

An approach to quantum gravity

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A model for quantum gravity is presented by treating the light-cone structure of spacetime as classical and the conformal factor as a quantum degree of freedom. The motivation and the details of the formalism are discussed. The approach is used to discuss the question of singularities in the cosmological models. It is shown that one can introduce the concept of stationary states for the quantum geometry, in analogy with the stationary states of simple quantum systems. The quantum stationary geometries (QSG's) avoid the classical singularities. The light-cone structure is determined by a set of equations involving the expectation values in the QSG concerned. The cosmological implications of the formalism, especially to matter creation, flatness, etc., are discussed. The theory is conformally invariant in the quantum level.

I. GRAVITY—CLASSICAL AND QUANTUM

Electromagnetism and gravity are the two long-range classical fields. Maxwell's equations describe electromagnetism while Einstein's theory of general relativity is now taken to describe gravitation adequately. However, classical physics is only a limiting case of quantum theory. There exists a host of experimental results (photoelectric effect, Compton effect, ...) which cannot be explained by classical electromagnetism. On the theoretical side, Maxwell's equations face problems of divergence when applied to discuss the self-force of a charged particle.² Thus Maxwell's equations are inadequate beyond a particular domain—both theoretically as well as experimentally.

The problem was tackled by quantizing the electromagnetic field. Earlier attempts³ were successful in describing the simple experimental results, but failed (due to divergences) when higher-order corrections were attempted. The problem was finally settled by the development of "renormalizable quantum electrodynamics" due to Feynman, Schwinger, and Tomonaga.⁴ The theory, often heralded as the most successful of physical theories, gives a prescription for computation of observable quantities.

What is the situation regarding gravity? Is a classical framework adequate or do we require a quantum version of the theory? Various considerations seem to indicate the need for "quantization of gravity."

To begin with, classical gravity is bedevilled by singularities, which shows an inconsistency of formalism. Powerful theorems, proved in the sixties,

almost conclusively rule out classical solutions to the crisis.⁵ Conceptually—if not experimentally—quantum gravity has become a necessity.

Cosmological considerations emphasize this need further. Classical general relativity leads automatically to the conservation of energy and momentum. But conventional "big-bang" models require the violation of energy-momentum conservation (at least) at one event, conventionally identified with the singularity. An extension of the theory, viz. quantum gravity, is needed to give meaningful answers to questions regarding the "creation of the universe."

Quantum gravity may be required from a purely operational point of view as well. When the matter is quantized but gravity is not, it is difficult to find a suitable generalization to Einstein's equations. (There is even a claim that the simplest possible extension may not be experimentally tenable; see Ref. 6.)

The above considerations merely establish the need for an extension of the classical theory of gravity but does not indicate any specific framework (except, of course, that the above problems must be solvable). Classical gravity is conceptually very different from other classical theories inasmuch as it plays the dual role of field and spacetime geometry. This has led to different approaches to quantum gravity which may be separated into two groups (i) attempts that treat gravity as spacetime geometry and proceed to "quantize the spacetime," (ii) attempts that treat gravity as a field in the flat-spacetime background and proceed to quantize this field. Unfortunately, both kinds of attempts lead to difficulties.

In the former approach,⁷ one reduces the Einstein

action to canonical form and attempts some variant of the canonical quantization. All the different methods face a certain level of operational difficulties, e.g., choice of variables, constraint equations, etc. and one major conceptual difficulty, viz. how to interpret a quantized spacetime geometry. The structure of physics demands the concept of well-defined spacelike, timelike, or null separation between the events (in other words, the light-cone structure must be well defined). When all the metric coefficients are treated as quantum variables, the light-cone structure becomes “fuzzy” and undergoes quantum fluctuations. (It is not possible to decide *a priori* which two events are connected by a spacelike interval, for example.) Since the concept of spacelike hypersurface itself is ill defined, even posing the problem of evolution becomes difficult.

One must also notice that this approach has not led to a clear solution to the problem of singularities. Various authors have expressed different views on this matter.⁸ Also, the questions regarding the big-bang singularity and creation of matter remain unexplained.

The second approach to quantize gravity, treating it as a field, is free from such conceptual difficulties but suffers serious setbacks of purely operational nature. The theory is perturbatively nonrenormalizable and a nonperturbative structure is largely unknown.⁹ Euclidean and lattice extensions of the theory are also not free from ambiguities.¹⁰ (Two recent formalisms, that of supergravity¹¹ and “induced gravity,”¹² however, show some promise.) It is even possible that the theory violates unitarity. No modification of Einstein’s theory—allowed within the classical tests of gravity—is known that is free from these objections.

