$$\vec{F}=(1/4\pi c)\int \vec{x}\times[\vec{E}(\vec{x})\times\vec{B}(\vec{x})]d\vec{x}$$

and to take account of the return flux.[2,4] Then the averaged kinetic momentum density operators $J(\vec{g})$ satisfy Eq. (5) rather than Eq. (6) when $\vec{g}$ is nonvanishing in the distant region of return flux. In a fully quantized theory, it is interesting that it is the total momentum density, defined by

$$\vec{P}(\vec{x})=\vec{J}(\vec{x})+(1/8\pi c)[\vec{E}(\vec{x})\times\vec{B}(\vec{x})-\vec{B}(\vec{x})\times\vec{E}(\vec{x})]\ , \tag{7}$$

which satisfies the Lie algebra of Eq. (6) when averaged with test functions.[9] For finite solenoids of arbitrary length, the total angular momentum obtained from $\vec{P}(\vec{x})$ is quantized in the usual integer multiples of $\hbar$, with the shift in orbital angular momentum exactly canceled by the angular momentum stored in the distant crossed fields. Representations in which the spectrum of total angular momentum is shifted can also be ruled out by a time-dependent analysis.[5] Like $\vec{J}(\vec{x})$, $\vec{P}(\vec{x})$ is manifestly gauge invariant. Lipkin and Peshkin point out that in the gauge $\nabla\cdot\vec{A}=0$ and $\vec{A}(\infty)=0$, the angular momentum in the crossed fields equals the additional angular momentum introduced by $\vec{A}$ when (following Ref. 5) we write the canonical angular momentum operator as $\vec{M}=\vec{L}+(e/c)\vec{x}\times\vec{A}$, $\vec{L}$ being the orbital angular momentum. The latter expression is, of course, gauge dependent.

The above considerations (return flux, or a condition at $t=-\infty$ in a time-dependent analysis) rule out the possibility that long particle-solenoid composites prepared in the laboratory, which are necessarily finite and whose history we know, obey unusual statistics. However, the unusual representations of the gauge-invariant Lie algebra exist. Thus we cannot rule out the unlikely possibility that actually infinite flux-tube—particle composites of cosmological origin will be discovered floating in space, with no return flux and undiscoverable history, obeying the unusual statistics suggested by Wilczek.

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*Visiting Fellow, 1982-83. Permanent address: Department of Mathematical Sciences, Northern Illinois University, De Kalb, Illinois 60115.

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