Classical scalar solitons in three space dimensions

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A systematic study of a class of covariant scalar field Lagrangians with derivative self-coupling is made. Methods are found which allow for the selection of classes of Lagrangian models with three-space-dimensional soliton solutions. These methods allow, in particular, for the elaboration of models with solutions which are finite-energy confining potentials.

I. INTRODUCTION

There is no universally accepted definition of the concept of soliton. The most restrictive definition requires the stability of the soliton in two (or more) soliton collision processes.[1] The existence of this kind of soliton has been proved for a restricted set of nonlinear field equations in one space dimension.[2,3] The mathematical difficulties present in the search for such solitons makes this definition too restrictive at present, at least, concerning its usefulness in field theory.

In classical field theory the following definition is generally accepted: A soliton is a static regular (C^m class) finite-energy solution of the field equations. This is the Coleman "lump."[4] In this paper, we shall adopt this definition, but we mention the existence of other ones.[5,6]

With this definition, one-dimensional models with soliton solutions can be easily elaborated, but their main interest is for testing useful properties with the hope of their eventual extrapolation to more realistic three-dimensional models.

When we consider three-(space-) dimensional models, Derrick's theorem[7] shows the nonexistence of scalar solitons for Lagrangians with nonderivative self-coupling.

Accordingly, many authors have checked for (and found) more complicated (and physically realistic) three-dimensional models with soliton solutions in spontaneously broken gauge field theories.[8–10] Here, the existence and stability of the soliton solution is due to the presence of topological conservation laws in the model.[11]

Another way for circumventing Derrick's theorem has been used by Deser, Duff, and Isham[12] (DDI). These authors start with the usual nonlinear chiral $SU(2)\times SU(2)$ Lagrangian for the pion field and construct a new model where the new Lagrangian is a function of the former one. If this function is properly chosen, the new model has soliton solutions and conserves the primitive internal symmetries.

In this paper we explore systematically the consequences of this last procedure in order to elaborate models with three-dimensional soliton solutions. Our study deals with the pure scalar fields only thereby avoiding the complexities introduced by any internal structure, but, in many cases, the procedure can be extended to more complicated fields, as is proved by the Born-Infeld[13] and DDI models.

In Sec. II, we analyze Lagrangians defined as arbitrary functions of the scalar $\partial_\mu\phi\partial^\mu\phi$ and we elaborate strategies for selecting those models with three-(space-) dimensional soliton solutions. Solitons obtained in this way are spherically symmetric, sharp-pointed "lumps"; that is to say, there is a jump in the first derivative of the potential $\phi(r)$ at the origin. As classical Lagrangian these are of no interest. We consider them as models for the effective Lagrangian of the self-interacting scalar field. It is an open question whether or not such Lagrangians, which include quantum effects, permit soliton solutions even though the corresponding classical Lagrangians do not.

In Sec. III we study the models that can be obtained from the former ones by changes in the field variables in the Lagrangian. In this way, we show the existence of well-behaved solitons.

The existence and preservation of all these solitons can be understood by the fact that some quantity (the first derivative of the Lagrangian) must have an infinite jump if the field is to decay on to the vacuum state.

In Sec. IV we analyze the linear stability of these solutions.

In Sec. V we briefly comment on the possibility of generalization of these techniques to a multicomponent scalar field.

In what follows we make use of the following conventions: the signature of the metric is $(+---)$. Dots on functions indicate differentiation with respect to their field arguments. Primes on functions indicate differentiation with respect to the coordinates.

II. A FAMILY OF SCALAR FIELD MODELS

Derrick's theorem asserts the nonexistence of solitons in three-dimensional scalar field models where the Lagrangian has the generic form

$$L=\partial_\mu\phi\partial^\mu\phi+V(\phi)\ , \tag{2.1}$$

$V(X)$ being an arbitrary function.

We will study the simplest scalar generalization of (2.1) avoiding Derrick's hypothesis, that is to say, Lagrangians of the general form

$$L=f(\partial_\mu\phi\partial^\mu\phi)\ , \tag{2.2}$$

$f(X)$ being an arbitrary function.

