Infrared coherence and the QED path integral

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The infrared coherent states developed by Faddeev and Kulish are used to construct the path-integral representation of quantum electrodynamics. The result shows that the time-ordered products of the asymptotic fields, previously evaluated within the operator formulation, can be derived from the functional formalism. This is accomplished by retaining a nonlocal piece of the original vertex when defining the perturbation series. The resulting path integral allows exact evaluation through a nonperturbative change of spinor variables and the use of the classical solution to the gauge field equation of motion in the presence of a point source moving at constant velocity. Extension of this functional technique to the renormalized theory and to massless nonlinear theories is briefly discussed.

I. INTRODUCTION

In the standard diagrammatic formulation of perturbation theory in quantum electrodynamics (QED) self-energy corrections are infrared divergent. A theorem by Lee and Nauenberg[1] demonstrates that this divergence can be removed from calculations of physical processes by summing over energetically degenerate initial and final states. As discussed by Yennie *et al.*,[2] such a procedure is realized by giving the photon a small mass in intermediate steps of calculation. The infrared-divergent terms are then canceled by summing over graphs with appropriate numbers of soft bremsstrahlung attached to the external lines, and returning the photon mass to zero. In effect, every scattering process changes the momenta of the participating charged particles. The accompanying electromagnetic field must reflect this change, and this gives rise to the necessity of the bremsstrahlung. This technique has been generalized to non-Abelian gauge theories.[3]

At a more fundamental level these infrared divergences arise from the failure to identify the correct asymptotic charged states. Bloch and Nordsieck[4] were first to point out that a physical charged particle state must be accompanied by an infinite number of coherent photons. In generalizing the work of Dollard[5] to the relativistic case, Faddeev and Kulish[6] were able to construct a pseudo-unitary operator which dresses the charged states of the standard theory with sufficient photons to represent the electromagnetic field associated with a particle moving at a constant velocity. In other work[7] the infrared states of Faddeev and Kulish, modified to satisfy the Gupta-Bleuler[8] condition, were used in a reduction formulation of scattering in QED. Such an approach leads to a modified interaction picture with a new set of Feynman rules. These rules give graphs free of infrared divergences, although the renormalization of the theory has remained obscure.[9]

Quite apart from these formal considerations, most current work in quantum field theory employs the path-integral formulation pioneered by Feynman.[10] Such an approach makes no direct reference to particle states, but instead gives a method to calculate time-ordered products of Heisenberg fields between the vacuum of the theory. Several authors[11] have used coherent states, which are complete when functionally integrated, to construct the path integral. In other work[12] the path-integral formulation of QED was derived by this method with special attention paid to the role of the Gupta-Bleuler condition. It was found that the Coulomb-gauge path integral represents the transition amplitude between the Fock vacuum dressed by the same unitary operator used to generate the interaction-picture representation. To make this clear, let A_μ be the interpolating photon field which obeys the interacting Feynman-gauge equation of motion

$$\Box A_\mu = e\bar{\psi}\gamma_\mu\psi \ , \quad (1.1)$$

where ψ is the interpolating bispinor field. If a_μ is the free Feynman-gauge field, the unitary operator $Z(t)$ connects A_μ to a_μ in the manner

$$Z(t)A_\mu(\vec{x},t)Z^{-1}(t)=a_\mu(\vec{x},t) \ . \quad (1.2)$$

The pseudo-unitary operator $V(t)$ (Ref. 13) is defined by its action on the positive-frequency part of $\partial_\mu A^\mu$, so that

$$V(t)Z(t)\partial_\mu A^{\mu(+)}(\vec{x},t)Z^{-1}(t)V^{-1}(t) = \partial_\mu a^{\mu(+)}(\vec{x},t) \ . \quad (1.3)$$

It was shown that

$$\langle 0|V(t_+)Z(t_+)Z^{-1}(t_-)V^{-1}(t_-)|0\rangle=\int[dA_\mu d\bar\psi d\psi]\delta(\vec\nabla\cdot\vec A)\exp\left[i\int_{t_-}^{t_+}dt\,d^3x(\mathscr{L}-J_\mu A^\mu-\bar\psi K-\bar K\psi)\right],\tag{1.4}$$

where J_μ and K are external sources, $\mathscr{L}$ is the gauge-invariant QED Lagrangian without a gauge-fixing term, and the measure in (1.4) will be exhibited in the next section.

In the standard evaluation of (1.4) the interaction terms in the Lagrangian are written as functional derivatives and the remaining path integral, quadratic and linear in the fields, is integrated. Such an evaluation results in the standard set of Feynman rules in the Coulomb gauge which exhibit the previously discussed infrared divergences. In other analyses[7] it was shown that the infrared coherent states consistent with the Gupta-Bleuler condition are created by a pseudo-unitary transformation which is the large-time limit of ZV. At first glance these two results seem contradictory.

It is the intent of this article to resolve this problem and to show how the path-integral formulation of QED may be evaluated *nonperturbatively* to obtain the Feynman rules consistent with the infrared structure of the theory. In Sec. II the infrared-coherent-state formalism is briefly reviewed and then used to define a complete set of infrared functional states. These states are used in turn to construct the path integral for QED with the same result as (1.4) again. It is shown how the contradiction mentioned previously is resolved. In Sec. III the correct method for nonperturbatively incorporating infrared effects into the Feynman rules is developed within the path-integral formalism. This method is demonstrated by deriving the modified bispinor propagators and vertices familiar from the operator formulation. In Sec. IV extension of this work to the full renormalization of QED in the modified interaction picture as well as to the nonperturbative evaluation of infrared effects in other massless nonlinear theories is discussed.

II. INFRARED COHERENCE AND THE PATH INTEGRAL

In the first part of this section the coherent-state formalism of Faddeev and Kulish will be reviewed and its renormalization will be outlined. In the second part a derivation of the path integral consistent with the infrared behavior of the theory will be presented.