We discuss in this paper an “in between” attempt at quantum gravity. We believe that the failure of the conventional approaches warrant the introduction of new physical assumptions. Classical gravity plays the dual role of field and geometry. We respect this duality in the quantum level as well and use a formalism which has the following feature. It treats the “field aspect” of the gravity as quantum mechanical and the “geometric aspect” as a classical c -number entity. Various aspects of the formalism and some of the results are presented below. A detailed discussion of the maximally symmetric cosmological solution is discussed in the following paper.¹³

The theory presented here appears to be capable of tackling the various questions which were raised earlier. In some places we shall use operational assumptions similar to those made in conventional attempts. We wish to mention two of these assumptions before proceeding further.

The first one relates to the imposition of symmetries on the quantum dynamics. Consider, for example, the homogeneous Bianchi cosmologies. These cosmologies can be represented, at the classical level, by a set of functions of time. To find the corresponding quantum theory one often treats these variables as q numbers. This “quantization of a homogeneous spacetime” is assumed to be the same as a “homogenized version of quantized spacetime.”¹⁴ One is forced to make this assumption because of the lack of complete knowledge about “quantized spacetime,” in general. We shall also resort to this assumption in our theory.

The second point is related to general covariance of quantum theory. Investigations about quantum fields, in curved spacetime and accelerated frames, have indicated the observer dependence of certain quantum processes.¹⁵ More precisely, though a quantum-theory Lagrangian may be generally covariant, the details of the processes can depend on the choice of time coordinate. One must notice that this feature has nothing to do with even gravity (let alone quantum gravity), and arises purely from the dependence of conventional field theory formalism on the choice of time coordinate. It is very doubtful whether one can avoid this dependence in quantum gravity (this dependence is noticed in conventional approaches; see Ref. 16). There can also be a physical reason as to why such a dependence need not be avoided. Loosely speaking, the quantum state of the spacetime is going to be produced by a “measurement” made by an observer. The setting up of the clocks and rods for the observer is certainly an integral part of the measurement. Thus the quantum state can very well depend on the coordinate system chosen, especially on the choice of the time coordinate. This problem is conventionally bypassed by making a “natural choice” for the time coordinate. We shall also resort to this operational technique when the need arises. We would like to stress that these two aspects are common to all approaches of quantum gravity and have no specific connection with our formalism.

II. BASIC FORMALISM

The causal relationship between events is decided by the light-cone structure. As we said before, we will follow an approach which quantizes the field aspect of spacetime geometry retaining the c -number formalism for the light-cone structure. What is this field aspect? In other words, what degree of freedom remains in the metric tensor after the light-cone structure is fixed? It is clear that two metrics g_{ik} and

$$\bar{g}_{ik} = \Omega^2(x^i)g_{ik} \quad (2.1)$$

will have the same light-cone structure. Thus the conformal degree of freedom of the spacetime geometry can be treated as a quantum variable, without affecting the light-cone structure. We shall treat Ω as the quantum variable and treat g_{ik} as a c-number metric. This choice has two major additional advantages. (i) The split up in Eq. (2.1) is generally covariant. When the coordinate system is transformed, Eq. (2.1) is retained as long as Ω transforms as a scalar and g_{ik} transforms as a tensor. (ii) As we will see below, the measure for the quantum functional integral is well defined for the conformal factor, removing one major mathematical difficulty. This allows an exact solution for many problems.

The physical interpretation of Eq. (2.1), of course, has to be modified when Ω is a quantum operator. One should use a suitable expectation value (we will discuss this in detail later) and write

$$\bar{g}_{ik} = \langle \Omega^2 \rangle g_{ik} . \quad (2.2)$$

The metrics in various quantum states are conformally related to g_{ik} and share the same light-cone structure.

