Supersymmetry on a lattice

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We develop Lagrangian and Hamiltonian lattice formulations of the Wess-Zumino model which preserve the superalgebra. We check that the lattice model passes into the continuum model in the zero-lattice-spacing limit, and obtain various perturbative results (nonrenormalization, effective-potential loop expansion) on the lattice. We study the phase structure with the one-loop effective potential and find dynamical symmetry breaking at large coupling in two dimensions, but not in four dimensions. Separately, we evaluate the strong-coupling limit in the Hamiltonian formulation and confirm the accuracy of the one-loop effective-potential predictions. Our methods should apply to certain other supersymmetric theories.

I. INTRODUCTION

Supersymmetric field theories[1,2] have become of interest to particle physicists for essentially two reasons. When attempting to go beyond the standard $SU(3)\times SU(2)\times U(1)$ theory, it is hoped that (global) supersymmetry might help to understand the vast difference between the scale of weak interactions and the Planck scale.[3] Furthermore (local) supersymmetry is the only known framework which allows one to combine gravity with the standard theory of strong and electroweak interactions.[4]

If supersymmetry applies to low-energy particle physics, it cannot be exact but must be broken. As a consequence of nonrenormalization theorems,[5] breaking of supersymmetry either has to be put in by hand (by choosing appropriate Higgs parameters), or it must come from nonperturbative effects. The second possibility certainly is more attractive, but not yet understood well enough. For two-dimensional models there is now strong evidence that such a dynamical breakdown is possible.[6,7] In four dimensions, however, the situation is less clear. Witten's index theorem[8] states that for a large class of models supersymmetry will not be broken by any nonperturbative effects. But there are still important cases left which are not covered by this argument, in particular, models with zero-mass fermions.[9] It is therefore important to investigate further the dynamics of supersymmetric theories beyond perturbation theory.

In recent years it has proven very useful to formulate quantum field theory on the lattice. Powerful techniques, such as the strong-coupling expansion, mean-field theory, or Monte Carlo computer simulations can then be used to obtain both quantitative and qualitative insights into the structure of the theory. Several attempts have also been made to put supersymmetry onto a lattice, but so far none of these has reached the stage where one could start to systematically investigate realistic supersymmetric models.

Difficulties arise from the fact that the (continuum) supersymmetry algebra contains the generators of translation and Lorentz rotations, e.g.,

$$\{Q_\alpha,\bar{Q}_\beta\}=-2\gamma^\mu_{\alpha\beta}P_\mu \ , \tag{1.1}$$

and such infinitesimal transformations are ill-defined on the lattice. Furthermore, it is well known that the proof of invariance under supersymmetry transformation requires the product rule of differentiation (Leibniz's rule), and, again, this rule is generally lost on the lattice.

Nicolai and Dondi[10] were the first to point out that, if one wants to define a supersymmetry algebra on the lattice, one is forced into nonlocal operators. Not only is one led to a version of the derivative operator which has long-range correlations, but also the interaction terms in the lattice action require some sort of nonlocality. Banks and Windey[11] and, later on, Rittenberg and Yankielowicz[12] tried to preserve, if not the full supersymmetry algebra, at least what seems to be the most crucial part of it:

$$H=\tfrac{1}{4}\sum_{\alpha=0}^{3}Q_\alpha{}^2 \ . \tag{1.2}$$

At the same time they attempted to avoid the nonlocal features mentioned before. Unfortunately, however, their lattice version does not fully recover Lorentz invariance, when the lattice spacing is taken to zero. Elitzur, Rabinovici, and Schwimmer[13] continued along these lines and succeeded in finding a class of models where these difficulties do not arise. Their method, however, so far only works for simple models, such as the Wess-Zumino model,[14] in two dimensions. In four dimensions the simplest model would be $N=2$ extended supersymmetry.

Very recently, the problem of putting supersymmetry onto a lattice has been investigated from a somewhat different angle.[15] Following Symanzik's idea of constructing an "improved" lattice action,[16] a lattice version of the Wess-Zumino model has been found which, by construction, agrees with the continuum theory in the limit of vanishing lattice spacing. Here, again, nonlocal operators were avoided. In the resulting lattice action, manifest supersymmetry is completely lost, and no way has been found so far to investigate this model outside of perturbation theory.

In this paper we describe a new attempt to find lattice versions of supersymmetric field theories. As a start we limit ourselves to the Wess-Zumino model, but we hope that in the future we will be able to handle supersymmetric gauge theories as well. Comparing our approach to the previous attempts mentioned before, we describe our strategy as follows. Our first goal is to preserve a manifest invariance under a set of transformations, which satisfy the same algebra as in the continuum case, even if all this leads us to the nonlocal features of Ref. 10. There is an obvious way to define such transformations. We require that the Fourier transform of our lattice version agrees with that of the continuum theory, apart from the constraint that, on the lattice, momenta range from $-\pi/a$ to π/a. In this way we satisfy, in Fourier space, the supersymmetry algebra [e.g., Eq. (1.1)]. Transforming back to configuration space we of course find the SLAC derivative[17] operator with long-range correlations. However, in accordance with Ref. 10 the interaction terms also become nonlocal. Typically we find

$$\int d^4x\, \phi^3(x) \rightarrow a^4 \sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3} V_{\vec{n}_1\vec{n}_2\vec{n}_3}\phi_{\vec{n}_1}\phi_{\vec{n}_2}\phi_{\vec{n}_3}\;, \quad (1.3)$$

where $\vec{n}_i$ denote the lattice points, and $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ will be specified below.

The main achievement of this procedure is that we can define transformations (supersymmetry, translation, Lorentz rotation) which, formally, satisfy the same commutation and anticommutation relations as in the continuum theory, and that we have a lattice action which is manifestly invariant under these transformations. It is, however, clear that what we call "translation" or "rotation" operator cannot be interpreted as an *infinitesimal* translation or rotation operator. Only after exponentiating do we obtain operators which correctly define *finite* translations or rotations. We also can show that our lattice theory has the correct continuum limit, quite in the spirit of Ref. 15.

After having established that our lattice theory has the correct symmetry properties, we than have to find methods of handling the nonlocal interaction terms (1.3). This is the second goal of our paper. We shall demonstrate that it is still possible to apply some of the well-known lattice approximations, in particular, mean-field theory and the strong-coupling expansion. The first approximation will be used in order to analyze the partition function (we actually calculate the effective potential in the one-loop approximation). For the strong-coupling expansion it turns out to be necessary to use the Hamiltonian formulation. We split the nonlocal interaction $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ [Eq. (1.3)] into a local piece and treat the nonlocal part as a perturbation. First, we solve the single-site problem, then study the influence of the nonlocal piece to all orders.

Applying all this to the Wess-Zumino model we find that supersymmetry is unbroken for all values of the coupling constant g (at fixed mass $m\neq 0$). The point $g=\infty$ turns out to be singular: two phases which for small g are very far apart from each other come together at $g=\infty$. We therefore interpret this limit as a phase-transition point. This analysis comes out of the mean-field-theory approximation. The behavior at $g=\infty$ is confirmed by our strong-coupling calculation: we prove that supersymmetry is still unbroken. We also compare these results with the two-dimensional case: there the phase transition from unbroken supersymmetry to broken supersymmetry seems to occur for some finite g, and at $g=\infty$ supersymmetry is broken.[6,7]

We organize our paper into essentially two parts. We first work in the Lagrangian formulation. After defining the action and the supersymmetry transformations on the lattice we make sure that our lattice action is invariant (Sec. II). We then (Sec. III) survey a variety of analyses of our partition function. The most attention is given to the one-loop effective potential, and we study the phase structure in that approximation. We do find, however, that the Lagrangian formulation is poorly adapted to the calculation of the strong-coupling limit. In the second part we therefore turn to the lattice Hamiltonian formulation (Sec. IV), and we use this formulation to calculate the ground-state energy in the strong-coupling limit (Sec. V). In Sec. VI we summarize and comment on our results.

II. THE LATTICE ACTION

We start from the continuum Lagrangian of the Wess-Zumino model[14]:

$$S_{\text{cont}} = \int d^4x(\mathscr{L}_{\text{kin}} + \mathscr{L}_m + \mathscr{L}_g)\;, \quad (2.1)$$

$$\mathscr{L}_{\text{kin}} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \frac{i}{2}\bar{\psi}\gamma\cdot\partial\psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2\;, \quad (2.2)$$

$$\mathscr{L}_m = m\left[FA + GB - \frac{i}{2}\bar{\psi}\psi\right]\;, \quad (2.3)$$

$$\mathscr{L}_g = g(FA^2 - FB^2 + 2GAB - i\bar{\psi}\psi A + i\bar{\psi}\gamma_5\psi B)\;. \quad (2.4)$$

Here ψ ($\bar{\psi}=\psi^T\gamma^0$) denotes the Majorana spinor field, A and B are the scalar and pseudoscalar fields, respectively, and F and G are auxiliary fields which we prefer to keep. We use the metric $(-1,+1,+1+1)$ (except for the following section where we find it more convenient to switch to the Euclidean metric), and our γ matrices are in the Majorana representation. (In Sec. V we will write down the explicit form of the γ_μ.)