The key step in the derivation presented by Faddeev and Kulish is to note the asymptotic behavior of the Z operator defined by (1.2). It follows that

$$\lim_{t\to\pm\infty} i\dot Z Z^{-1}\equiv H_I^{\rm as}(t)=\int d^3x\, j_\mu^{\rm as}(\vec x,t)a^\mu(\vec x,t)\,,\tag{2.1}$$

where a_μ is the free Feynman-gauge photon field with the decomposition

$$a_\mu(x)=\int\frac{d^3k}{(2\pi)^{3/2}}(2\omega_k)^{-1/2}[a_\mu(\vec k)e^{ikx}+a_\mu^\dagger(\vec k)e^{-ikx}],\quad \omega_k=|\vec k|\,,\tag{2.2}$$

while the current $j_\mu^{\rm as}$ has the form

$$\begin{aligned}j_\mu^{\rm as}(\vec x,t)&=e\int\frac{d^3p}{(2\pi)^{3/2}}\sum_{s=1}^{2}(b^\dagger_{\vec p,s}b_{\vec p,s}-d^\dagger_{\vec p,s}d_{\vec p,s})\frac{p_\mu}{\epsilon_p}\delta\left[\vec x-\frac{\vec p}{\epsilon_p}t\right]\\&\equiv e\int\frac{d^3p}{(2\pi)^{3/2}}\rho(\vec p)\frac{p_\mu}{\epsilon_p}\delta\left[\vec x-\frac{\vec p}{\epsilon_p}t\right].\end{aligned}\tag{2.3}$$

The $b^\dagger$ and $d^\dagger$ are creation operators for particles and antiparticles familiar from the Fourier decomposition of the free Dirac field

$$\phi(x)=\int\frac{d^3p}{(2\pi)^{3/2}}\left[\frac{m}{\epsilon_p}\right]^{1/2}[u_{\vec p,s}(x)b^\dagger_{\vec p,s}+v_{\vec p,s}(x)d_{\vec p,s}],\quad {\epsilon_p}^2=|\vec p|^2+m^2\,.\tag{2.4}$$

If the free fields (2.2) and (2.4) are used as interaction-picture fields, then the interaction-picture Hamiltonian consistent with this choice will not automatically switch off at asymptotic times. This result follows directly from the fact that (2.1) has nontrivial interactions in that limit due to the masslessness of the photon.

The solution to this problem is to select different interaction-picture fields so that the associated Hamiltonian does vanish at asymptotic times. In the method of Faddeev and Kulish this is accomplished by the assumption that the correct interaction-picture fields, denoted $a_\mu^{\rm as}$ and $\phi^{\rm as}$, are related to the free fields by a unitary operator U, so that

$$a_\mu^{\rm as}=Ua_\mu U^{-1},\quad \phi^{\rm as}=U\phi U^{-1}\ . \tag{2.5}$$

Combining (2.5) with (2.1) and (1.2) gives

$$U(t_2)U^{-1}(t_1)=T\left\{\exp\left[i\int_{t_1}^{t_2}dt\,H_I^{\rm as}(t)\right]\right\}. \tag{2.6}$$

Expression (2.6) has been evaluated elsewhere[7] to obtain

$$U(t)=\exp[-R(t)]\exp[-i\beta(t)]\ , \tag{2.7}$$

where

$$R(t)=e\int\frac{d^3k}{(2\pi)^{3/2}}d^3p(2\omega_k)^{1/2}\rho(\vec{\rm p})\frac{p_u}{(pk)}\left[a_\mu^\dagger(\vec{\rm k})e^{ikpt/\epsilon_p}-a_\mu(\vec{\rm k})e^{-ikpt/\epsilon_p}\right] \tag{2.8a}$$

and

$$\beta(t)=e^2\int d^3p\,d^3q\rho(\vec{\rm p})\rho(\vec{\rm q})f(\vec{\rm p},\vec{\rm q},t)\ , \tag{2.8b}$$

with

$$f(\vec{\rm p},\vec{\rm q},t)=\int\frac{d^3k}{(2\pi)^3}pq\left[2\omega_k\epsilon_p kq\left[\frac{kp}{\epsilon_p}-\frac{kq}{\epsilon_q}\right]\right]^{-1}\sin\left[\left[\frac{kp}{\epsilon_p}-\frac{kq}{\epsilon_q}\right]t\right]. \tag{2.8c}$$

When (2.7) is applied in (2.5), the free photon field is modified by the Lienard-Wiechert potential of any charges present. The charged particles develop an eikonal phase which represents a distortion of the plane wave due to the presence of other charges.

When dressed by the U operator the charged Fock states become orthogonal to the original Fock spectrum. This manifests itself in the form of divergences present in the time-ordered products of the $\phi^{\rm as}$, necessitating a renormalization procedure. This can be effected by defining a momentum cutoff version[9] of the Hamiltonian (2.1),

$$H_i^{\rm as}(t,\alpha)\equiv e\int\frac{d^3p}{(2\pi)^{3/2}}\frac{d^3k}{(2\omega_k)^{1/2}}\rho(\vec{\rm p})\frac{p^\mu}{\epsilon_p}[a_\mu^\dagger(-\vec{\rm k})e^{-ikpt/\epsilon_p}+a_\mu(\vec{\rm k})e^{ikpt/\epsilon_p}]e^{-\alpha kp/m^2}\ . \tag{2.9}$$