The transition from classical to quantum theory is conventionally made by using Feynman's path-integral approach.¹⁷ In the case of gravity one normally proceeds as follows. Suppose the spacetime is foliated by a family of spacelike hypersurfaces parametrized by the "time" coordinate t . The transition amplitude from a given three-geometry ${}^3\mathcal{G}_1$ at t_1 to ${}^3\mathcal{G}_2$ at t_2 is (postulated to be) given by

$$K({}^3\mathcal{G}_2 t_2 ; {}^3\mathcal{G}_1 t_1) = \sum_{\text{paths}} \exp \left[\frac{i}{\hbar} J \right], \quad (2.3)$$

where J is the classical action for Einstein's theory and the "sum" is over all metrics with correct boundary conditions. One minor problem arises because of the fact that the Einstein action J contains second derivatives of the metric. To avoid this it is better to use the Einstein action along with the Hawking counterterm¹⁸ in the form

$$J = \frac{1}{16\pi} \int_{\nu} R \sqrt{-g} d^4x + \frac{1}{8\pi} \int_{\partial\nu} K \sqrt{\pm h} d^3x + J_m, \quad (2.4)$$

where K is the trace of the second fundamental form induced on the boundary $\partial\nu$, $h_{\mu\nu}$ is the induced metric on the surface, and J_m is the matter action.

In our formalism the quantum geometries can differ from g_{ik} only in the conformal factor. Thus one can ask for the probability amplitude for transi-

tion from a conformal factor $\Omega_1(\vec{x})$ at t_1 to $\Omega_2(\vec{x})$ at t_2 . This is given by

$$K[\Omega_2(\vec{x})t_2 ; \Omega_1(\vec{x})t_1] = \sum_{\text{all}\Omega} \exp \left[\frac{i}{\hbar} J[\Omega] \right], \quad (2.5)$$

where $J(\Omega)$ has the form¹⁹

$$J[\Omega] = \frac{1}{16\pi} \int (R\Omega^2 - 6\Omega_i\Omega^i) \sqrt{-g} d^4x + J_m \quad (2.6)$$

(the surface term arising from the second derivative of Ω is canceled with the Hawking surface term). The sum over paths can be rigorously defined because of the quadratic nature of the action in Eq. (2.6), provided J_m is also quadratic in Ω . It is difficult to decide on a suitable formalism for treating J_m , and each type of source must be treated separately. We shall be mostly concerned with sources which are conformally invariant (like electromagnetic radiation) for which J_m will be independent of Ω . There is a basic conceptual difference between Eqs. (2.3) and (2.5). Strictly speaking, Eq. (2.3) is not in the proper form because the three-geometry ${}^3\mathcal{G}$ carries the information about the time coordinate as well (at least in a wide class of manifolds; see Ref. 1, Chap. 21). Thus there will be cases in which it is inappropriate to add the time labels separately. However, in Eq. (2.5) we consider the conformal part as a scalar degree of freedom, which cannot "carry" any further information. Thus the time labels in Eq. (2.5) are quite necessary. This is only a technical distinction because, in the future, we will be concerned only with the developments based on Eq. (2.5).

We also have to determine the equations satisfied by the metric g_{ik} . This is done as follows. We quantize the conformal factor Ω in an arbitrary background metric g_{ik} . Once this is done, one can calculate the expectation values $\langle \Omega^2 \rangle$ and $\langle \partial_i \Omega \partial^i \Omega \rangle$ in the quantum state of the system, which will allow one to treat the action J as an effective action for the classical background as²⁰

$$J_{\text{eff}} = \frac{1}{16\pi} \int (R \langle \Omega^2 \rangle - 6 \langle \Omega^i \Omega_i \rangle) \sqrt{-g} d^4x + \langle J_m \rangle \quad (2.7)$$

The variation of this action with respect to the metric g_{ik} gives the equations for the "background" metric g_{ik} as

$$\langle \Omega^2 \rangle (R_{ik} - \frac{1}{2} g_{ik} R) + 6t_{ik} = -8\pi G T_{ik} + (g_{ik} \square - \nabla_i \nabla_k) \langle \Omega^2 \rangle, \quad (2.8)$$

where

$$t_{ik} = -\langle \Omega_i \Omega_k \rangle + \frac{1}{2} g_{ik} \langle \Omega_a \Omega^a \rangle \quad (2.9)$$

and ∇_i represent covariant differentiation with respect to x^i . The quantization has to be performed in a given metric g_{ik} , which in turn depends on the quantum state via Eq. (2.8). It is clear that we are involved with a complicated set of coupled equations whose consistent solutions will determine the nature of quantum geometries.