The associated field equation is

$$\partial_\mu[\dot{f}(\partial_\nu\phi\partial^\nu\phi)\partial^\mu\phi]=0 \tag{2.3a}$$

or

$$\dot{f}(\partial_\nu\phi\partial^\nu\phi)\partial_\mu\partial^\mu\phi+2\ddot{f}(\partial_\nu\phi\partial^\nu\phi)\partial^\mu\phi\partial^\sigma\phi\partial_\mu\partial_\sigma\phi=0 \tag{2.3b}$$

and the corresponding canonical energy tensor is

$$T^{\mu\nu}=-f(\partial_\alpha\phi\partial^\alpha\phi)g^{\mu\nu}+2\dot{f}(\partial_\alpha\phi\partial^\alpha\phi)\partial^\mu\phi\partial^\nu\phi\ . \tag{2.4}$$

If we look for static spherically symmetric (elementary) solutions of (2.3), we have to solve the equation

$$\frac{\phi''}{\phi'}+2\frac{\ddot{f}(-\phi'^2)}{\dot{f}(-\phi'^2)}\phi'\phi''+\frac{2}{r}=0 \tag{2.5}$$

which has one first integral

$$r^2\phi'\dot{f}(-\phi'^2)=\Lambda\ , \tag{2.6}$$

Λ being an arbitrary constant.

The energy density for this solution is

$$T^{00}=-f(-\phi'^2) \tag{2.7}$$

and the total energy is

$$E=-4\pi\int_0^\infty r^2 f(-\phi'^2(r))dr\ . \tag{2.8}$$

From Eq. (2.6) we obtain

$$\frac{df}{dr}=-2\Lambda\frac{\phi''(r)}{r^2} \tag{2.9}$$

and

$$f(r)=-2\Lambda\int\frac{\phi''(r)}{r^2}dr+\text{const}\ . \tag{2.10}$$

Equations (2.6) to (2.9) suggest strategies for elaborating models with finite-energy solutions. As a first strategy, we have the following:

(a) Choose a monotonic (invertible) function $\phi'(r)$. If its derivative $\phi''(r)$ vanishes at infinity faster than $1/r$ [and $f(0)=0$], Eq. (2.10) shows the convergence of the integral (2.8). We also choose $\phi'(r)$ so that $\phi''(0)=0$ in order to have regularity for the fields everywhere.

(b) The knowledge of the inverse function $r(\phi')$ and the integration of (2.6) allows us to obtain the function $f(x)$ and, thus, the form of the Lagrangian.

In this way, we can construct good sharp-pointed solitons, but the corresponding field theory is not always admissible, because the Lagrangian functional is not defined outside the class of solutions[14] or the energy functional is not positive definite everywhere.

As an example, we start with the function

$$\phi'(r)=\left[\frac{1}{r^4+\mu^2}\right]^{1/2} \tag{2.11}$$

satisfying the requirements of (a); following (b) we obtain the corresponding Lagrangian

$$L=\frac{1}{\mu^2}[(1+\mu^2\partial_\mu\phi\partial^\mu\phi)^{1/2}-1]\ . \tag{2.12}$$

This is the scalar version of the well-known Born-Infeld generalization of electrodynamics.[13] As the parameter μ goes to zero, (2.12) reduces to the d'Alembertian Lagrangian.

The corresponding energy functional (2.4) is positive definite in the set on which the Lagrangian is well defined, and the total energy of the elementary solution (2.11) is finite. But the Lagrangian functional is not defined when

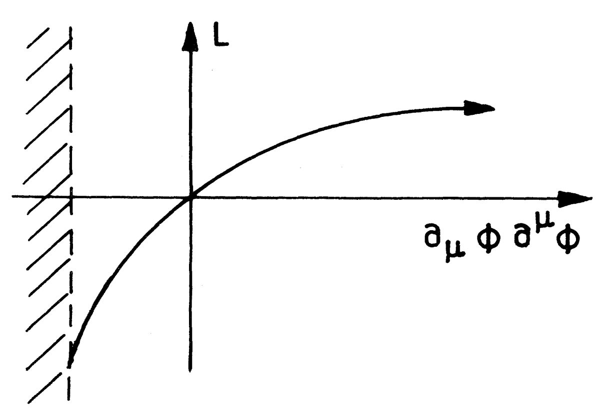

FIG. 1. Functional form of the scalar Born-Infeld Lagrangian.