Using (2.9) to derive (2.7) allows the definition of finite propagators for finite α. It is necessary to augment (2.9) with a masslike counterterm

$$H^c(\alpha)=\delta m\int\frac{d^3p}{\epsilon_p}n(\vec{\rm p})\ , \tag{2.10}$$

where

$$n(\vec{\rm p})=\sum_{s=1}^{2}(b_{\vec{\rm p},s}^\dagger b_{\vec{\rm p},s}+d_{\vec{\rm p},s}^\dagger d_{\vec{\rm p},s}) \tag{2.11a}$$

and

$$\delta m=\frac{e^2m^2}{8\pi^2\alpha}\ . \tag{2.11b}$$

The nonlocal structure of (2.10) is due to the nonlocality of (2.1) [see (2.32) of this section]. As a result, the Coulomb phase operator $\beta(t)$, which appears in (2.7), causes the spinor operators, when normal ordered, to develop the phase

$$:\exp[i\beta(t)]b_{\vec{\rm k},s}\exp[-i\beta(t)]:=:\exp\left[-2ie^2\int d^3q\,f(\vec{\rm k},\vec{\rm q},t)\rho(\vec{\rm q})\right]:b_{\vec{\rm k},s}+b_{\vec{\rm k},s}\exp[-ie^2f(\vec{\rm k},\vec{\rm k},t)]\ . \tag{2.12}$$

Using the form of (2.8c) developed from (2.9), it follows that

$$e^2 f(\vec{p},\vec{p},t)=\frac{1}{\alpha}\frac{e^2m^2}{8\pi^2\epsilon_p}t\ , \tag{2.13}$$

hence, the forms (2.10) and (2.11). The fact that $n(\vec{p})$ and $\rho(\vec{p})$ commute allows the counterterm (2.11a) to cancel the phase (2.12) exactly. It is also necessary to define renormalized spinor fields

$$\phi_R^{\text{as}}=Z_{\text{IR}}^{-1/2}\phi^{\text{as}} \tag{2.14}$$

in order to remove divergences from the spinor propagator. This will be discussed in greater detail in the next section.

In order to derive the path integral it is necessary to introduce functional coherent states.[14] These are defined in the Fock basis by the formal statements

$$|f_\mu,\eta,\xi,t\rangle=\exp\left\{i\int d^3x[(f_\mu^\alpha+f_\mu^{\alpha *})\dot{a}_\mu-(\pi_\mu^f+\pi_\mu^{f*})a_\mu]\right\}\exp\left\{\int d^3x[(\xi^\dagger-\eta^\dagger)\phi-\phi^\dagger(\xi-\eta)]\right\}|0\rangle\ , \tag{2.15}$$

where a_μ and ϕ are given by (2.2) and (2.4), while $|0\rangle$ is the Fock vacuum. The functions f_μ^α and π_μ^f are assumed to have Fourier transforms with arbitrary time dependence,

$$f_\mu^\alpha=\int\frac{d^3k}{(2\pi)^{3/2}}(2\alpha_k)^{-1/2}f_\mu(\vec{k})e^{-\vec{k}\cdot\vec{x}+i\alpha_k t},\quad \pi_\mu^f=i\int\frac{d^3k}{(2\pi)^{3/2}}\frac{\omega_k}{(2\alpha_k)^{1/2}}f_\mu(\vec{k})e^{-i\vec{k}\cdot\vec{x}+i\alpha_k t} \tag{2.16}$$

while the ξ and η are Grassmann variables[15] with the expansions

$$\eta(x)=\int\frac{d^3p}{(2\pi)^{3/2}}\left[\frac{m}{\epsilon_p}\right]^{1/2}\sum_{s=1}^{2}c_{p,s}^*u_{p,s}(x),\quad \xi(x)=\int\frac{d^3p}{(2\pi)^{3/2}}\left[\frac{m}{\epsilon_p}\right]^{1/2}\sum_{s=1}^{2}d_{p,s}v_{p,s}(x)\ . \tag{2.17}$$

Insertion of these expansions into (2.15) allows the inner product to be evaluated with the result that

$$\langle f_\mu,\eta_1,\xi_1,t\,|\,g_\mu,\eta_2,\xi_2,t\rangle=\exp\left[\tfrac{1}{2}i\int d^3x(f_\mu^\alpha\pi^{\mu f*}-f_\mu^{\alpha *}\pi^{\mu f}+g_\mu^\beta\pi^{\mu g*}-g_\mu^{\beta *}\pi^{\mu g})-2f_\mu^{\alpha *}\pi^{\mu g}+2g_\mu\pi^{\mu f*})\right]$$
$$\times\exp\left[-\tfrac{1}{2}\int d^3x(\xi_1^\dagger\xi_1+\eta_1^\dagger\eta_1+\xi_2^\dagger\xi_2+\eta_2^\dagger\eta_2-2\xi_1^\dagger\xi_2-2\eta_1^\dagger\eta_2)\right]\ . \tag{2.18}$$

It is possible to define a form of completeness for these states. This is accomplished by breaking the integral in (2.19) into a sum over lattice cells of volume $\epsilon_{\vec{x}}^{\,3}$ and treating the value of each function at the lattice cell as an independent variable. If this is done, then the formal statement

$$\int[df_\mu d\eta\, d\xi]\,|f_\mu,\eta,\xi,t\rangle\langle f_\mu,\eta,\xi,t\,|=1 \tag{2.19}$$

follows, where the measure is given by

$$[df_\mu d\eta\, d\xi]=\prod_{\vec{x}}[(\epsilon_{\vec{x}}^{\,3})^{10}d\eta_x d\eta_x^\dagger d\xi_x d\xi_x^\dagger df_{\mu x}^* df_x^\mu d\pi_{\mu x}^* d\pi_x^\mu]\ . \tag{2.20}$$

The product runs over all lattice sites in the volume.

The measure defined in (2.20) implicitly sums over the negative-norm ghost states present in the Feynman-gauge formulation of QED. This can be seen explicitly by looking at the Fourier transform of (2.18). These states cause the inner product to become arbitrarily large, and so must be excluded from the measure (2.20). This is achieved by constraining the measure to only those states which satisfy the Gupta-Bleuler condition and calculating transition elements only between similarly constrained states. Such a procedure is straightforward in the noninteracting case, amounting to insertion of δ functions of the form $\delta(\vec{\nabla}\cdot\vec{f}+\pi_0^f)$ into the measure. In the interacting case complications arise and it is necessary to dress these states with the pseudo-unitary transformation defined by (1.3) and exhibited explicitly elsewhere.[7,12,13] This transformation decouples the transverse and ghost modes when charge is present and must be implemented for both the full interacting theory[12] and the abridged infrared version[7] of Faddeev and Kulish. The net effect of this is to cause the covariant formulation of QED to be equivalent dynamically to the Coulomb-gauge formulation.