We shall now proceed to discuss various aspects of this formalism, especially in the context of quantum cosmology. In order to make headway with the equations we shall proceed in three stages. First of all, we shall treat the metric tensor as a given classical solution and study the effect of quantum conformal fluctuations (QCF) in this spacetime. This corresponds to neglecting the back reaction of QCF on the metric g_{ik} . We shall show that near any singularity of g_{ik} , the QCF diverges. Thus the back reaction cannot be neglected near the singularity and one has to use the full formalism. In the second stage, we shall study the nature of quantum states, which are important near the singularity. In the last stage we discuss the self-consistent solutions for some simplified cosmological models. A full discussion of homogeneous, isotropic universe is presented in the accompanying paper.¹³

III. STAGE ONE—QCF IN A SINGULAR SPACETIME

Classical gravity gives adequate description up to length scales of the order of 10^{-33} cm. Thus we expect QCF to be significant only near the strong gravitational field regime, especially near singularities. The physical concept of QCF can be illustrated nicely by considering a radiation-filled spacetime. Such a spacetime (homogeneous, isotropic, radiation filled) has a metric

$$ds^2 = \langle \Omega^2(t) \rangle S^2(t) \left[dt^2 - \frac{dr^2}{1 - kr^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (3.1)$$

Here we have already imposed the assumptions of homogeneity and isotropy on $\Omega(x^i)$ and have made it a function of time alone, in accordance with the discussion in Sec. I. The expansion factor $S(t)$ has the classical value

$$S(t) = \begin{cases} S_0 t & (k=0), \\ S_0 \sin t & (k=+1), \\ S_0 \sinh t & (k=-1). \end{cases} \quad (3.2)$$

The evolution of QCF is determined by the kernel

$$K[\Omega_2 t_2; \Omega_1 t_1] = \int \mathcal{D} \Omega(t) \exp \left\{ \frac{i}{\hbar} \left[\left[-\frac{3}{8\pi} \right] \times \int_{t_1}^{t_2} dt S^2(t) \dot{\Omega}^2 V \right] \right\}. \quad (3.3)$$

We have used Eq. (2.6) and the facts that (i) R is zero for the given metric, (ii) J_m is independent of Ω for the conformally invariant radiation field. We have written

$$V = 4\pi \int_0^L \frac{r^2 dr}{(1 - kr^2)^{1/2}} \quad (3.4)$$

for the region of space under consideration. (In the case of a closed model this can be taken to be the total volume of the universe; otherwise one can limit at the particle horizon. Our main results are independent of this choice.) This path integral can be easily evaluated¹⁷ to give

$$K[\Omega_2, \Omega_1] = F(\tau_2 \tau_1) \exp \left[\frac{im}{\hbar^2} \frac{(\Omega_2 - \Omega_1)^2}{\tau_2 - \tau_1} \right], \quad (3.5)$$

where F is an arbitrary function, and

$$m = -\frac{3V}{8\pi}; \quad \tau_2 - \tau_1 = \int_{t_1}^{t_2} \frac{dt}{S^2(t)}. \quad (3.6)$$

To understand the physics behind this kernel, consider how it propagates a wave function $\psi(\Omega_1)$ in time. We shall assume that the wave packet was a Gaussian with the classical mean $\langle \Omega \rangle = 1$ and a given dispersion. That is,

$$\psi[\Omega_1] = \left[\frac{1}{2\pi\sigma^2} \right]^{1/4} \exp \left[-\frac{(\Omega_1 - 1)^2}{4\sigma^2} \right]. \quad (3.7)$$

The wave function at any other time t_2 , found by integrating

$$\psi[\Omega_2, t_2] = \int_{-\infty}^{+\infty} d\Omega_1 K[\Omega_2 t_2; \Omega_1 t_1] \psi[\Omega_1] \quad (3.8)$$

is given by

$$|\psi[\Omega_2 t_2]|^2 = \left[\frac{1}{2\pi\sigma^2(t_2)} \right]^{1/2} \exp \left[-\frac{(\Omega_2 - 1)^2}{2\sigma^2(t_2)} \right], \quad (3.9)$$

where

$$\sigma^2(t_2) = \sigma^2(t_1) \left[1 + \frac{\hbar^2}{4m^2\sigma_1^4} \left(\int_{t_1}^{t_2} \frac{dt}{S^2(t)} \right)^2 \right]. \quad (3.10)$$