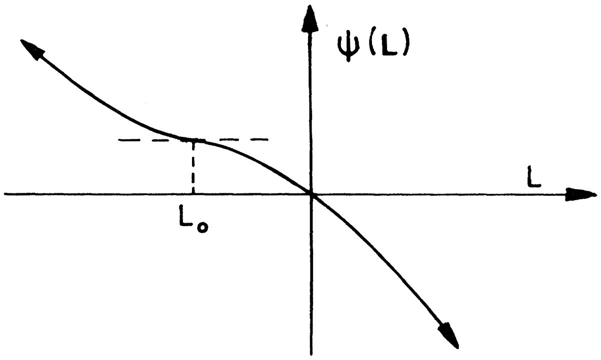

FIG. 2. Implicit form of the Lagrangian function for scalar models with soliton solution.

$$\partial_\nu\phi\partial^\nu\phi < -\frac{1}{\mu^2}$$

(see Fig. 1). In order to overcome these difficulties, we shall set up a second strategy which leads directly to models with a good Lagrangian and good energetic behavior [though there is always an unavoidable singularity of the first derivative of the Lagrangian at the center of the soliton, as can easily be seen from (2.6) and (2.9)]:

(a) First of all, we choose L as a function of r: $L(r)=f(-\phi'^2(r))$ in the implicit form,

$$r^2=-\frac{d}{dL}[\sqrt{\psi(L)}]=-\frac{\dot{\psi}(L)}{2\sqrt{\psi(L)}} \tag{2.13}$$

with $\psi(L)$ being a monotonic (invertible) function defined everywhere. Moreover, we impose $\psi(0)=0$; $\dot{\psi}(0)<0$ and $\dot{\psi}(L_0)=\ddot{\psi}(L_0)=0$, L_0 being a negative number (see Fig. 2). This last condition guarantees that $L(r)$ decreases from L_0 when $r=0$ and vanishes as $1/r^4$ when $r\to\infty$ making convergent the energy integral (2.8).

(b) Using Eq. (2.9) in the form

$$\int r^2 dL=-2\Lambda\phi'(r)$$

we obtain

$$\psi(L)=4\Lambda^2\phi'^2(r)=-4\Lambda^2\partial_\mu\phi\partial^\mu\phi \tag{2.14}$$

(we can now choose $\Lambda=\frac{1}{2}$ without loss of generality) and

$$L=\psi^{-1}(-\partial_\mu\phi\partial^\mu\phi) \tag{2.15}$$

gives us the form of the Lagrangian.

(c) By elimination of L between (2.13) and (2.14), we obtain the solution depending on the arbitrary constant Λ and, thus, the general elementary solution of the model.

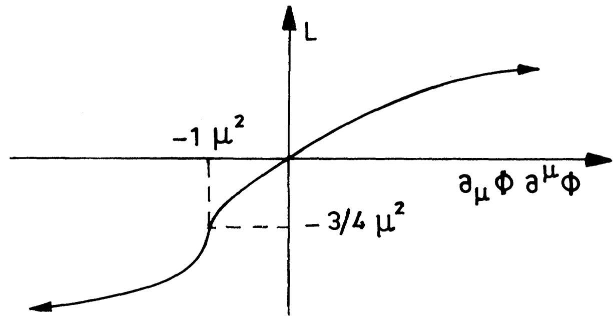

FIG. 3. Functional form of the Lagrangian of Eq. (2.17)

To see one example of how the method works, choose the function

$$\psi(L)=\frac{1}{\mu^2}\left[1-\left[1+\frac{4\mu^2}{3}L\right]^{3/2}\right] \tag{2.16}$$

which satisfies the conditions of (a).