It has been shown,[13] however, that failure to implement this procedure does not harm QED if the time-ordered product is used to calculate a process on the energy shell. This is because the ghost modes decouple in the Abelian case due to the fact that the photon does not carry electric charge. In order to simplify the remainder of this section procedure will be ignored. The diligent reader can verify that an analysis where this program is followed will simply result in a path integral with a Coulomb-gauge constraint, as in (1.4).

The path integral consistent with the choice of infrared coherent states as intermediate states will now be derived. Some preliminary considerations need to be addressed first. The derivation of the Hamiltonian which governs the time development of the asymptotic fields (2.5) is straightforward. If H_0 is the free Feynman-gauge Hamiltonian which satisfies

$$\frac{\partial}{\partial t}a_\mu = i[H_0, a_\mu] \,, \tag{2.21}$$

then the Hamiltonian H^{as}, which satisfies

$$\frac{\partial}{\partial t}a_\mu^{\text{as}} = i[H^{\text{as}}, a_\mu^{\text{as}}] \,, \tag{2.22}$$

is given by

$$H^{\text{as}} = UH_0U^{-1} - i\dot{U}U^{-1} \,. \tag{2.23}$$

The time development of the U operator is given by

$$U(t) = e^{iH^{\text{as}}t}U(0)e^{-iH_0 t} \,. \tag{2.24}$$

The completeness relation for the functional infrared coherent states is written formally as

$$\int [df_\mu d\eta\, d\xi]U(t)\,|\,f_\mu,\eta,\xi,t\,\rangle\langle\, f_\mu,\eta,\xi,t\,|\,U^{-1}(t) = 1 \,. \tag{2.25}$$

The path integral is defined as the vacuum-to-vacuum transition element. In the first case only the infrared form of the interaction will be considered, so that only the interaction given by (2.1) is present. In such a case the object of interest is

$$Z_{\text{IR}}(t_+, t_-) = \langle 0\,|\,U^{-1}(t_+)U(t_-)\,|\,0\rangle \,, \tag{2.26}$$

where $|\,0\rangle$ is the Fock vacuum. From the form of U given by (2.7) it follows that Z_{IR} is unity. The path-integral representation of (2.26) is obtained by partitioning the time interval $(t_+ - t_-)$ into N steps of arbitrarily small duration ϵ and inserting the projection operator of (2.25) at the respective times. This leaves $(N+1)$ matrix elements to be evaluated, where the jth matrix element then takes the form

$$Z_{j\epsilon} = \langle f_j^\mu, \eta_j, \xi_j, t_j\,|\,U^{-1}(t_j)U(t_j - \epsilon)\,|\,f_{j-1}^\mu, \eta_{j-1}, \xi_{j-1}, t_{j-1}\rangle \,. \tag{2.27}$$

Using (2.23) and (2.24) gives

$$U^{-1}(t)U(t-\epsilon) = \exp[-i\epsilon(H_0 - iU^{-1}\dot{U})]\exp(i\epsilon H_0) \,. \tag{2.28}$$

From the fact that

$$-iU^{-1}\dot{U} = i\dot{U}^{-1}U = H_I^{\text{as}} \tag{2.29}$$

and

$$\exp(i\epsilon H_0)\,|\,f_{\mu,j-1}, \eta_{j-1}, \xi_{j-1}, t_{j-1}\rangle = |\,f_{j-1}^\mu, \eta_{j-1}, \xi_{j-1}, t_j\rangle \,, \tag{2.30}$$

the matrix element (2.27) takes the form

$$Z_{j\epsilon} = \langle f_j^\mu, \eta_j, \xi_j, t_j\,|\,\exp[-i\epsilon(H_0 + H_I^{\text{as}})]\,|\,f_{j-1}^\mu, \eta_{j-1}, \xi_{j-1}, t_j\rangle \,. \tag{2.31}$$

In order to evaluate (2.31) H_I^{as} is written in configuration space as

$$H_I^{\text{as}} - :e\int d^3x\, d^3y\, d^3z\, a_\mu(\vec{x},t)Q^\mu(\vec{x},\vec{y},\vec{z},t)\phi_\alpha^\dagger(\vec{y},t)\phi_\alpha(\vec{z},t): \,, \tag{2.32}$$

where

$$Q_\mu(\vec{x},\vec{y},\vec{z},t) = \int \frac{d^3p}{(2\pi)^{9/2}}\frac{p_\mu}{\epsilon_p}\delta\left[\vec{x} - \frac{\vec{p}}{\epsilon_p}t\right]\exp[i\vec{p}\cdot(\vec{y}-\vec{z})] \,. \tag{2.33}$$