Substituting the form of $S(t)$, we get

$$\begin{aligned} \sigma_2^2 &= \sigma_1^2 \left[1 + \frac{\hbar^2}{4m^2\sigma_1^4} \frac{1}{S_0^4} \left(\frac{1}{t_2} - \frac{1}{t_1} \right)^2 \right] \quad (k=0) \\ &= \sigma_1^2 \left[1 + \frac{\hbar^2}{4m^2\sigma_1^4} \right. \\ &\quad \left. \times \frac{1}{S_0^4} (\cot t_2 - \cot t_1)^2 \right] \quad (k=+1) \\ &= \sigma_1^2 \left[1 + \frac{\hbar^2}{4m^2\sigma_1^4} \right. \\ &\quad \left. \times \frac{1}{S_0^4} (\coth t_2 - \coth t_1)^2 \right] \quad (k=-1). \end{aligned} \quad (3.11)$$

One major feature is common to all these dispersions: they diverge as $t \rightarrow 0$. In other words, the wave function becomes more and more delocalized and spread out as the singularity is approached. Since the mean value of a distribution has a meaning only when the dispersion is finite, one can conclude that the classical solution ceases to have any significance near the singularity. The classical evolution, so to say, is drowned in the sea of quantum geometries. One can no longer neglect the back reaction on g_{ik} via Eq. (2.8). Near the singularities, the full equation must be considered.

The question may arise as to whether the result is sufficiently general or whether it is a consequence of the particular symmetries that are present in the example. Though one can easily show that the result extends to various other simple systems,²¹ a general proof would be comforting. Such a proof, indeed, can be given provided one can get a handle on either the source term J_m or on the form of the metric near the singularity. We shall briefly indicate the line of proof below (see, for details, Refs. 19 and 20).

Consider the Green's function associated with the classical variational equation, for the action in Eq. (2.6),

$$\square G + \frac{1}{6} R G = \delta(X, Y) (-g)^{-1/2}. \quad (3.12)$$

Assuming that the source term J_m is also quadratic in Ω , one can perform the path integration and obtain

$$K[\Omega_2, \Omega_1] = F \exp \left[\frac{i}{\hbar} \bar{J}[\Omega_2, \Omega_1] \right], \quad (3.13)$$

where \bar{J} is the classical value of the action.¹⁷ By a straightforward but lengthy analysis one can express the classical value of the action J in terms of $G(x, y)$ as (see Ref. 19 for details)

$$\begin{aligned} J &= \int \int A_{11}(\bar{x}_1, \bar{x}'_1) \Omega_1(\bar{x}_1) \Omega_1(\bar{x}'_1) d^3 \bar{x}_1 d^3 \bar{x}'_1 \\ &\quad + \int \int A_{22}(\bar{x}_2, \bar{x}'_2) \Omega_2(\bar{x}_2) \Omega_2(\bar{x}'_2) d^3 \bar{x}_2 d^3 \bar{x}'_2 \\ &\quad + 2 \int \int A_{12}(\bar{x}_1, \bar{x}_2) \Omega_1(\bar{x}_1) \Omega_2(\bar{x}_2) d^3 \bar{x}_1 d^3 \bar{x}_2, \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} A_{11}(\bar{x}_1, \bar{x}'_1) &= \frac{3}{8\pi} \int \sqrt{-g'_1} d^3 \bar{x}_2 G(\bar{x}_2, \bar{x}_1)^{-1} \\ &\quad \times \frac{\partial}{\partial t_1} G(\bar{x}_2, \bar{x}'_1), \\ A_{22}(\bar{x}_2, \bar{x}'_2) &= -\frac{3}{8\pi} \int \sqrt{-g'_2} d^3 \bar{x}_1 G(\bar{x}, \bar{x}_1)^{-1} \\ &\quad \times \frac{\partial}{\partial t_2} G(\bar{x}'_2, \bar{x}_1), \\ A_{12}(\bar{x}_1, \bar{x}_2) &= \frac{3}{8\pi} G(x_1, x_2)^{-1}. \end{aligned} \quad (3.15)$$

In the above equations we suppressed the time coordinate and indicated by G^{-1} the inverse of the Green's function G .