Our lattice version of the Wess-Zumino model is obtained most easily if we introduce interpolating fields. They are defined to have the same Fourier transform as the lattice field variables. As an example, we write for the A field:

$$A_{\vec{n}} = \frac{1}{(2\pi)^2}\int_{-\pi/a}^{\pi/a} d^4k\, e^{i\vec{k}\cdot\vec{n}a}\tilde{A}(k)\;, \quad (2.5)$$

$$A(x) = \frac{1}{(2\pi)^2}\int_{-\pi/a}^{\pi/a} d^4k\, e^{i\vec{k}\cdot\vec{x}}\tilde{A}(k)\;. \quad (2.6)$$

$A_{\vec{n}}$ stands for the field A of the lattice point $\vec{n}=(n_0,n_1,n_2,n_3)$; a is the lattice spacing, $\tilde{A}(k)$ denotes the Fourier transform, and $A(x)$ is our interpolating field. By inverting (2.5)

$$\tilde{A}(k) = \frac{a^4}{(2\pi)^2}\sum_{\vec{n}} e^{-i\vec{k}\cdot\vec{n}a}A_{\vec{n}}\;, \quad (2.7)$$

and substituting into (2.6), we find

$$A(x) = \sum_{\vec{n}} K_{\vec{n}}(x)A_{\vec{n}}\;, \quad (2.8)$$

where

$$K_{\vec{n}}(x)=\frac{a^4}{(2\pi)^4}\int_{-\pi/a}^{\pi/a}d^4k\, e^{i\vec{k}\cdot(\vec{x}-\vec{n}a)} . \tag{2.9}$$

One easily sees that $A(x)$ interpolates between the lattice sites, i.e., for $\vec{x}=\vec{m}a$ we have $A(x)=A_{\vec{m}}$. Interpolating fields for the other field variables of the Wess-Zumino model are defined in a similar way. Inserting (2.8) into the action integral (2.1) and carrying out the x integration, we arrive at our lattice action:

$$S_{\text{lattice}}=a^4\sum\Bigg\{\tfrac{1}{2}A_{\vec{n}_1}D_{\vec{n}_1,\vec{n}_2}A_{\vec{n}_2}+\tfrac{1}{2}B_{\vec{n}_1}D_{\vec{n}_1,\vec{n}_2}B_{\vec{n}_2}-\frac{i}{2}\bar{\psi}_{\vec{n}_1}\gamma_\mu\Delta^\mu_{\vec{n}_1,\vec{n}_2}\psi_{\vec{n}_2}$$
$$+\tfrac{1}{2}F_{\vec{n}}{}^2+\tfrac{1}{2}G_{\vec{n}}{}^2+m\left[F_{\vec{n}}A_{\vec{n}}+G_{\vec{n}}B_{\vec{n}}-\frac{i}{2}\bar{\psi}_{\vec{n}}\psi_{\vec{n}}\right]$$
$$+gV_{\vec{n}_1\vec{n}_2\vec{n}_3}[F_{\vec{n}_1}A_{\vec{n}_2}A_{\vec{n}_3}-F_{\vec{n}_1}B_{\vec{n}_2}B_{\vec{n}_3}+2G_{\vec{n}_1}A_{\vec{n}_2}B_{\vec{n}_3}-i\bar{\psi}_{\vec{n}_1}(A_{\vec{n}_2}-\gamma_5B_{\vec{n}_2})\psi_{\vec{n}_3}]\Bigg\} . \tag{2.10}$$

Here the summation extends over all repeated indices, and the derivative matrices Δ^μ, D are defined as

$$\Delta^\mu_{\vec{m}\,\vec{n}}=\frac{a^4}{(2\pi)^4}\int_{-\pi/a}^{\pi/a}d^4k(ik^\mu)e^{i\vec{k}\cdot(\vec{m}-\vec{n})a} , \tag{2.11}$$

$$D_{\vec{m},\vec{n}}=\frac{a^4}{(2\pi)^4}\int_{-\pi/a}^{\pi/a}d^4k(-k^2)e^{i\vec{k}\cdot(\vec{m}-\vec{n})a} . \tag{2.12}$$

Obviously they satisfy the relation

$$\sum_{\vec{n}_1}\Delta^\mu_{\vec{m}\,\vec{n}_1}\Delta_{\mu\,\vec{n}_1\vec{n}}=D_{\vec{m}\,\vec{n}} . \tag{2.13}$$

The vertex $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ stands for

$$V_{\vec{n}_1\vec{n}_2\vec{n}_3}=\frac{a^8}{(2\pi)^8}\int_{-\pi/a}^{\pi/a}d^4k_1d^4k_2d^4k_3\delta(k_1+k_2+k_3)\exp\left[-i\sum_j\vec{k}_j\cdot\vec{n}_ja\right] . \tag{2.14}$$

It is important to note that the δ function in (2.14) is nonperiodic. This vertex $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ satisfies the relation

$$V_{\vec{n}_1\vec{n}_2\vec{m}}\Delta^\mu_{\vec{m}\,\vec{n}_3}+V_{\vec{n}_2\vec{n}_3\vec{m}}\Delta^\mu_{\vec{m}\,\vec{n}_1}+V_{\vec{n}_3\vec{n}_1\vec{m}}\Delta^\mu_{\vec{m}\,\vec{n}_2}=0 , \tag{2.15}$$

which is nothing but the Leibniz rule for differentiating products

$$\Delta^\mu_{\vec{n}_3\vec{m}}V_{\vec{m}\,\vec{n}_1\vec{n}_2}f_{\vec{n}_1}g_{\vec{n}_2}$$
$$=V_{\vec{n}_3\vec{n}_1\vec{n}_2}[(\Delta^\mu_{\vec{n}_1\vec{m}}f_{\vec{m}})g_{\vec{n}_2}+f_{\vec{n}_1}(\Delta^\mu_{\vec{n}_2\vec{m}}g_{\vec{m}})] . \tag{2.16}$$

We will find that maintenance of the Leibniz rule is all that is required to make the lattice action invariant under supersymmetric transformations.

The disadvantage of our lattice action (2.10) lies, of course, in the fact that the interaction term $gV_{\vec{n}_1\vec{n}_2\vec{n}_3}$ has become nonlocal. For large separation of lattice points $\vec{n}_1$ and $\vec{n}_2$ the vertex $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ goes to zero with an inverse power of $|\vec{n}_1-\vec{n}_2|$, and one might fear that this nonlocality presents a severe difficulty. As we will show, however, it is possible to apply standard approximation techniques to this nonlocal type of interaction. The first is the stationary phase approximation (or mean field theory, to be discussed in the following section), and it makes use of the fact that for field configurations which are independent of space and time the nonlocal vertex becomes simple. The relevant property of $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ reads

$$\sum_{\vec{n}_1}V_{\vec{n}_1\vec{n}_2\vec{n}_3}=\delta_{\vec{n}_2\vec{n}_3} . \tag{2.17}$$

The second approximation scheme is the strong-coupling expansion (to be discussed in Sec. V, within the Hamiltonian formalism), and the basic idea is to treat the off-diagonal part of $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ as a perturbation. Because of the rather slow fall-off of $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ as a function of the separation of lattice points it is not obvious at all why such a treatment should work. As we will see, it is the supersymmetric structure of our model which allows us to control this perturbation expansion to all orders.