Following (b) we obtain the Lagrangian

$$L=\frac{3}{4\mu^2}[(1+\mu^2\partial_\nu\phi\partial^\nu\phi)^{2/3}-1] \tag{2.17}$$

(see Fig. 3). This Lagrangian functional has the following properties:

(i) It is defined and regular for every field ϕ (except in $\partial_\mu\phi\partial^\mu\phi=-1/\mu^2$ where $\dot{L}$ diverges).

(ii) The energy functional obtained from (2.17) is positive definite everywhere as can be easily shown from (2.4) and (2.17).

(iii) As in the scalar Born-Infeld model, (2.17) can be interpreted as a nonlinear generalization of the d'Alembertian Lagrangian:

$$\lim_{\mu\to 0} L=\tfrac{1}{2}\partial_\nu\phi\partial^\nu\phi\ .$$

This suggests the possibility of elaborating a nonlinear electromagnetic theory from (2.17) by just following the inverse way which leads from the original Born-Infeld theory to the scalar one (2.12).

Now, following (c) we obtain the expression of the soliton field in implicit form

$$\frac{r^2}{\Lambda}=\frac{(1-\mu^2\phi'^2)^{1/3}}{\phi'}\ . \tag{2.18}$$

Obtaining the explicit expression requires the resolution of a third-degree irreducible algebraic equation. The final form is

$$\phi'(r)=\frac{\Lambda}{\text{P.R.}\left\{\left[\frac{r^6}{2}+\left[\frac{r^{12}}{4}-\frac{\mu^6\Lambda^6}{27}\right]^{1/2}\right]^{1/3}+\left[\frac{r^6}{2}-\left[\frac{r^{12}}{4}-\frac{r^6\Lambda^6}{27}\right]^{1/2}\right]^{1/3}\right\}}\ . \tag{2.19}$$

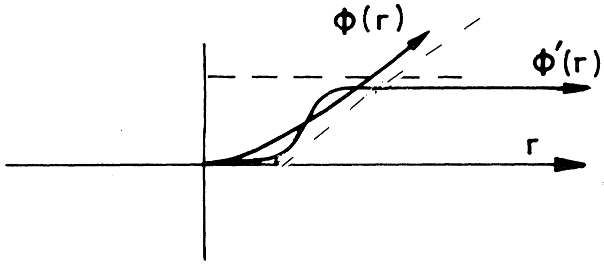

FIG. 4. Qualitative form of the potential and the field of a scalar confining potential.

Here, the symbol P.R. means "the only positive real root" of the expression between curly brackets.

As it should $\phi'(r)$ reduces to the d'Alembertian elementary solution (Λ/r^2) in the limit $\mu=0$. The soliton potential $\phi(r)$ is obtained by simply integrating (2.19). After a straightforward manipulation, we now obtain the soliton energy from (2.8) and (2.13):

$$E=2.456\pi\left\{\left|\frac{\Lambda^3}{\mu}\right|\right\}^{1/2} . \qquad (2.20)$$

As it should this energy diverges when we go to the d'Alembertian limit $(\mu=0)$.

We now explore the possibility of constructing in this way models with a finite-energy confining potential as elementary solutions. Now, the potential $\phi(r)$ diverges asymptotically as r increases and the field $\phi'(r)$ approaches asymptotically to a constant value $\phi'(\infty)$ (Fig. 4). The potential becomes a confining one by a suitable coupling with the matter field (for example, a Yukawa coupling). If the energy is to be finite, we see from (2.8) and (2.10) that $\phi'(r)$ must approach its asymptote faster than $1/r$. Moreover, Eq. (2.6) shows that the Lagrangian function must have a vertical tangent at the origin and a horizontal one at $\phi'(\infty)$, the asymptotic value of

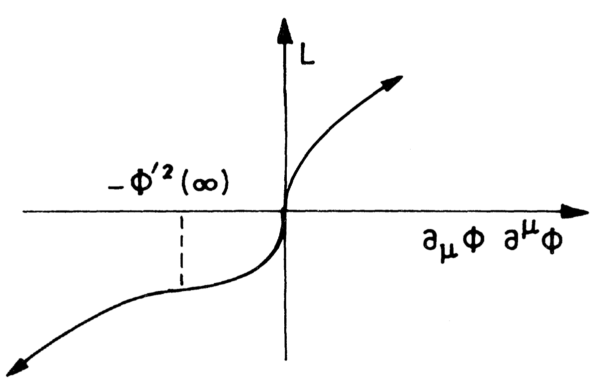

FIG. 5. Functional form of a Lagrangian with confining potential as elementary solution.