When the kernel is used to propagate the wave functional (since Ω now can depend on the space coordinates) by the equation

$$\begin{aligned} \psi[\Omega_2(\bar{x})] &= \int \mathcal{D}\Omega_1[\bar{x}] K[\Omega_2(\bar{x})t_2; \Omega_1(\bar{x})t_1] \\ &\quad \times \psi[\Omega_1(\bar{x})], \end{aligned} \quad (3.16)$$

it is the "cross term" A_{12} of Eq. (3.14) that retains the "memory" of the initial state. Thus the final state will have total uncertainty about Ω_2 if A_{12} vanishes at some event. In other words, the dispersion will diverge at the singularity if the Green's function diverges at that event.

If one assumes that the source consists of dust [a slightly modified formalism is required with $(\Omega - 1)$ replacing Ω ; see Ref. 19], one can prove this fact by considering the conformal invariance of Eq. (3.12). If no specific form is assumed for the source, it is necessary to use a sufficiently general form of the metric near the singularity. Such a metric is given by Belinskii *et al.*²² One can explicitly solve for the Green's function near the singularity and demonstrate the divergence.²³

Either way, it is clear that QCF diverges near the classical singularity necessitating the use of our full formalism. In other words, the classical metric which is obtained by neglecting the back reaction of the conformal fluctuation terms in Eq. (2.7) is not valid near the singularity.

IV. STAGE TWO—QUANTUM STATIONARY GEOMETRIES

Classically, a radiation-filled Friedmann universe is described by a single function, $S(t)$. This goes to zero at $t=0$, making the spacetime singular. However, we have just now reasoned out that the classical picture is not valid near $t=0$. What kind of quantum states are relevant to our problem?

An analogy might be helpful. Consider the electron in the hydrogen atom. Classically, it is described by a function $q(t)$ that spirals down to the singularity. But quantum mechanics avoids this difficulty by introducing a set of well-defined stationary states. The quantum uncertainty prevents the electron from reaching the origin and provides a well-defined ground state. Can this analogy be used in the case of the collapsing universe? A simple argument shows that it may be possible. Consider the metric of the collapsing universe with the quantum corrections $\langle \Omega^2 \rangle$. This correction has the effect of replacing $S^2(t)$ by

$$S_{\text{eff}}^2 = \langle \Omega^2 \rangle S^2(t) = [1 + \sigma^2(t)] S^2(t). \quad (4.1)$$

It is easy to see that near $t=0$

$$S_{\text{eff}}^2(t) \sim \frac{\hbar^2}{4m^2} \frac{1}{S_0^2 \sigma^2(t_1)}. \quad (4.2)$$

In other words, S^2 is bounded from below by a purely quantum-mechanical term, which provides, in some sense, the “ground state for the geometry.”

This suggests that one should look at the stationary states for the quantum geometry, rather than the kernel. Given the action for the system one can at once write down the Hamiltonian. However, as long as we treat $S(t)$ and $\Omega(t)$ separately, the Hamiltonian will contain $S(t)$ and will be a function of time, excluding the existence of stationary states. (Moreover, we will still be “tied” to the classical solution.) In order to tackle these difficulties, we shall treat the overall conformal factor as the quantum variable, and consider the metric to be

$$ds^2 = \langle \Omega^2 \rangle \left[c^2 dt^2 - \frac{dr^2}{1-r^2/a^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (4.3)$$

(we have switched over to normal units; note that Ω is dimensionless and r and a have dimensions of length). The action governing the conformal factor is

$$\begin{aligned} J &= \frac{Vc^4}{16\pi G} \int_{t_1}^{t_2} dt (\bar{R}\Omega^2 - 6\dot{\Omega}^2) \\ &= -\frac{1}{2}M \int_{t_1}^{t_2} dt (\dot{q}^2 - \omega^2 q^2), \end{aligned} \quad (4.4)$$

where we have made the substitutions

$$\begin{aligned} q &= a\Omega, \quad M = \frac{3}{2}\pi \left[\frac{ac^2}{G} \right], \\ \omega &= \frac{c}{a}, \quad V = \int_0^L d^3\vec{x} \sqrt{-g}. \end{aligned} \quad (4.5)$$

This is just the action for the harmonic oscillator of frequency ω . The classical solution is

$$\delta J = 0 \implies q(t) = q_0 \sin \omega t \quad (4.6)$$

(with a suitable origin for time). This gives the form of solution for the classical radiation-filled model, as it should. The quantum dynamics, of course, can be analyzed by calculating the path-integral kernel. But since we are now interested in the stationary states of the system we shall go directly to the Schrödinger equation, which reads

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial q^2} - \frac{1}{2} M \omega^2 q^2 \psi. \quad (4.7)$$