After having discussed the unpleasant features of our lattice action (2.10), we now turn to the advantageous ones. We first define the following supersymmetry transformations[1]:

$$\delta A_{\vec{n}}=i\bar{\eta}\psi_{\vec{n}} ,$$
$$\delta B_{\vec{n}}=i\bar{\eta}\gamma_5\psi_{\vec{n}} ,$$
$$\delta F_{\vec{n}}=\Delta^\mu_{\vec{n}\,\vec{m}}i\bar{\eta}\gamma_\mu\psi_{\vec{m}} , \tag{2.18}$$
$$\delta G_{\vec{n}}=\Delta^\mu_{\vec{n}\,\vec{m}}i\bar{\eta}\gamma_5\gamma_\mu\psi_{\vec{m}} ,$$
$$\delta\psi_{\vec{n}}=\Delta^\mu_{\vec{n},\vec{m}}(A_{\vec{m}}-\gamma_5B_{\vec{m}})\gamma_\mu\eta+(F_{\vec{n}}+\gamma_5G_{\vec{n}})\eta$$

(here $\eta,\bar{\eta}=\eta^T\gamma^0$ denote anticommuting Grassmann variables). Using the Leibniz rule (2.15), a straightforward (though lengthy) calculation shows that our lattice action

(2.10) is invariant under these transformations. As in the continuum theory, one finds that each of the three pieces of the action is separately invariant: the kinetic part, the m part, and the interaction part. Furthermore, we define generators of the Poincaré's group. For the translation operators we define

$$P_\mu \colon (\Delta_\mu)_{\vec{m}\,\vec{n}}\ , \tag{2.19}$$

and for homogeneous Lorentz transformations

$$M_{\mu\nu}\colon (M_{\mu\nu})_{\vec{m}\,\vec{n}} = V_{\vec{m}\,\vec{n}_1\vec{n}_2}(a n_{1\mu}\Delta_{\nu\vec{n}_2\vec{n}} - a n_{1\nu}\Delta_{\mu\vec{n}_2\vec{n}})\ . \tag{2.20}$$

Again the Leibniz rule (2.15) is crucial in proving the correct commutation relations

$$[P_\mu, P_\nu] = 0\ , \tag{2.21}$$

$$[P_\mu, M_{\lambda\nu}] = g_{\mu\lambda}P_\nu - g_{\mu\nu}P_\lambda\ , \tag{2.22}$$

$$[M_{\mu\nu}, M_{\lambda\sigma}] = g_{\mu\lambda}M_{\sigma\nu} + g_{\mu\sigma}M_{\nu\lambda} + g_{\nu\lambda}M_{\mu\sigma} + g_{\nu\sigma}M_{\lambda\mu}\ . \tag{2.23}$$

Combining the supersymmetry transformations (2.18) with the Poincaré generators $P_\mu, M_{\mu\nu}$, one also verifies the superalgebra part

$$\{Q_\alpha, \bar{Q}_\beta\} = -2\gamma^\mu_{\alpha\beta}P_\mu\ , \tag{2.24}$$

$$[P_\mu, Q_\alpha] = 0\ , \tag{2.25}$$

$$[Q_\alpha, M^{\mu\nu}] = i\sigma^{\mu\nu}_{\alpha\beta}Q_\beta\ , \tag{2.26}$$

where $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ and Q is defined by (2.18). We therefore can conclude that our lattice action (2.10) is manifestly supersymmetric. A lattice version of the Wess-Zumino model can thus be defined through the following partition function:

$$Z = \int \prod_{\vec{n}} \frac{dA_{\vec{n}}\,dB_{\vec{n}}\,dF_{\vec{n}}\,dG_{\vec{n}}\,d\psi_{\vec{n}}}{(2\pi)^2 i} e^{iS_{\text{lattice}}}\ . \tag{2.27}$$

We conclude this section by mentioning that our definition of P_μ and $M_{\mu\nu}$ allow us to define a *finite* shift and a *finite* rotation, respectively. After some calculations one finds that

$$(e^{a\Delta_\mu})_{\vec{m}\,\vec{n}} = \delta_{m_0 n_0} \cdots \delta_{m_\mu, n_\mu+1} \cdots \delta_{m_3 n_3} \tag{2.28}$$

and

$$(e^{(\pi/2)M_{\mu\nu}})_{\vec{m}\,\vec{n}} = \delta_{m_0 n_0} \cdots \delta_{n_\nu m_\mu}\delta_{n_\mu, -m_\nu} \cdots \delta_{m_3 n_3}\ . \tag{2.29}$$

Equation (2.28) defines a shift by one lattice spacing in the μ direction, Eq. (2.29) a 90° rotation in the μ-ν plane. We see that finite transformations generated by our superalgebra give the expected results when the transformation is permitted on the lattice.

This is a good place to remark why we chose to maintain the superalgebra on the lattice. One alternative would be to require that the Lagrangian be invariant only under those finite transformations permitted on the lattice. However, the finite transformations do not form a dense subset of the continuum transformations even in the limit of zero lattice spacing. It is therefore difficult to establish the connection with continuum supersymmetry.

III. ANALYSIS OF THE PARTITION FUNCTION

In this section we explore the lattice partition function (2.27) by a variety of simple methods. Our conclusion is that there is no sign of dynamical symmetry breaking of supersymmetry near weak coupling in four dimensions, but that a more detailed study is required in the strong-coupling regime. The quantity which signals the breakdown of supersymmetry is a nonzero ground-state energy density. Working in Euclidean space we expect that

$$\lim_{V\to\infty}\left[-\frac{1}{V}\ln Z\right] = E_0 > 0 \tag{3.1}$$

is a necessary and sufficient condition for the breakdown of supersymmetry.

It will be convenient to switch (for this section only) to the Euclidean metric. The partition function then reads

$$Z = \int \prod_{\vec{n}} \frac{dA_{\vec{n}}\,dB_{\vec{n}}\,dF_{\vec{n}}\,dG_{\vec{n}}\,d\psi_{\vec{n}}}{(2\pi)^2} e^{S_{\text{lattice}}}\ , \tag{3.2}$$

and the lattice action differs from (2.10) only in a few coefficients in front of fermionic terms:

$$S_{\text{lattice}} = \text{bosonic part} + a^4 \sum \left[\frac{i}{2}\bar{\psi}_{\vec{m}}\gamma_\mu\Delta^\mu_{\vec{m}\,\vec{n}}\psi_{\vec{n}} - \tfrac{1}{2}m\bar{\psi}_{\vec{n}}\psi_{\vec{n}} - gV_{\vec{n}_1\vec{n}_2\vec{n}_3}(\bar{\psi}_{\vec{n}_1}A_{\vec{n}_2}\psi_{\vec{n}_3} - i\bar{\psi}_{\vec{n}_1}\gamma_5 B_{\vec{n}_2}\psi_{\vec{n}_3})\right]\ . \tag{3.3}$$

(Our metric is now $g_{\mu\nu} = -\delta_{\mu\nu}$, $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$, $\gamma_5{}^2 = 1$, and ψ is a Majorana spinor analytically continued from Minkowski to Euclidean space.)

In order to make contact with the continuum theory, we first look at the weak-coupling region of (3.2). At $g=0$ the fermionic and bosonic integrations decouple from each other and can be done analytically. As expected, the fermions provide a negative contribution to the ground-state energy, which then is exactly canceled by the bosonic contributions. As a result, $Z=1$. In lowest nontrivial order of g^2 we have the contributions shown in Fig. 1. In the continuum theory, these diagrams sum up to zero, and in fact these cancellations continue so that $Z=1$ is correct to all orders of perturbation theory. For our lattice model we also find that the order g^2 corrections to Z vanish (for all a). This can be seen quite easily by noticing that our lattice-Feynman rules are the same as in the continuum, except that the momentum along each propagator line is restricted to the interval $(-\pi/a, \pi/a)$. At each vertex we have a momentum-conserving δ function. [As we have remarked after (2.14), it is important that this δ function is nonperiodic. Otherwise we would have "umklapp" contributions which would spoil the cancellations.] With these modifications of the Feynman rules, the cancellations among the diagrams of Fig. 1 still work, and we expect that this will continue to be true in higher order in g^2. If

FIG. 1. Two-loop graphs for the partition function Z. The sum extends over all possible internal particle lines.

so, $Z=1$ in our model to all orders perturbation theory, and supersymmetry remains unbroken, except for possible nonperturbative terms.

Next we want to make sure that our lattice model in the limit $a \to 0$ agrees with the continuum theory. To this end we have considered all potentially divergent vertex functions in the one-loop approximation, in the same spirit as in Ref. 15. In the limit $a \to 0$ our renormalized vertex functions agree with those of the continuum theory, and the necessary counterterms are supersymmetric. We have not yet fully explored the two-loop approximation, but expect that the limit $a \to 0$ agrees with the continuum theory. We mention that the difficulties in using the SLAC derivative, which have been discussed in connection with nonsupersymmetric gauge theories, do not apply to our case.[18,19] In particular, the supersymmetric structure of our model improves the situation in that ultraviolet divergences are never worse than logarithmic and counterterms of the constant $|p_\mu|$ never appear.[5]