$\phi'(r)$. Furthermore, these points must be inflection points if the energy functional is to be positive definite (Fig. 5). By expanding this Lagrangian around the horizontal tangent point and using (2.6) we can easily show that $\phi'(r)$ approaches its asymptote as $1/r$ at best and, thus, the energy associated with the elementary solution is not finite. The conclusion is the incompatibility between finite energy and good behavior of the energy and Lagrangian functionals for confining potentials as elementary solutions of models of the family (2.2). In the next section we elaborate on methods allowing the construction of good models with a confining-potential soliton solution.

III. FIELD TRANSFORMATIONS

In this section we shall see the possibilities of construction of new models from Lagrangians of the form (2.2) by field transformations, that is to say, by making transformations to new fields obtained as functions of the previous ones.

If we define the field $\bar{\phi}$ by

$$\bar{\phi}(x)=\eta^{-1}(\phi(x)) , \qquad (3.1)$$

where η is a well-behaved arbitrary invertible function, the Lagrangian (2.2) becomes

$$L=f[\dot{\eta}^2(\bar{\phi})\partial_\mu\bar{\phi}\partial^\mu\bar{\phi}] . \qquad (3.2)$$

The associated field equations are obtained by varying (3.2) or from (2.3) by simply substituting (3.1), and the corresponding solutions are, thus, related by the same formula. It is easily seen that the character of the energy functional is conserved after the transformation. The elementary solution corresponding to the Lagrangian (3.2) is thus directly obtained from

$$\bar{\phi}(r)=\eta^{-1}(\phi(r)) \qquad (3.3)$$

and the associated total energy is the same in both cases [all these properties can be easily proved by combining (3.1) and the elementary properties of the functional derivative]. Thus the construction of Lagrangian scalar models of the form (3.2) with soliton solutions is automatically reduced to the same problem as in case (2.2).

Further, we can choose the function $\eta(\bar{\phi})$ in order to obtain models with solitons of any prescribed form. In this way, suppose we want to obtain a model where the elementary solution is the regular invertible function $\bar{\phi}(r)$. We start with a good model of the (2.2) family with elementary solution $\phi(r)$ which is a soliton [for example, (2.17)], $\phi(r)$ being the primitive of (2.19). We call $R(\bar{\phi})$ the inverse function of $\bar{\phi}(r)$ and we construct the transformation

$$\phi(x)=\eta(\bar{\phi}(x))=\phi\{R[\bar{\phi}(x)]\} . \tag{3.4}$$

By substitution, the Lagrangian (2.17) becomes

$$L=\frac{3}{4\mu^2}\{[1+\mu^2\dot{\eta}^2(\bar{\phi})\partial_\nu\bar{\phi}\partial^\nu\bar{\phi}]^{2/3}-1\} \tag{3.5}$$

and the associated elementary solution is the prescribed function $\bar{\phi}(r)$. This solution is a soliton of the model (3.5) and the associated total energy is the same (2.20) as in the starting case.

In particular, we can choose $\phi(r)$ as any confining potential and thus find field-theoretical models for the hadronic potentials with finite energy. (For the use of confining potentials in hadron physics see Ref. 15 and references quoted therein.)

In order to restrict the generality of this field transformation we shall impose on the transformed Lagrangian the condition of semilinearity; that is to say, the final Lagrangian must be invariant under the transformations

$$\phi\to\alpha\phi , \tag{3.6}$$

α being an arbitrary real constant.