Next, the relations

$$\langle f_\mu,\eta_1,\xi_1,t\,|\,:[a_\mu(\vec{x},t)]^m:\,|\,g_\mu,\eta_2,\xi_2,t\rangle=[f_\mu^{\alpha *}(x)+g_\mu^{\beta}(x)]^m\langle f_\mu,\eta_1,\xi_1,t\,|\,g_\mu,\eta_2,\xi_2,t\rangle\ ,$$
$$\langle f_\mu,\eta_1,\xi_1,t\,|\,:[\dot{a}_\mu(\vec{x},t)]^m:\,|\,g_\mu,\eta_2,\xi_2,t\rangle=[\pi_\mu^{f *}(x)+\pi_\mu^{g}(x)]^m\langle f_\mu,\eta_1,\xi_1,t\,|\,g_\mu,\eta_2,\xi_2,t\rangle\ , \tag{2.34}$$
$$\langle f_\mu,\eta_1,\xi_1,t\,|\,\phi(\vec{x},t)\,|\,g_\mu,\eta_2,\xi_2,t\rangle=[\eta_2(x)+\xi_1(x)]\langle f_\mu,\eta_1,\xi_1,t\,|\,g_\mu,\eta_2,\xi_2,t\rangle\ ,$$
$$\langle f_\mu,\eta_1,\xi_1,t\,|\,\phi^\dagger(\vec{x},t)\,|\,g_\mu,\eta_2,\xi_2,t\rangle=[\eta_1^\dagger(x)+\xi_2^\dagger(x)]\langle f_\mu,\eta_1,\xi_1,t\,|\,g_\mu,\eta_2,\xi_2,t\rangle$$

are coupled with (2.32) to evaluate (2.31) in the limit that ϵ is arbitrarily small so that terms of $O(\epsilon^2)$ can be ignored. The result, using a shorthand notation, is

$$Z_{j\epsilon}\approx\exp[-i\epsilon H(f_\mu,\eta,\xi)]\langle f_{\mu j},\eta_j,\xi_j,t_j\,|\,f_{j-1}^{\mu},\eta_{j-1},\xi_{j-1},t_j\rangle\ , \tag{2.35}$$

where

$$H=H_0+H_I^{\rm as}\ . \tag{2.36}$$

Next, the π_μ integrations are performed and the new variables

$$A_\mu^j=f_\mu^j+f_\mu^{j*},\quad \tilde{A}_\mu^j=f_\mu^j-f_\mu^{j*},\quad \psi_j^\dagger=\xi_j^\dagger+\eta_j^\dagger,\quad \tilde{\psi}_j^\dagger=\xi_j^\dagger-\eta_j^\dagger,\quad \psi_j=\xi_j+\eta_j,\quad \tilde{\psi}_j=\xi_j-\eta_j \tag{2.37}$$

are defined. The tilde variables are integrated out and the formal definitions

$$A_{j+1}^\mu-A_j^\mu=\epsilon\dot{A}_j^\mu,\quad \psi_{j+1}-\psi_j=\epsilon\dot{\psi}_j \tag{2.38}$$

are made. The final result is

$$Z_{\rm IR}(t_+,t_-)=\int[dA_\mu d\psi^\dagger d\psi]\exp\left[i\int_{t_-}^{t_+}dt\int d^3x(\mathscr{L}_0+\mathscr{L}_I^{\rm as})\right]\ , \tag{2.39}$$

where

$$\int d^3x\,\mathscr{L}_I^{\rm as}=-e\int d^3x\,d^3y\,d^3z\,A_\mu(\vec{x},t)Q^\mu(\vec{x},\vec{y},\vec{z},t)\psi_a^\dagger(\vec{y},t)\psi_a(\vec{z},t) \tag{2.40}$$

and

$$[dA_\mu d\psi^\dagger d\psi]=\prod_{\vec{x},t}[\epsilon^{9/2}\pi^{-1/2}dA_\mu(\vec{x},t)d\psi^\dagger(\vec{x},t)d\psi(\vec{x},t)]\ , \tag{2.41}$$

with $\mathscr{L}_0$ the standard free Feynman-gauge Lagrangian.

In order to construct the path-integral representation of the full theory, the W operator must be introduced. W maps the interpolating fields into the infrared fields in the manner

$$W\psi W^{-1}=\phi^{\rm as},\quad WA_\mu W^{-1}=a_\mu^{\rm as}\ . \tag{2.42}$$

It then follows that the projection operator for the interpolating theory is given by

$$\int[df_\mu d\eta\, d\xi]W^{-1}U\,|\,f_\mu,\eta,\xi,t\rangle\langle f_\mu,\eta,\xi,t\,|\,U^{-1}W=1\ . \tag{2.43}$$

Using (1.2) and (2.5) it follows that

$$W=UZ^{-1}\ , \tag{2.44}$$

so that

$$U^{-1}W=Z^{-1}\ . \tag{2.45}$$

Result (2.45) shows that the projection operator (2.43) reduces to the one used previously[12] (modulo the implementation of the Gupta-Bleuler condition) to develop the QED path integral, so that

$$Z(t_+,t_-)\equiv\langle 0\,|\,Z(t_+)Z^{-1}(t_-)\,|\,0\rangle=\int[dA_\mu d\psi^\dagger d\psi]\exp\left[i\int_{t_-}^{t_+}dt\int d^3x(\mathscr{L}_0+\mathscr{L}_I)\right]\ , \tag{2.46}$$

where $\mathscr{L}_I$ is the standard interaction

$$\mathscr{L}_I = -eA^\mu \bar{\psi}\gamma_\mu \psi \ . \tag{2.47}$$

From the fact that, by construction,

$$\underset{t\to\pm\infty}{\text{weak-lim}}\, ZU = \lambda_\pm \ , \tag{2.48}$$

where $\lambda_\pm$ is a phase factor, it follows that the path integral in (2.46) will result regardless of the intermediate states used to construct it as long as those states can be related to the Fock basis by a unitary transformation.

Of course, the infrared divergences will appear in the path-integral formalism if (2.46) is expanded about $\mathscr{L}_0$. The solution is to perform the expansion of (2.46) about the nonquadratic path integral (2.39). This is accomplished by writing (2.46) as

$$Z(t_+,t_-) = \int [dA_\mu d\psi^\dagger d\psi] \exp\left\{ i \int_{t_-}^{t_+} dt \int d^3x [\mathscr{L}_0 + \mathscr{L}_I^{\text{as}} + (\mathscr{L}_I - \mathscr{L}_I^{\text{as}})] \right\} , \tag{2.49}$$

and treating $(\mathscr{L}_I - \mathscr{L}_I^{\text{as}})$ as the interaction. This exhibits a key aspect of the path-integral formalism: a change of interaction-picture basis states is manifested by splitting the interaction into two separate pieces and including one of these pieces in the "basis" path integral. At least in the case of QED the removal of infrared divergences has nothing to do with the measure appearing in the path integral.