Leaving the domain of perturbation theory, it is most tempting to try a strong-coupling expansion around $g=\infty$. Such an attempt, however, runs into severe difficulties which stem from the nonlocality of the interaction vertex $gV_{\vec{n}_1\vec{n}_2\vec{n}_3}$. Usually, when putting a field theory model onto a lattice the interaction terms remain local and the only nonlocality resides in terms involving the lattice gradient operators. In the strong-coupling limit, such terms are suppressed by an inverse power of the coupling constant, and one is left with a product of single-site integrals. In our case, terms with gradient operators are also down by some inverse power of g, but the remaining interaction terms involve $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ and hence couple different sites together. One then might try to treat the off-diagonal parts of $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ as a perturbation, such that in the zeroth-order approximation one has a product of single-site integrals. However, it turns out that such a single-site integral treats fermions and bosons slightly asymmetrically. The negative contribution of the fermions to the ground-state energy wins over the positive energy of the bosons, and the resulting ground-state energy density equals $E_0=-\ln 2$. In order to see how this result emerges it is convenient to define the partition function (3.2) in a finite volume (with periodic boundary conditions). The volume is shrunk until it contains just one point:

$$Z_{\text{single site}}=\int \frac{dA\,dB\,d\psi}{2\pi}\exp[-\tfrac{1}{2}m\bar{\psi}\psi-g\bar{\psi}(A-i\gamma_5 B)\psi-\tfrac{1}{2}(mA+gA^2-gB^2)-\tfrac{1}{2}(mB+2gAB)^2]\ . \tag{3.4}$$

This integral, as a function of $m/\sqrt{g}$, can be studied in detail, and one finds the following properties: $Z=1$ at $g=0$ and $Z=2$ at $g=\infty$. Moreover, performing a perturbation expansion around $g=0$ one finds that $Z=1$ to all orders in g. It thus has one of the most remarkable properties of supersymmetry (nonrenormalization to all orders of perturbation theory), but it is badly suited as a starting point for a strong-coupling expansion. The reason for this lies in the fact that our neglect of the nondiagonal pieces of $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$ destroys the property of H being the sum of squares of the supersymmetric charges. This is one of the reasons why, in the following sections, we shall switch to the Hamiltonian formulation and try a similar perturbative treatment of the off-diagonal part of $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$. As we shall demonstrate, there it turns out that the "local" forms $Q_\alpha^{(0)}$, $H^{(0)}$ of the charges Q_α and the Hamiltonian H are such that $H^{(0)}=[Q_\alpha^{(0)}]^2$ is preserved.

We finally mention another attempt to evaluate our partition function in the limit $g=\infty$. Having done the fermion integration and taken the limit $g\to\infty$ we found that the action as a function of A and B has a maximum away from $A=B=0$. Expanding around this stationary point we again found a negative ground-state energy, which indicates that our treatment still violates the balance between fermionic and bosonic contributions to the ground-state energy.

In the remainder of this section we shall evaluate the partition function (3.2) in another approximation: Taking the field variables $A_{\vec{n}}=A$, $B_n=B$, $F_n=F$, and $G_n=G$ to be constant in space-time, we calculate the effective potential in the one-loop approximation and search for minima with respect to F, G, A, and B. The difference between this calculation and the strong-coupling approximations above is that in a systematic expansion in loops the fermions and bosons are on the same footing. In the two strong-coupling calculations the fermions were completely integrated out as the first step, but approximations were made in evaluating the remaining partition function.

Knowledge of the one-loop effective potential will give us some information about the phase structure of the theory. The reason why this approximation is well suited for our nonlocal interaction is that for constant field configurations the nonlocality becomes harmless. [See the discussion around (2.17).] In momentum space our expressions will be the same as those in the continuum theory, except that our momentum integration ranges only from $-\pi/a$ to π/a. The virtue of our lattice version then lies in the fact that all our expressions are regulated in a supersymmetric way. Using standard techniques we find for the effective potential up to one loop:

$$V(A,B,F,G)=-\tfrac{1}{2}F^2-\tfrac{1}{2}G^2-F(mA+gA^2-gB^2)-G(mB+2gAB)$$
$$-\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\ln[k^2+(m+2gA)^2+(2gB)^2]$$
$$+\tfrac{1}{2}\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\ln\{[k^2+(m+2gA)^2+(2gB)^2]^2-4g^2(F^2+G^2)\}\ . \quad (3.5)$$

(Here we have taken out overall volume factor, and we have put $a=1$.) If we find an extremum of V with $V=0$, supersymmetry will be unbroken. If $V>0$ at this extremum, supersymmetry is broken.

The easiest way to decide whether supersymmetry is broken or not is the following[7]: In order that V has an extremum with respect to F and G, we must fulfill the two conditions

$$0=\frac{\partial V}{\partial F}=-F-mA-gA^2+gB^2-4g^2F\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\frac{1}{\phi}\ , \quad (3.6)$$

$$0=\frac{\partial V}{\partial G}=-G-mB-2gAB-4g^2G\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\frac{1}{\phi}\ , \quad (3.7)$$

where ϕ stands for the argument of the second logarithm in (3.5):

$$\phi=[k^2+(m+2gA)^2+(2gB)^2]^2-4g^2(F^2+G^2)\ . \quad (3.8)$$

As long as $\phi\geq 0$ (for negative ϕ the potential becomes complex valued), the matrix of second derivatives $\partial^2V/\partial F^2$, $\partial^2V/\partial G^2$, and $\partial^2V/\partial F\partial G$ can be shown to be negative definite. So, for fixed A and B, V can have only one stationary point as a function of F and G, where V is a maximum. Since $V(A,B,0,0)=0$, it must be that $V>0$ at this maximum, unless the maximum sits at $F=G=0$. Therefore, supersymmetry is unbroken if Eqs. (3.6) and (3.7) are satisfied for $F=G=0$. This is the case if

$$0=mA+g(A^2-B^2)\ , \quad (3.9)$$

$$0=mB+2gAB\ . \quad (3.10)$$

For real B we find two solutions:

$$\text{I: } (A,B)=(0,0)\ , \quad (3.11)$$

$$\text{II: } (A,B)=\left[-\frac{m}{g},0\right]\ . \quad (3.12)$$

We conclude that supersymmetry is unbroken for all g, but there are two fixed points which coincide at $g=\infty$. The two phases correspond to a discrete symmetry of the Wess-Zumino model[8]: $A\to A+m/g$, $m\to -m$, and in perturbation theory they are infinitely far apart. The point $g=\infty$ is a phase transition point and thus singular. It follows from (3.5) that the same conclusion could have been drawn already from the tree approximation: the fact that one-loop quantum corrections do not change the situation at all, gives rise to some confidence in the validity of this result.

In order to compare with earlier work on two-dimensional models, we briefly describe how the situation changes when we go from four to two dimensions.[7,8,13] The two-dimensional model analogous to the Wess-Zumino model in four dimensions has the form

$$\mathcal{L}=-\tfrac{1}{2}(\partial_\mu A)^2+\frac{i}{2}\bar{\psi}\gamma\cdot\partial\psi+\tfrac{1}{2}W''\bar{\psi}\psi+\tfrac{1}{2}F^2-FW'\ , \quad (3.13)$$

with the superpotential

$$W=-\left[\frac{m}{2}A^2+\frac{g}{3}A^3\right]\ . \quad (3.14)$$

The lattice model is set up in the same way as in the four-dimensional case, with appropriate changes in the definitions of Δ^{μ}_{mn}, D_{mn}, and $V_{\vec{n}_1\vec{n}_2\vec{n}_3}$. The one-loop effective potential is

$$V(A,F)=-\tfrac{1}{2}F^2+FW'+\tfrac{1}{2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\ln[k^2+FW''+(W'')^2]-\tfrac{1}{2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\ln[k^2+(W'')^2]\ . \quad (3.15)$$

Supersymmetry is unbroken only if the extremum condition

$$0=\frac{\partial V}{\partial F}=-F+W'+\tfrac{1}{2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{W'''}{k^2+FW''+(W'')^2} \quad (3.16)$$

has a solution for $F=0$. Equation (3.16) with $F=0$ reads

$$0=-A(m+gA)-g\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{1}{k^2+(m+2gA)^2}\ , \quad (3.17)$$

or

$$0=f_1(y)-f_2(y)\ , \quad (3.18)$$

with

$$f_1(y)=y-\frac{m^2}{4g^2}\ , \quad (3.19)$$

$$f_2(y)=-\int_{-\pi/g}^{\pi/g}\frac{d^2k}{(2\pi)^2}\frac{1}{k^2+4y}\ , \quad (3.20)$$

$$y=\left[A+\frac{m}{2g}\right]^2\ . \quad (3.21)$$

The two functions f_1 and f_2 are shown in Fig. 2 and it is evident that for small g we have two solutions near $A=0$ and $A=-m/g$, respectively, whereas for large g there are no solutions at real values A. We conclude that at some finite $g_{\rm crit}$ a phase transition occurs, such that supersym-

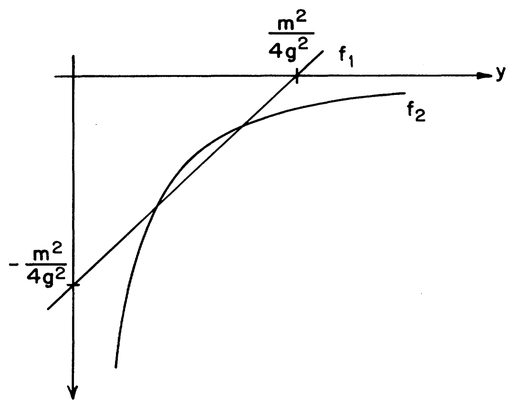

FIG. 2. Graphical illustration of Eq. (3.18).

metry is unbroken for small g but broken for $g > g_{\text{crit}}$. From (3.16) and (3.17) we see that in the tree approximation supersymmetry appears to be unbroken for *all g*. In contrast to the four-dimensional case, quantum corrections significantly change this situation and supersymmetry is broken dynamically. Higher-order loop corrections to the effective potential should be expected to change position and behavior near the phase-transition point g_{crit}, but not the qualitative picture.