If we start with a Lagrangian (2.2) with associated elementary solution $\phi(r)$, after a change of the generic form (3.4), the new Lagrangian is

$$L=f(\phi'^2(R(\bar{\phi}))\dot{R}^2(\bar{\phi})\partial_\mu\bar{\phi}\partial^\mu\bar{\phi}) . \tag{3.7}$$

The new Lagrangian is invariant under (3.6) if

$$\phi'(R(\bar{\phi}))\dot{R}(\bar{\phi})=\frac{1}{K\bar{\phi}} , \tag{3.8}$$

K being an arbitrary real constant. Equation (3.8) implies

$$K\phi(R(\bar{\phi}))=\ln\left[\frac{\bar{\phi}}{\bar{\phi}_0}\right] ,$$

$\bar{\phi}_0$ being a new arbitrary real constant.

Thus, the new Lagrangian is

$$L=f\left[\frac{\partial_\mu\bar{\phi}\partial^\mu\bar{\phi}}{K^2\bar{\phi}^2}\right] \tag{3.9}$$

and the associated elementary solution is obtained from the initial one by

$$\bar{\phi}(r)=\bar{\phi}_0e^{K\phi(r)} . \tag{3.10}$$

The transformation (3.10) is the generalization to a general Lagrangian (2.2) of Bel semilinearity transformation defined in the d'Alembertian case.[15]

IV. STABILITY

A detailed study of the linear stability of the elementary solutions of the field equations of the type (2.3) requires the solution of the linear equation

$$\partial_\mu[\dot{f}(\partial_\nu\phi_0\partial^\nu\phi_0)\partial^\mu\phi + \ddot{f}(\partial_\nu\phi_0\partial^\nu\phi_0)\partial_\sigma\phi_0\partial^\sigma\phi\partial^\mu\phi_0]=0 \tag{4.1}$$

obtained by the perturbation $\phi_0(r)+\epsilon\phi(x)$ of the elementary solution and neglecting the quadratic terms in the small parameter ϵ.

Equation (4.1) must be integrated in every particular case and the soliton will be stable if the basic solutions (normal modes) are purely oscillating (see for details Refs. 2 and 4). The general study of stability from (4.1) for all the family (2.2) is difficult, but we can obtain general conditions for static stability using the variational method.[16]

Consider a small perturbation of the soliton $\phi(r)$ of the form

$$\phi+\delta\phi=\phi(r)+\epsilon\phi_1(\vec{x}) , \tag{4.2}$$

where ϵ is the small parameter of the variation and $\phi_1(\vec{x})=\phi_1(r,\theta,\varphi)$ is a time-independent function which vanishes at infinity faster than $1/r^2$.

The first variation of the energy functional (linear in ϵ) is

$$\delta_1E=-2\epsilon\int\dot{L}[-\phi'^2(r)]\partial_\mu\phi(r)\partial^\mu\phi_1(\vec{x})d\Omega . \tag{4.3}$$

If the derivative of the Lagrangian function is regular everywhere, it is easily shown that the condition $\delta_1E=0$ is satisfied for arbitrary variations of the static solution. But in our models, $\dot{L}$ diverges at the center of the soliton and the energy is stationary, only, if the variation vanishes at $r=0$.

The second variation of the energy functional (quadratic in ϵ) is

$$\delta_2E=-\epsilon^2\int\left\{2\ddot{L}[-\phi'^2(r)]\phi'^2(r)\frac{\partial\phi_1}{\partial r} - \dot{L}[-\phi'^2(r)]\nabla\phi_1\cdot\nabla\phi_1\right\}d\Omega . \tag{4.4}$$

The stability criterion amounts to the requirement

$$\delta_1E=0,\quad \delta_2E\geq 0 \tag{4.5}$$

and from (4.4) we obtain the static stability conditions

$$\ddot{L}[-\phi'^2(r)]\leq 0,\quad \dot{L}[-\phi'^2(r)]\geq 0 . \tag{4.6}$$

These conditions imply for the Lagrangian function a form like that of Fig. 3 between its values at the vacuum and at the center of the soliton. They are satisfied, in particular, by our solutions (2.19). It is thus statically stable against variations which vanish at $r=0$.