In the next section it will be shown how the path integral (2.39) is evaluated nonperturbatively to obtain the infrared propagators.

III. NONPERTURBATIVE EVALUATION

The path integral is used primarily to evaluate time-ordered products of field operators. For example, the spinor propagator is given by

$$S_{\alpha\beta}(x-y) = \langle 0 | T[\bar{\psi}_\alpha(x)\psi_\beta(y)] | 0 \rangle = \int [dA_\mu d\psi^\dagger d\psi] \bar{\psi}_\alpha(x)\psi_\beta(y) \exp\left[i \int d^4x \mathscr{L} \right] . \tag{3.1}$$

For the purposes of this section consideration will be limited to the Lagrangian

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I^{\text{as}} , \tag{3.2}$$

where $\mathscr{L}_I^{\text{as}}$ is given by (2.40) and $\mathscr{L}_0$ is the free Feynman-gauge Lagrangian, given explicitly by

$$\mathscr{L}_0 = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}(\partial_\mu A^\mu)^2 + i\bar{\psi}(\gamma^\mu\partial_\mu + im)\psi \ . \tag{3.3}$$

It is understood that the results of path integrals such as (3.1) are to be used in a perturbative expansion of the full theory as expressed by (2.46).

The method usually employed to evaluate integrals such as (3.1) consists of adding external source terms, rewriting the interaction as a functional power series in these sources, and performing the remaining path integral to obtain a set of propagators and Feynman rules. Such an approach yields a power-series representation of time-ordered products whose summation is not straightforward.

The integral (3.1) can be analyzed nonperturbatively by the change of spinor variables given by

$$\psi_\alpha(\vec{x},t) = \int d^3y\, k(\vec{x},\vec{y},t)\phi_\alpha(\vec{y},t) , \tag{3.4}$$

where

$$k(\vec{x},\vec{y},t) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} e^{iC_{\vec{p}}(t)} , \tag{3.5}$$

with

$$C_{\vec{p}}(t) = \tfrac{1}{2}e \int d^3x'dt' [\theta(t-t') - \theta(t'-t)] \delta\left[\vec{x}' - \frac{\vec{p}}{\epsilon_p}t' \right] \left[\frac{p_\mu}{\epsilon_p} \right] A^\mu(\vec{x}',t') . \tag{3.6}$$

For later convenience $C_{\vec{p}}(t)$ will be written

$$C_{\vec{p}}(t) = \int d^3x'dt' C^\mu_{\vec{p}}(\vec{x}',t',t) A_\mu(\vec{x}',t') , \tag{3.7}$$

where

$$C^{\mu}_{\vec{p}}(\vec{x}\,',t',t)=\tfrac{1}{2}e[\theta(t-t')-\theta(t'-t)]\left[\frac{p^{\mu}}{\epsilon_p}\right]\delta\left[\vec{x}\,'-\frac{\vec{p}}{\epsilon_p}t'\right] . \tag{3.8}$$

The transformation (3.4) does not change the form of $\mathscr{L}_I^{\rm as}$, rather causing it to become

$$\int d^4x\,\mathscr{L}_I^{\rm as}=-e\int d^3x\,d^3y\,d^3z\,dt\,A_\mu(\vec{x},t)Q^{\mu}(\vec{x},\vec{y},\vec{z},t)\psi^{\dagger}(\vec{y},t)\psi(\vec{z},t)$$
$$\rightarrow -e\int d^3x\,d^3y\,d^3z\,dt\,A_\mu(\vec{x},t)Q^{\mu}(\vec{x},\vec{y},\vec{z},t)\phi^{\dagger}(\vec{y},t)\phi(\vec{z},t)\equiv\int d^4x\,\mathscr{L}_I^{\rm as'} , \tag{3.9}$$

while the kinetic spinor terms become

$$\int d^4x[i\bar{\psi}(\gamma^{\mu}\partial_\mu+im)\psi]\rightarrow\int d^4x[i\bar{\phi}(\gamma^{\mu}\partial_\mu+im)\phi-\mathscr{L}_i^{\rm as'}] . \tag{3.10}$$

The transformation (3.4) decouples the spinor field from the photon field, so that the Lagrangian becomes

$$\mathscr{L}'=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}(\partial_\mu A^{\mu})^2+i\bar{\phi}(\gamma^{\mu}\partial_\mu+im)\phi . \tag{3.11}$$

The measure appearing in (3.1) becomes

$$[dA_\mu d\psi^{\dagger}d\psi]=[dA^{\mu}d\phi^{\dagger}d\phi](\det k\,\det k^*)^{-1} . \tag{3.12}$$

It is easy to show that

$$\det k\,\det k^*=\det kk^*=\det\delta(\vec{x}-\vec{y})=1 , \tag{3.13}$$

so that the measure is invariant. This is simply a reflection of the fact that the transformation (3.4) mimics the action of the transformation (2.7) which is formally unitary.