All these results have been derived so far in a framework which is not too far away from perturbation theory. In the second part of our paper to which we now turn we will confirm all this in a strong-coupling expansion.

IV. HAMILTONIAN FORMULATION

In this section we will set up a lattice version of the Hamiltonian of the Wess-Zumino model. As we have explained before, our main motivation for developing this formalism comes from the desire to calculate the strong-coupling limit of the ground-state energy. It will be shown in the subsequent section that in the Hamiltonian formalism such an attempt is successful. The easiest way to arrive at the lattice Hamiltonian is again via interpolating fields. We start with the continuum theory. Here we have

$$H=\int d^3x \left[\tfrac{1}{2}(\pi_A{}^2+\pi_B{}^2)+\tfrac{1}{2}(\vec{\nabla}A)^2+\tfrac{1}{2}(\vec{\nabla}B)^2+\frac{i}{2}\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi+V \right] , \tag{4.1}$$

$$V=\tfrac{1}{2}(F^2+G^2)+\frac{i}{2}\bar{\psi}[m+2g(A-\gamma_5 B)]\psi , \tag{4.2}$$

$$F=-mA-g(A^2-B^2) , \tag{4.3}$$

$$G=-mB-2gAB , \tag{4.4}$$

$$P_k=-\int d^3x \left[\pi_A \nabla_k A+\pi_B \nabla_k B+\frac{i}{2}\bar{\psi}\gamma_0 \nabla_k \psi \right] , \tag{4.5}$$

$$Q=\int d^3x[-(\pi_A+\gamma_5\pi_B)\psi+\vec{\gamma}\cdot\vec{\nabla}(A-\gamma_5 B)\gamma^0\psi-(F+\gamma_5 G)\gamma^0\psi] . \tag{4.6}$$

The commutator algebra reads

$$[\pi_A(\vec{x},t),A(\vec{x}\,',t)]=-i\delta^3(\vec{x}-\vec{x}\,') , \tag{4.7}$$

$$[\pi_B(\vec{x},t),B(\vec{x}\,',t)]=-i\delta^3(\vec{x}-\vec{x}\,') , \tag{4.8}$$

$$\{\psi_\alpha(\vec{x},t),\psi_\beta(\vec{x}\,',t)\}=\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{x}\,') , \tag{4.9}$$

$$[\pi_A(\vec{x},t),F(\vec{x}\,',t)]=i\delta^3(\vec{x}-\vec{x}\,')[m+2gA(\vec{x},t)] , \tag{4.10}$$

$$[\pi_B(\vec{x},t),G(\vec{x}\,'t)]=i\delta^3(\vec{x}-\vec{x}\,')[m+2gA(\vec{x},t)] , \tag{4.11}$$

$$[\pi_B(\vec{x},t),F(\vec{x},t)]=[\pi_A(\vec{x},t),G(\vec{x}\,',t)]=-2ig\delta^3(\vec{x}-\vec{x}\,')B(\vec{x},t) . \tag{4.12}$$

Substituting the interpolating fields into (4.1)—(4.6) and performing x integrations, we arrive at the lattice version of the operators H, P_k, Q:

$$H=a^3\sum \left[\tfrac{1}{2}(\pi_{A\vec{n}}{}^2+\pi_{B\vec{n}}{}^2)+\tfrac{1}{2}(\Delta^k_{\vec{n},\vec{r}}A_{\vec{r}})(\Delta^k_{\vec{n},\vec{s}}A_{\vec{s}})+\tfrac{1}{2}(\Delta^k_{\vec{n},\vec{r}}B_{\vec{r}})(\Delta^k_{\vec{n},\vec{s}}B_{\vec{s}})+\frac{i}{2}\bar{\psi}_{\vec{m}}\gamma_k\Delta^k_{\vec{m}\,\vec{n}}\psi_{\vec{n}} \right]+V , \tag{4.13}$$

with

$$V=a^3\sum \left\{ \tfrac{1}{2}(F_{\vec{n}}{}^2+G_{\vec{n}}{}^2)+\frac{i}{2}\psi_{\vec{m}}[m\delta_{\vec{m}\,\vec{n}}+2gV_{\vec{m}\,\vec{n}\,\vec{s}}(A_{\vec{s}}-\gamma_5 B_{\vec{s}})]\psi_{\vec{n}} \right\} , \tag{4.14}$$

$$F_{\vec{n}}=-mA_{\vec{n}}-gV_{\vec{n}\,\vec{r}\,\vec{s}}(A_{\vec{r}}A_{\vec{s}}-B_{\vec{r}}B_{\vec{s}}) , \tag{4.15}$$

$$G_{\vec{n}}=-mB_{\vec{n}}-2gV_{\vec{n}\,\vec{r}\,\vec{s}}A_{\vec{r}}B_{\vec{s}} , \tag{4.16}$$

$$P_k=-a^3\sum\left[\pi_{A\vec{n}}\Delta^k_{\vec{n}\vec{m}}A_{\vec{m}}+\pi_{B\vec{n}}\Delta^k_{\vec{n}\vec{m}}B_{\vec{m}}+\frac{i}{2}\bar{\psi}_{\vec{n}}\gamma_0\Delta^k_{\vec{n}\vec{m}}\psi_{\vec{m}}\right], \tag{4.17}$$

$$Q=a^3\sum[-(\pi_{A\vec{n}}+\gamma_5\pi_{B\vec{n}})\psi_{\vec{n}}+\gamma_k\Delta^k_{\vec{n}\vec{m}}(A_{\vec{m}}-\gamma_5B_{\vec{m}})\gamma^0\psi_{\vec{n}}-(F_{\vec{n}}+\gamma_5G_{\vec{n}})\gamma^0\psi_{\vec{n}}]\ . \tag{4.18}$$

The commutation relations are

$$[\pi_{A\vec{n}},A_{\vec{m}}]=-ia^{-3}\delta_{\vec{m},\vec{n}}\ , \tag{4.19}$$

$$\{\psi_{\alpha\vec{n}},\psi_{\beta\vec{m}}\}=a^{-3}\delta_{\alpha\beta}\delta_{\vec{m},\vec{n}}\ , \tag{4.20}$$

$$[\pi_{A\vec{n}},F_{\vec{m}}]=ia^{-3}(m\delta_{\vec{n}\vec{m}}+2gV_{\vec{n}\vec{m}\vec{s}}A_{\vec{s}})\ , \tag{4.21}$$

and analogous formulas hold for the remaining relations. It is now a matter of straightforward, though tedious, algebra to verify the superalgebra:

$$[\bar{\eta}Q,A_{\vec{n}}]=i\bar{\eta}\psi_{\vec{n}}\ , \tag{4.22}$$

$$[\bar{\eta}Q,B_{\vec{n}}]=i\bar{\eta}\gamma_5\psi_{\vec{n}}\ , \tag{4.23}$$

$$[\bar{\eta}Q,F_{\vec{n}}]=i\bar{\eta}[-m\delta_{\vec{n},\vec{s}}-2gV_{\vec{n}\vec{r}\vec{s}}(A_{\vec{r}}-\gamma_5B_{\vec{r}})]\psi_{\vec{s}}\ , \tag{4.24}$$

$$[\bar{\eta}Q,G_n]=i\bar{\eta}[-m\delta_{\vec{n},\vec{s}}-2gV_{\vec{n}\vec{r}\vec{s}}(A_{\vec{r}}-\gamma_5B_{\vec{r}})]\gamma_5\psi_{\vec{s}}\ , \tag{4.25}$$

$$[\bar{\eta}Q,\psi_{\vec{n}}]=(\pi_{A\vec{n}}-\gamma_5\pi_{B\vec{n}})\gamma^0\eta+\Delta^k_{\vec{n},\vec{m}}(A_{\vec{m}}-\gamma_5B_{\vec{m}})\gamma_k\eta+(F_{\vec{n}}+\gamma_5G_{\vec{n}})\eta\ , \tag{4.26}$$

$$\{Q_\alpha,\bar{Q}_\beta\}=2\gamma_{\alpha\beta}{}^0H-2\gamma_{k,\alpha\beta}P^k\ , \tag{4.27}$$