If the variation is not null at the origin, our solutions appear to be unstable. Nevertheless, as in the Prasad-Sommerfield solution,[9] our solitons have an integration constant [Λ in Eq. (2.6)] which sets the scale of length. Thus solutions for different values of Λ are identical with respect to the appropriate length scale. Then, a "non-null at the origin" variation becomes a "null at the origin" variation after a change of length scale and the solitons can be considered stable.

The same considerations on stability can be easily extended to the transformed Lagrangian of Sec. III.

V. THE MULTICOMPONENT SCALAR FIELD

The existence of internal degrees of freedom for the scalar field introduces some qualitatively new features as to the construction of models with soliton solutions. These new features are, essentially, the topological properties associated with the internal structure and internal symmetries of these models.

We first consider the multicomponent scalar fields governed by Lagrangian densities of the form

$$L = f(\partial_\mu \pi_i \partial^\mu \pi_i) , \qquad (5.1)$$

$f(X)$ being an arbitrary function. The n-component scalar field is now a vector in an n-dimensional Euclidean space and the model is invariant under the transformations of the internal symmetry group O_n. All the methods developed in Sec. II, in order to elaborate models with spherically symmetric soliton solutions, can be generalized to the multicomponent fields (5.1) in a straightforward way. Moreover, the field transformations developed in Sec. III can easily be generalized to the Lagrangian (5.1). These transformations lead to Lagrangians of the form

$$L = f[g_{ij}(\pi)\partial_\mu \pi^i \partial^\mu \pi^j] , \qquad (5.2)$$

where the metrics in the internal space are Euclidean.

There is nevertheless a more general family of models defined by Lagrangians of the form (5.2) whose metric $g_{ij}(\pi)$ is, however, Riemannian. The classical example of the latter case is the nonlinear chiral $SU(2)\times SU(2)$ model for the pion field.

The Lagrangian is

$$L_\sigma = g_{ij}(\pi)\partial_\mu \pi^i \partial^\mu \pi^j \qquad (5.3)$$

with

$$g_{ij}(\pi) = \delta_{ij} + \frac{\pi^i \pi^j}{f^2 - \pi^2} ,$$

where

$$\pi^2 = \sum_K \pi^K \pi^K$$

and f is a positive constant.

The model (5.3) does not have static soliton solutions, but we can ask the following question: Is there some function of the Lagrangian L_σ leading to a model with static soliton solutions? The answer is yes. It is the DDI model[11] defined by the Lagrangian

$$L = -(-L_\sigma)^{3/2} . \qquad (5.4)$$

Although our method cannot be immediately generalized to these multicomponent scalar models with internal Riemannian spaces, it is enlightening to try to understand the role of the functional transformations such as (5.4) in the existence of the soliton. This can be done by avoiding other causes of existence of solitons in the model like the internal structure and, thus, by studying the one-component scalar model whose Lagrangian has the same functional form.

Thus, in the case of the DDI model we must study the pure scalar Lagrangian

$$L = -\left[-\left[1 + \frac{\phi^2}{f^2 - \phi^2} \right] \partial_\mu \phi \partial^\mu \phi \right]^{3/2} . \qquad (5.5)$$

The static solution which is, now, the analog of the DDI soliton is the spherically symmetric one. However, the simple application of the methods of Secs. II and III shows that this solution is not a soliton. This result points out that the existence of the DDI soliton is not only due to the functional form of the Lagrangian (5.4). In fact, Isham[17] has shown the presence of conserved topological currents in the model which are responsible, also, for the existence and stability of the soliton.

VI. CONCLUSION

We have systematically explored the scalar covariant field theories with Lagrangians which are arbitrary functions of the scalar $\partial_\mu \phi \partial^\mu \phi$ (avoiding the consequences of Derrick's theorem). In this way we have succeeded in elaborating strategies for constructing models with three-space-dimensional soliton solutions (in the sense of field theory) and we have given some explicit examples. Some of these examples are nonlinear generalizations of the d'Alembertian linear model.

The method of Lagrangian transformations allows the elaboration of very rich classes of solitons. We note in particular the possibility of constructing models which admit semilinearity and models which are a field-theoretical basis for finite-energy confining potentials.

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