The phase (3.7), which is proportional to A_μ, *may now be absorbed into the action to yield an exactly integrable path integral.* Relation (3.1) becomes

$$S_{\alpha\beta}(x-y)=\int d^3x'd^3y'd^3p\,d^3k\,e^{-i\vec{p}\cdot(\vec{x}-\vec{x}\,')}e^{i\vec{k}\cdot(\vec{y}-\vec{y}\,')}$$
$$\times\int[dA_\mu d\phi^{\dagger}d\phi]\bar{\phi}_\alpha(\vec{x}\,',t_x)\phi_\beta(\vec{y}\,',t_y)$$
$$\times\exp\left[i\int d^4z\{\mathscr{L}'+[C^{\mu}_{\vec{k}}(\vec{z},t_z,t_y)-C^{\mu}_{\vec{p}}(\vec{z},t_z,t_x)]A_\mu(z)\}\right] , \tag{3.14}$$

where C^{μ} is defined by (3.8). First the spinor variables are integrated, and then the integrations over the x',y', and k variables are performed. The result is

$$S_{\alpha\beta}(x-y)=-i\int\frac{d^3p}{(2\pi)^3}\left[\frac{m}{\epsilon_p}\right]\{\theta(t_x-t_y)\Lambda^{(+)}_{\alpha\beta}(p)\exp[-ip(x-y)+g_{\vec{p}}(t_x,t_y)]$$
$$+\theta(t_y-t_x)\Lambda^{(-)}_{\alpha\beta}(p)\exp[ip(x-y)+g_{-\vec{p}}(t_x,t_y)]\} , \tag{3.15}$$

where $\Lambda^{(+)}$ and $\Lambda^{(-)}$ are the standard spinor energy projection operators

$$\Lambda^{(+)}_{\alpha\beta}(p)=\left[\frac{\not p+m}{2m}\right]_{\alpha\beta} , \quad \Lambda^{(-)}_{\alpha\beta}(p)=\left[\frac{-\not p+m}{2m}\right]_{\alpha\beta} , \tag{3.16}$$

while the phase $g_{\vec{p}}$ is given by

$$g_{\vec{p}}(t_x,t_y)=\ln\int[dA_\mu]\exp\left[i\int d^4z\{-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}(\partial_\mu A^{\mu})^2+[C^{\mu}_{\vec{p}}(z,t_y)-C^{\mu}_{\vec{p}}(z,t_x)]A_\mu(x)\}\right] . \tag{3.17}$$

The path integral (3.17) is evaluated by the change of variable

$$A_\mu(z)=A'_\mu(z)-\int d^4z'\Delta_{\mu\nu}(z-z')[C^{\nu}_{\vec{p}}(\vec{z}\,',t'_z,t_y)-C^{\nu}_{\vec{p}}(\vec{z}\,',t'_z,t_x)] , \tag{3.18}$$

where $\Delta_{\mu\nu}$ is the causal Green's function which satisfies

$$\Box\Delta_{\mu\nu}(z-z')=g_{\mu\nu}\delta^4(z-z') , \tag{3.19}$$

with the explicit representation

$$\Delta_{\mu\nu}(z-z')=ig_{\mu\nu}\int \frac{d^3k}{(2\pi)^3}(2\omega_k)^{-1}[\theta(t-t')e^{ik(z-z')}+\theta(t'-t)e^{-ik(z-z')}] . \tag{3.20}$$

The phase then becomes

$$g_{\vec{p}}(t_x,t_y)=\tfrac{1}{2}i\int d^4z\, d^4z'[C^{\mu}_{\vec{p}}(z,t_y)-C^{\mu}_{\vec{p}}(z,t_x)]\Delta_{\mu\nu}(z-z')[C^{\nu}_{\vec{p}}(z',t_y)-C^{\nu}_{\vec{p}}(z',t_x)] . \tag{3.21}$$

Expression can be simplified by using (3.8) and (3.20) to obtain

$$\begin{aligned} g_{\vec{p}}(t_x,t_y)=&-\frac{e^2}{(2\pi)^3}\int \frac{d^3k}{2\omega_k}\frac{m^2}{(kp)^2}[\theta(t_y-t_x)e^{i(kp/\epsilon_p)(t_x-t_y)}+\theta(t_x-t_y)e^{i(kp/\epsilon_p)(t_x-t_y)}-1] \\ &+\int \frac{d^3k}{(2\pi)^3}\frac{e^2m^2}{2\omega_k\epsilon_p(kp)}[i(t_x-t_y)\theta(t_x-t_y)-i(t_x-t_y)\theta(t_y-t_x)] . \end{aligned} \tag{3.22}$$

The second term on the right side of (3.22) is recognized as the cutoff-free version of (2.13), so that it is necessary to add the counterterm (2.10) to the path-integral version of the theory to remove this divergent phase. Adding the counterterm (2.10) has the effect of inserting the same phase with opposite sign into the spinor transformation (3.4), so that, in the cutoff case, the second term in (3.22) is canceled. This term will be excluded from further consideration.

The first term on the right side of (3.22), when combined with the step functions appearing in (3.15), yields the standard result known from the operator approach. The integrals are divergent and are evaluated by the introduction of the covariant cutoff discussed in the previous section. Omitting the details, the result is

$$\exp[\theta(t_y-t_x)g_{\vec{p}}(t_x,t_y)]=(i\alpha)^{\beta}\left[\frac{\epsilon_p}{m^2}\right]^{-\beta}\exp[i\pi\beta\theta(t_y-t_x)]\,|\,t_x-t_y\,|^{\beta},\quad \beta=\frac{e^2}{4\pi^2} . \tag{3.23}$$

This shows the well-known result that the multiplicative renormalization (2.14) of the spinor fields gives an ultraviolet-finite result when

$$Z_{\rm IR}=\alpha^{\beta} . \tag{3.24}$$

This shows the standard result that $Z_{\rm IR}\to 0$ as the cutoff is removed. Recalling that Z gives the probability of finding a single bare particle in the physical single-particle state this result simply reflects the fact that U is an improper transformation.