$$[P_k,Q_\alpha]=[H,Q_\alpha]=0\ . \tag{4.28}$$

In proving these relations a vital role is played by the Leibniz rule (2.10). In the following section we shall use the Hamiltonian (4.13), the supercharges (4.18), and the relation (4.27). The latter implies that

$$H=\tfrac{1}{4}\sum_{\alpha=0}^{3}Q_\alpha{}^2\ . \tag{4.29}$$

V. SUPERSYMMETRY IN THE STRONG-COUPLING LIMIT

It is convenient to scale the dynamical variables

$$A_{\vec{n}}=g^{-1/3}a^{-1}A'_{\vec{n}},\quad \pi_{A\vec{n}}=g^{1/3}a^{-2}\pi'_{A\vec{n}},\quad \psi_{\vec{n}}=a^{-3/2}\psi'_{\vec{n}}\ , \tag{5.1}$$

with the field B scaled like A. Then $\{\psi'_{m\alpha},\psi'_{n\beta}\}=\delta_{mn}\delta_{\alpha\beta}$ and $\pi'_{An}=-i\partial/\partial A'_n$. Dropping primes, the Hamiltonian and charges take the form

$$H=g^{2/3}a^{-1}\sum\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial A_{\vec{n}}{}^2}+\frac{\partial^2}{\partial B_{\vec{n}}{}^2}\right)+V_{\vec{n}\vec{r}\vec{s}}[i\psi_{\vec{n}}\gamma^0(A_{\vec{r}}-\gamma_5B_{\vec{r}})\psi_{\vec{s}}+\tfrac{1}{2}V_{\vec{n}\vec{t}\vec{u}}(A_{\vec{r}}A_{\vec{s}}-B_{\vec{r}}B_{\vec{s}})(A_{\vec{t}}A_{\vec{u}}-B_{\vec{t}}B_{\vec{u}}) + 2V_{\vec{n}\vec{t}\vec{u}}A_{\vec{r}}B_{\vec{s}}A_{\vec{t}}B_{\vec{u}}]\right]+O(g^0)\ , \tag{5.2}$$

$$Q=g^{1/3}a^{-1/2}\sum\left[i\frac{\partial}{\partial A_{\vec{n}}}+i\gamma_5\frac{\partial}{\partial B_{\vec{n}}}+V_{\vec{n}\vec{r}\vec{s}}(A_{\vec{r}}A_{\vec{s}}-B_{\vec{r}}B_{\vec{s}}+2\gamma_5A_{\vec{r}}B_{\vec{s}})\gamma^0\right]\psi_{\vec{n}}+O(g^{-1/3})\ .$$

In this section we drop subordinate terms in g^{-1}. The supersymmetry algebra in the strong-coupling limit is

$$\{Q_\alpha,Q_\beta\}=2\delta_{\alpha\beta}H\ . \tag{5.3}$$

Our program is to see if Q_1 has eigenstates with eigenvalue zero. If it does, it follows that these states have energy zero and are annihilated by all the charges. They then comprise the supersymmetric ground-state multiplet of the theory, and it is plausible that supersymmetry is unbroken in our model all the way from $g=0$ to $g=\infty$.

Unfortunately, the strong-coupling charges are nonlocal, owing to the coupling V which we had to introduce to maintain the supersymmetry algebra on the lattice. As a result, the strong-coupling limit is nontrivial. Our approach is to partition V into local and nonlocal parts:

$$V_{\vec{n}\vec{r}\vec{s}}=V_0\delta_{\vec{n}\vec{r}}\delta_{\vec{n}\vec{s}}+\lambda(V_1)_{\vec{n}\vec{r}\vec{s}}\ . \tag{5.4}$$

λ is an ordering parameter to be set $\lambda=1$ at the end. There is a corresponding partition of Q and H:

$$Q=Q^{(0)}+\lambda Q^{(1)}\ ,\qquad H=H^{(0)}+\lambda H^{(1)}+\lambda^2H^{(2)}\ . \tag{5.5}$$

We will show that to all orders in the perturbation λ there are eigenfunctions of Q_1 having eigenvalue zero. Since the dimensions of the operators in $Q_1^{(0)}$ and $Q_1^{(1)}$ are the same, we believe our perturbation series in λ has a finite radius of convergence. This argument rules our nonperturbative contributions at $\lambda \sim 0$, and makes it highly plausible that Q_1 continues to have eigenstates with zero eigenvalue at $\lambda=1$.

$Q_1^{(0)}$ is a sum of single-site operators, and we begin by finding eigenfunctions of the single-site summands. We use the Majorana matrices

$$\gamma^0=\begin{bmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix},\ \gamma^1=\begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix},$$
$$\gamma^2=\begin{bmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix},\ \gamma^3=\begin{bmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix}, \tag{5.6}$$
$$\gamma_5=\begin{bmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{bmatrix}.$$

We introduce fermion operators

$$b_+=\frac{1}{\sqrt{2}}(\psi_2+i\psi_3),\quad b_-=\frac{1}{\sqrt{2}}(-\psi_1+i\psi_4) \tag{5.7}$$

and their adjoints. (The site labels are suppressed.) These operators satisfy anticommutation relations

$$\{b_l,b_{l'}\}=0,\quad \{b_l,b_{l'}^\dagger\}=\delta_{l,l'}\ (l,l'=+,-)\ , \tag{5.8}$$

so we can construct single-site wave functions in the form

$$\begin{aligned} |\psi\rangle &= f_{00}(A,B)\,|0\rangle + f_{10}(A,B)b_-^\dagger\,|0\rangle \\ &\quad + f_{01}(A,B)b_+^\dagger\,|0\rangle + f_{11}(A,B)b_-^\dagger b_+^\dagger\,|0\rangle \\ &\equiv F\,|0\rangle\ , \end{aligned} \tag{5.9}$$

where $|0\rangle$ is defined by

$$b_l\,|0\rangle=0\ . \tag{5.10}$$

The spin operator is

$$S_z=\frac{i}{4}\bar{\psi}\gamma^1\gamma^2\psi=\tfrac{1}{2}(b_+^\dagger b_+ - b_-^\dagger b_-)\ , \tag{5.11}$$

so in the continuum theory $b_+^\dagger(b_-^\dagger)$ would create a Majorana fermion having $s_z=\frac{1}{2}\ (-\frac{1}{2})$.

The single-site eigenvalue equation $Q_{n1}^{(0)}\,|\psi\rangle=q\,|\psi\rangle$ takes the form

$$\begin{aligned} -i\frac{\partial f_{10}}{\partial A}-i\frac{\partial f_{01}}{\partial B}+iV_0(A^2-B^2)f_{10}+2iV_0ABf_{01}&=\mu f_{00}\ , \\ -i\frac{\partial f_{00}}{\partial A}+i\frac{\partial f_{11}}{\partial B}-iV_0(A^2-B^2)f_{00}-2iV_0ABf_{11}&=\mu f_{10}\ , \\ -i\frac{\partial f_{11}}{\partial A}-i\frac{\partial f_{00}}{\partial B}+iV_0(A^2-B^2)f_{11}-2iV_0ABf_{00}&=\mu f_{01}\ , \\ -i\frac{\partial f_{01}}{\partial A}+i\frac{\partial f_{10}}{\partial B}-iV_0(A^2-B^2)f_{01}+2iV_0ABf_{10}&=\mu f_{11}\ , \end{aligned} \tag{5.12}$$

where $\mu=q(2a)^{1/2}g^{-1/3}$. Note that if $|\psi\rangle$ results from these equations with eigenvalue μ, then

$$|\tilde{\psi}\rangle=f_{00}\,|0\rangle-f_{10}b_-^\dagger\,|0\rangle-f_{01}b_+^\dagger\,|0\rangle+f_{11}b_-^\dagger b_+^\dagger\,|0\rangle \tag{5.13}$$

is an associated eigenfunction with eigenvalue $-\mu$.

Equations (5.12) can be simplified by three changes. First, write them in terms of the linear combinations

$$\begin{aligned} \Sigma&=f_{00}+if_{11},\quad \Delta=f_{00}-if_{11}\ , \\ S&=f_{10}+if_{01},\quad D=f_{10}-if_{01}\ . \end{aligned} \tag{5.14}$$

The function Σ and Δ are amplitudes for components having an even number of fermions (bosonic amplitudes) whereas S and D are amplitudes of components having one fermion (fermionic amplitudes). We shall see that the ground state is purely bosonic, $S=D=0$, but all other states have contributions from all four amplitudes.