It is easy to see that the photon propagator is unchanged, so that

$$\langle 0\,|\,T[A_\mu(x)A_\nu(x')]\,|\,0\rangle=i\Delta_{\mu\nu}(x-x') . \tag{3.25}$$

Higher-order time-ordered products can be analyzed by the same transformations (3.4) and (3.18). For example, it is straightforward to show that

$$G_{\mu\alpha\beta}(x,y,z)=\langle 0\,|\,T[\bar\psi_\alpha(x)\psi_\beta(y)A_\mu(z)]\,|\,0\rangle=\int [dA_\mu d\psi^\dagger d\psi]\bar\psi_\alpha(x)\psi_\beta(y)A_\mu(z)\exp\left[i\int d^4w\,\mathscr{L}(w)\right] \tag{3.26}$$

can be evaluated to obtain

$$\begin{aligned} G_{\mu\alpha\beta}=i\int \frac{d^3p}{(2\pi)^3}\left[\frac{m}{\epsilon_p}\right]\{&\,\theta(t_x-t_y)\Lambda^{(+)}_{\alpha\beta}(p)\Gamma_\mu(z,t_x,t_y,\vec{p})\exp[-ip(x-y)+g_{\vec{p}}(t_x,t_y)] \\ &+\theta(t_y-t_x)\Lambda^{(-)}_{\alpha\beta}(p)\Gamma_\mu(z,t_x,t_y,-\vec{p})\exp[ip(x-y)+g_{-\vec{p}}(t_x,t_y)]\} , \end{aligned} \tag{3.27}$$

where

$$\Gamma_\mu(z,t_x,t_y,\vec{p})=\int d^4z'\Delta_{\mu\nu}(z-z')[C^{\nu}_{\vec{p}}(z',t_y)-C^{\nu}_{\vec{p}}(z',t_x)] . \tag{3.28}$$

All others follow similarly.

IV. CONCLUSIONS

It has been the intent of this paper to relate the infrared-coherent-state formalism developed by Faddeev and Kulish to the path-integral formulation of QED. The results of this program will now be summarized and discussed.

It was shown that the path-integral representation of QED developed by using intermediate infrared coherent states is identical to the path integral developed when the standard Fock states are employed. There are two reasons for such a result. The first is that the transformation (2.7) on the charged states is formally unitary. This guarantees that the measure appearing in the path integral will be unchanged. The second reason is that this transformation leaves the vacuum unchanged, so that the same transition element is calculated by either representation. Of course, these aspects are intimately related to the Abelian structure of QED which prevents nonlinear terms in the gauge field from appearing in the action. The absence of nonlinearity allows exact evaluation of (2.6) and a determination of the *on-shell* infrared structure of QED. Neither reason would be present if the ground state of the system was composed of pairs, as in the BCS theory of superconductivity.

The second major result is that the path integral allows a nonperturbative evaluation by a change of spinor variables which reduces the action to a quadratic form. This nonperturbative analysis should allow an extension of the infrared program to the full renormalized theory. Such a procedure would begin with the usual path integral written in terms of bare fields. The interaction (2.40) would then be added and subtracted from the action and the renormalized fields could be defined. It is assumed, although unverified, that the wave-function renormalization constant would factorize into two pieces, one removing the familiar ultraviolet divergence of the standard theory while the other would remove the divergences introduced by the new vertex. Such a procedure should leave the perturbation series both infrared and ultraviolet finite. Explicit verification of this remark will be deferred.

At a deeper level the phase which the spinor propagator develops due to the infrared behavior of the theory is related to the *classical* solution for the gauge field in the presence of a point source moving at a constant velocity. The current associated with the point source is given by (3.8) and the classical solution appears in the transformation (3.18). It is easy to see that the spinor—gauge-boson vertex in gauge-invariant theories always takes the asymptotic form (2.1), modulo complications due to isospin. It is then clear that the classical solution in the presence of a point source will dominate the functional. It is hoped that such an analysis will produce an insight into the infrared structure of massless nonlinear theories.

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[1]T. D. Lee and M. Nauenberg, Phys. Rev. 133, 1549 (1964).

[2]D. Yennie, S. Frautschi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961).

[3]J. Cornwall and G. Tiktopoulos, Phys. Rev. D 13, 3370 (1976); V. Ganapathi and G. Sterman, *ibid.* 23, 2408 (1981).

[4]F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).

[5]J. D. Dollard, J. Math. Phys. 5, 729 (1964).

[6]L. D. Faddeev and P. P. Kulish, Teor. Mat. Fiz. 4, 153 (1970) [Theor. Math. Phys. 4, 745 (1970)].

[7]M. S. Swanson, Phys. Rev. D 25, 2086 (1982).

[8]S. N. Gupta, Proc. Phys. Soc. London 63, 681 (1950); K. Bleuler, Helv. Phys. Acta 23, 567 (1950).

[9]For a discussion of many aspects of the coherent state formalism see N. Papanicolaou, Phys. Rep. 24C, 229 (1976).

[10]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path-Integrals* (McGraw-Hill, New York, 1965); E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

[11]J. R. Klauder, Ann. Phys. (N.Y.) 11, 123 (1960); S. S. Schweber, J. Math. Phys. 3, 831 (1962); L. D. Faddeev, Teor. Mat. Fiz. 1, 3 (1969) [Theor. Math. Phys. 1, 1 (1969)]; F. A Berezin, *ibid.* 6, 146 (1971) [*ibid.* 6, 194 (1971); C. L. Hammer, J. C. Schrauner, and B. DeFacio, Phys. Rev. D 18, 373 (1978); 19, 667 (1979).

[12]M. S. Swanson, Phys. Rev. D 24, 2132 (1981).

[13]K. Haller, Nucl. Phys. B57, 589 (1973); Acta Phys. Austriaca 42, 163 (1975).

[14]R. J. Glauber, Phys. Rev. 131, 2766 (1963); J. R. Klauder and E. C. G. Sudarshan. *Fundamentals of Quantum Optics* (Benjamin, New York, 1968); F. Rohrlich, in *Analytical Methods in Mathematical Physics* (Gordon and Breach, New York, 1970); B. DeFacio and C. L. Hammer, J. Math. Phys. 17, 267 (1976).

[15]H. Grassmann, *Hermann Grassmanns Gesammelte Mathematische and Physikalischi Werke* (Teubner, Leipzig, 1894), Vols. 11 and 12; M. Berezin, *The Method of Second Quantization* (Academic, New York, 1966); J. Rzewuski, *Quantum Field Theory II* (Hefner, New York, 1968).