Second, introduce polar coordinates in field space: $A=R\sin\theta$, $B=R\cos\theta$. Third, introduce Fourier expansions in θ. For example,

$$\Delta(R,\theta)=\sum_{m=-\infty}^{\infty}\Delta_m(R)e^{im\theta}\ . \tag{5.15}$$

We then obtain coupled ordinary differential equations for the radial amplitudes:

$$\begin{aligned} \Sigma_m'+\frac{m}{R}\Sigma_m+iV_0R^2\Delta_{m+1}&=\mu S_{m-1}\ , \\ -\Delta_{m+1}'+\frac{m+1}{R}\Delta_{m+1}+iV_0R^2\Sigma_m&=\mu D_{m+2}\ , \\ -S_{m-1}'+\frac{m-1}{R}S_{m-1}-iV_0R^2D_{m+2}&=\mu\Sigma_m\ , \\ D_{m+2}'+\frac{m+2}{R}D_{m+2}-iV_0R^2S_{m-1}&=\mu\Delta_{m+1}\ . \end{aligned} \tag{5.16}$$

Fortunately, only four radial functions are involved in Eqs. (5.16). In addition, we can deduce from them two uncoupled differential equations which are satisfied by the fermionic components of $|\psi\rangle$:

$$\begin{aligned} S_{m-1}''+\frac{1}{R}S_{m-1}'+\left[\mu^2-\frac{(m-1)^2}{R^2}-V_0{}^2R^4\right]S_{m-1}&=0\ , \\ D_{m+2}''+\frac{1}{R}D_{m+2}'+\left[\mu^2-\frac{(m+2)^2}{R^2}-V_0{}^2R^4\right]D_{m+2}&=0\ . \end{aligned} \tag{5.17}$$

These equations are instances of the generic equation

$$U_m''+\frac{1}{R}U_m'+\left[\mu_m{}^2-\frac{m^2}{R^2}-V_0{}^2R^4\right]U_m=0\ . \tag{5.18}$$

In order that $|\psi\rangle$ have finite norm, we require solutions finite at $R=0$ and $R=\infty$. Indicial and asymptotic analyses of Eq. (5.18) shows that this requirement leads to a discrete spectrum of eigenvalues $\mu_{m,s}{}^2$ $(s=1,2,\ldots)$ and associated orthonormal eigenfunctions. We thus obtain two sets of solutions of Eq. (5.16) designated by $r=-1$ and $r=2$:

$$\Sigma_{m,s,-1}=N_1\left[-U'_{m-1,s}+\frac{m-1}{R}U_{m-1,s}\right]e^{im\theta}\,,$$
$$\Delta_{m,s,-1}=-iN_1V_0R^2U_{m-1,s}e^{i(m+1)\theta}\,,$$
$$S_{m,s,-1}=N_1(\mu_{m-1,s}{}^2)^{1/2}U_{m-1,s}e^{i(m-1)\theta}\,,\tag{5.19}$$
$$D_{m,s,-1}=0,\quad \mu=(\mu_{m-1,s}{}^2)^{1/2}$$

and

$$\Sigma_{m,s,2}=-iN_2V_0R^2U_{m+2,s}e^{im\theta}\,,$$
$$\Delta_{m,s,2}=N_2\left[U'_{m+2,s}+\frac{m+2}{R}U_{m+2,s}\right]e^{i(m+1)\theta}\,,$$
$$S_{m,s,2}=0\,,\tag{5.20}$$
$$D_{m,s,2}=N_2(\mu_{m+2,s}{}^2)^{1/2}U_{m+2,s}e^{i(m+2)\theta}\,,$$
$$\mu=(\mu_{m+2,s}{}^2)^{1/2}\,.$$

Solutions having $\mu\to-\mu$ can be generated by changing the signs of the fermionic components S and D in Eqs. (5.19) and (5.20). These additional solutions we designate by changing the index $s\to-s$.

There are two special solutions having zero fermionic components S and D. It follows from Eq. (5.16) that $\mu=0$ and

$$\Sigma''_m-\frac{3}{R}\Sigma'_m-\frac{m(m+4)}{R^2}\Sigma_m-V_0{}^2R^4\Sigma_m=0\,.\tag{5.21}$$

Solutions bounded at infinity are obtained from Eqs. (5.21) and the first of (5.16):

$$\Sigma_{m,0,0}=N_3R^2K_{(m+2)/3}\left[\frac{V_0R^3}{3}\right]e^{im\theta}\,,$$
$$\Delta_{m,0,0}=-iN_3R^2K_{(m-1)/3}\left[\frac{V_0R^3}{3}\right]e^{i(m+1)\theta}\,.\tag{5.22}$$

These solutions are finite at $R=0$ only for $m=0,-1$. We have set the indices $s=r=0$ for them. The fact that we have found two normalizable solutions to our zero-energy eigenvalue problem implies that, at the single-site level and in the limit $g\to\infty$, supersymmetry is unbroken. As to the degeneracy of our solutions, we believe it reflects the two supersymmetric vacua identified in Eqs. (3.11) and (3.12).

Now consider the lattice as a whole. For each single-site state there is an operator $F_{\vec{n},m,s,r}$ which creates that state on site $\vec{n}$, as in Eq. (5.9). Using these we create lattice states

$$|\{m_{\vec{n}},s_{\vec{n}},r_{\vec{n}}\}\rangle=\prod_{\vec{n}}F_{\vec{n},m_{\vec{n}},s_{\vec{n}},r_{\vec{n}}}|0\rangle\,.\tag{5.23}$$

Since the F's contain fermion operators, the lattice product must be taken in a fixed order. These states are orthonormal. To see this, note that our solutions of the single-site problem are orthonormal. It follows from Eq. (5.9) that

$$\int_{-\infty}^{\infty}dA_{\vec{n}}dB_{\vec{n}}\{F^\dagger_{\vec{n},m',s',r'},F_{\vec{n},m,s,r}\}=\delta_{mm'}\delta_{ss'}\delta_{rr'}+K\,,\tag{5.24}$$

where K is normal ordered in fermion operators. Orthonormality of the lattice states follows from this anticommutation relation. Completeness of the lattice states can be proved starting with the completeness of the eigenfunctions of Eq. (5.18).

Our lattice states (5.23) are eigenstates of the Hamiltonian $H^{(0)}$:

$$H^{(0)}=[\sum_{\vec{n}}Q^{(0)}_{\vec{n},1}]^2=\sum_{\vec{n}}[Q^{(0)}_{\vec{n},1}]^2\,,\tag{5.25}$$

$$H^{(0)}|\{m_{\vec{n}},s_{\vec{n}},r_{\vec{n}}\rangle=E_{\{m_{\vec{n}},s_{\vec{n}},r_{\vec{n}}\}}|\{m_{\vec{n}},s_{\vec{n}},r_{\vec{n}}\}\rangle\,,\tag{5.26}$$

where

$$E_{\{m_{\vec{n}},s_{\vec{n}},r_{\vec{n}}\}}=\sum_{\vec{n}}\mu_{m_{\vec{n}}+r_{\vec{n}},s_{\vec{n}}}{}^2\,.\tag{5.27}$$

To prove Eq. (5.26), we use a result following from the single site problem:

$$[Q^{(0)}_{\vec{n},1}]^2F_{\vec{n},ms,r}=\mu_{m+r,s}{}^2F_{\vec{n},m,s,r}+\bar{K}\,,\tag{5.28}$$

whre $\bar{K}|0\rangle=0$. In addition, it is important that $[Q^{(0)}_{\vec{n},1}]^2$ is even in fermion operators so

$$[[Q^{(0)}_{\vec{n},1}]^2,F_{\vec{n}',m,s,r}]=0\ (\vec{n}\neq\vec{n}')\,.\tag{5.29}$$

The action of $Q_1^{(0)}$ (and also $Q_1^{(1)}$) on the lattice states is more complicated. According to Eq. (5.2), these operators are sums of fermion operators $\psi_{\vec{n}}$. When acting on a lattice state, a Q operator maps fermionic components at site $\vec{n}$ onto bosonic components and vice versa. At all other sides, each component is mapped into itself.

Since $[H^{(0)},Q_1^{(0)}]=0$, linear combinations of eigenfunctions of $H^{(0)}$ can be formed that are also eigenstates of $Q_1^{(0)}$. In particular, appropriate linear combinations of $|\{m_{\vec{n}},s_{\vec{n}},r_{\vec{n}}\}\rangle$ and $Q_1^{(0)}|\{m_{\vec{n}},s_{\vec{n}},r_{\vec{n}},\}\rangle$ are eigenfunctions of $Q_1^{(0)}$ with eigenvalues $\pm[E_{\{m_{\vec{n}},s_{\vec{n}},r_{\vec{n}}\}}]^{1/2}$.

The supersymmetric ground states on the lattice are $|\{m_{\vec{n}},0,0\}\rangle$:

$$Q_1^{(0)}|\{m_{\vec{n}},0,0\}\rangle=0\tag{5.30}$$

for $m_{\vec{n}}=0$ or -1. These 2^N degenerate states are presumably split by the gradient interactions, which drop out in the strong-coupling limit.

Now introduce the nonlocal perturbation $\lambda Q_1^{(1)}$. We will use perturbation theory to construct states

$$|\Psi_0\rangle=|\{m_{\vec{n}},0,0\}\rangle+\sum_{p=1}^{\infty}\sum_{\{m'_{\vec{n}},s'_{\vec{n}},r'_{\vec{n}}\}}\lambda^pC^{(p)}(\{m'_{\vec{n}},s'_{\vec{n}},r'_{\vec{n}}\})|\{m'_{\vec{n}},s'_{\vec{n}},r'_{\vec{n}}\}\rangle\,,\tag{5.31}$$

which satisfy

$$[Q_1^{(0)}+\lambda Q_1^{(1)}]|\Psi_0\rangle=0\,.\tag{5.32}$$

The sum in Eq. (5.31) is over all nonzero-energy states. Our proof is by induction. Assume that Eq. (5.32) is satisfied for orders λ^p, where $p \le p_0$, by the coefficients

$$C^{(p)}=(-1)^p\sum_{\{k\}}\frac{\langle \{m'_{\vec n},s'_{\vec n},r'_{\vec n}\} \mid Q_1^{(0)}Q_1^{(1)} \mid \{k_1\}\rangle \cdots \langle \{k_{p-1}\} \mid Q_1^{(0)}Q_1^{(1)} \mid \{m_{\vec n},0,0\}\rangle}{E_{\{k_1\}}E_{\{k_2\}}\cdots E_{\{k_{p-1}\}}} \quad (5.33)$$

The intermediate state sums are over all nonzero-energy states. We will show that Eq. (5.32) is satisfied in order λ^{p_0+1} provided Eq. (5.33) is extended to $p=p_0+1$.

The requirement that the coefficient of λ^{p_0+1} vanish on the left-hand side of Eq. (5.32) is

$$\sum_{\{m'_{\vec n},s'_{\vec n},r'_{\vec n}\}}[C^{(p_0+1)}(\{\,\})Q_1^{(0)}+C^{(p)}(\{\,\})Q_1^{(1)}] \mid \{m'_{\vec n},s'_{\vec n},r'_{\vec n}\}\rangle=0\,. \quad (5.34)$$

We must show that $C^{(p_0+1)}$ can be chosen so that the projection of the left-hand side on every lattice state vanishes. First consider zero-energy states by forming the matrix element with $\langle \{\overline{m}_{\vec n},0,0\} \mid$. Since $\langle \{\overline{m}_{\vec n},0,0\} \mid Q_1^{(0)}=0$, we must show that the matrix element given by the second term in (5.34) vanishes. This matrix element is

$$M=(-1)^{p_0}\sum_{\{k\}}\frac{\langle \{\overline{m}_{\vec n},0,0\} \mid Q_1^{(1)} \mid \{k_0\}\rangle\langle \{k_0\} \mid Q_1^{(0)}Q_1^{(1)} \mid \{k_1\}\rangle \cdots \langle \{k_{p_0-1}\} \mid Q_1^{(0)}Q_1^{(1)} \mid \{m_{\vec n},0,0\}\rangle}{E_{\{k_1\}}E_{\{k_2\}}\cdots E_{\{k_{p_0-1}\}}}\,. \quad (5.35)$$

Consider the leftmost factor in the sum, the matrix element $\langle \{\overline{m}_{\vec n},0,0\} \mid Q_1^{(1)} \mid \{k_0\}\rangle$, and compare terms in the sums over states which differ solely by $s_i \to -s_i$ in the intermediate states. These simultaneous reflections do not affect the energy denominators, but they do change the sign of the designated matrix element. We recall that $Q_1^{(1)}$ is a sum of fermion operators $\psi_{\vec n}$, and at site $\vec n$ the matrix element is an integral over a product of a bosonic amplitude from the ground state and a fermionic amplitude from the intermediate state $\mid \{k_0\}\rangle$. The reflection $s_i \to -s_i$ just changes the sign of the fermionic amplitudes [cf. Eq. (5.13)]. All other sites are unaffected because the matrix element involves an integral over products of bosonic amplitudes at those sites. As a result, the designated matrix element changes sign. All other matrix elements in Eq. (5.35) are unchanged under the reflection $s_i \to -s_i$. This is because the operator $Q_1^{(0)}Q_1^{(1)}$ is bilinear in the fermion operators, i.e., it is a sum of terms $\psi_{\vec n}\psi_{\vec m}$. When $s_i \to -s_i$, factors in the matrix elements coming from sites $\vec m$ and $\vec n$ change signs, but those from all other sites are unchanged. The overall sign of matrix elements of $Q_1^{(0)}Q_1^{(1)}$ therefore is unchanged by $s_i \to -s_i$. Altogether, we see there is a pairwise cancellation of contributions to M by states defined by $s_i \to -s_i$, and therefore $M=0$.

Next consider the subspace of finite-energy states, and the projection of the left-hand side of Eq. (5.34) onto this subspace. On the subspace the operator $Q_1^{(0)}$ is invertible, so it suffices to take projections $\langle \{\overline{m}_n,\overline{s}_n,\overline{r}_n\} \mid Q_1^{(0)}$. Taking this projection, we find the λ^{p_0+1} term in Eq. (5.32) vanishes provided we take Eq. (5.33) to hold for $p=p_0+1$. Thus $Q_1 \mid \Psi_0\rangle=0$ to all orders of perturbation theory.

It is worth noting that very little has gone into our demonstration. First, we required the existence of supersymmetric single-site states. Then we used the fact that energies of single-site eigenstates are not altered by change of sign of fermionic amplitudes. It followed that $M=0$, which immediately led to the construction of the perturbative supersymmetric lattice states. These requirements are so simple that we believe our version of lattice supersymmetry could be used to examine dynamical symmetry breaking at $g=\infty$ in a variety of models.

We conclude this section with a brief comment on the two-dimensional version of the Wess-Zumino model. The single-site problem is identical with Witten's supersymmetric quantum-mechanics model,[6] and in our case

$$W(A)=-mA-gA^2\,. \quad (5.36)$$

As was shown in Ref. 6, there is no normalizable zero-energy solution, since the function

$$\psi(A)=\exp\left[\int_0^A \frac{dx}{\hbar}W(x)\sigma_3\right]\psi(0) \quad (5.37)$$

becomes infinite either at $A\to\infty$ or at $A\to-\infty$. Hence supersymmetry is broken at the single-site level, and for the moment we do not have any evidence that the off-diagonal part of $V_{\vec n_1\vec n_2\vec n_3}$ will restore it. This is quite consistent with the picture which we have gained in Sec. III.

VI. SUMMARY

We have seen that it is possible to give both Lagrangian and Hamiltonian formulations of the Wess-Zumino model on a lattice in which supersymmetry is manifest. The perturbative properties of these formulations are very close to the continuum theory. The reason for this is that, in momentum space, the lattice formulation simply introduces a cutoff in a way that preserves supersymmetry. As a result, we were able to show in Sec. III that in the zero-spacing limit, all potentially divergent amplitudes agree with continuum perturbation theory. We also verified, through two loops, that perturbative nonrenormalization also occurs in the lattice theory. Finally, we studied the effective potential through one loop. We found that in four dimensions there are two supersymmetric vacua which coalesce as $g\to\infty$, and in two dimensions there is dynamical breaking of supersymmetry where g exceeds a critical coupling.

These perturbative results indicate that our method of putting supersymmetry on the lattice does not mutilate the Wess-Zumino model, at least as far as we have checked. However, the real interest in a lattice formulation of supersymmetry lies in whether one can do nontrivial calculations outside the perturbative regime. Here Wilson's lattice gauge theories define the standard: for such theories, strong coupling and Monte Carlo methods have been distinct advances. Our formulation has two obvious technical problems, the presence of fermions (which is inevitable), and the nonlocal couplings. But we showed in Sec. V that a degree of computability survives in the strong-coupling limit, and we give strong evidence that supersymmetry is not dynamically broken in the Wess-Zumino model in four dimensions, but in two dimensions it is.

Our main interest has been in the question whether supersymmetry will be broken for strong coupling. Our conclusion agrees with Witten's argument based on counting of zero-energy states.[8] This argument has been applied to a large class of theories, where it rules out dynamical symmetry breaking. However, there are exceptions. One is supersymmetric gauge theories with matter fermions in complex representations, and in fact Peskin has given an example where supersymmetry is broken.[9] Another is massless theories which cannot be obtained as the limit of a massive theory.

Given these results, we would like to extend our lattice formulation to supersymmetric gauge theories and other theories to which Witten's argument cannot be applied. Beyond that, lattice supersymmetry offers a means of studying supersymmetry far from the perturbative regime. For example, in models where there is dynamical breaking of supersymmetry, our formulation might be used to study the nature and dynamics of these phase transition.

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