Incorporating the extra minus sign into the “energy” (which has no physical meaning in our case), we can write the solution

$$\psi_n(q, t) = e^{i\epsilon_n t/\hbar} \phi_n(q), \quad (4.8)$$

$$\begin{aligned} \phi_n(q) &= (2^n n!) \left[\frac{M\omega}{\pi\hbar} \right]^{1/4} \\ &\times H_n \left[q \left[\frac{M\omega}{\hbar} \right]^{1/2} \right] \exp \left[-\frac{M\omega q^2}{2\hbar} \right], \end{aligned} \quad (4.9)$$

where H_n is the Hermite polynomial,

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}. \quad (4.10)$$

Thus the stationary states of the quantum geometry are the same as the stationary states of a harmonic oscillator. In particular, the conformal factor has the expectation value

$$\langle \Omega^2 \rangle_n = \left[\frac{2}{3\pi} \right] \left[\frac{L_p}{a} \right]^2 \left(n + \frac{1}{2} \right). \quad (4.11)$$

Obviously the classical collapse cannot proceed all the way and must stop at the lower bound,

$$\langle \Omega^2 \rangle_{\text{min}} = \left[\frac{1}{3\pi} \right] \left[\frac{L_p}{a} \right]^2. \quad (4.12)$$

The spacetime geometry (when the universe is in the n th stationary state) is given by

$$ds^2 = \langle \Omega^2 \rangle_n \left[c^2 dt^2 - \frac{dr^2}{1-r^2/a^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (4.13)$$

$$= L_p^2 \left[\frac{2}{3\pi} \right] \left(n + \frac{1}{2} \right) [d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (4.14)$$

With this we have completed the analogy between an electron in the hydrogen atom and the collapsing universe.

There remains the question as to what the present quantum state is of the universe. In other words, how can one go from Eq. (4.14) to the classical limit? Mathematically speaking, the problem is the same as that for any harmonic oscillator, say, a bob oscillating at the end of the string. The classical limit can be achieved by postulating that it is in a large- n stationary state, or in a coherent state of the harmonic oscillator with the probability function^{24,25}

$$|\psi(q,t)|^2 = N \exp \left[-\frac{M\omega}{\hbar} (q - q_0 \sin \omega t)^2 \right]. \quad (4.15)$$

In the latter case the metric has the form

$$ds^2 = \left[q_0^2 \sin^2 \eta + \frac{2}{3\pi} L_p^2 \right] \times [d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)], \quad (4.16)$$

which again has the same lower bound.

The above mathematical answer does not, however, really help one to decide the quantum state of the universe. This conceptual problem is related to the question of what is meant by a measurement of the state of the universe—since it is the measurement that produces the state. Notice, however, that we have not yet used the “back reaction” equation to determine g_{ik} . This will put some restriction on the wave function $\Psi(q,t)$ but the basic problem must await a conceptual advance for its solution.

We have solved the quantum dynamics of the radiation-filled universe exactly. How does the introduction of some extra matter affect the system? For example, can an addition of a part in the action, represented by

$$J_{\text{int}} = V \int_{t_1}^{t_2} dt \rho(t), \quad (4.17)$$

cause transitions between energy levels?

Notice that since the length scales associated with the universe is $\sim 10^{27}$ cm and $L_p \sim 10^{-33}$ cm, the

quantum states have

$$n \approx \frac{a^2}{L_p^2} \approx 10^{120}. \quad (4.18)$$

If the proper length scales in a region of size L change because of a transition from n to m , we have

$$\frac{\Delta L^2}{L^2} = \frac{m-n}{n}. \quad (4.19)$$

Even if one assumes that $\Delta L^2/L^2 \sim 10^{-20}$ are observable, we need transitions by

$$m-n = 10^{-20} \times 10^{120} \sim 10^{100}. \quad (4.20)$$

This requires the $\rho(t)$ to vary extremely rapidly. In fact, it must have significant Fourier components at frequencies of the order of

$$\nu = \omega(m-n) = \frac{c}{a}(m-n) \sim 10^{83} \text{ Hz}. \quad (4.21)$$

[For comparison, $\nu(\sim \dot{\rho}/\rho)$ of matter in the early universe is of the order of unity.] One can be reasonably sure of the stability of stationary states as far as macroscopic astrophysics is concerned. Of course, at the microscopic level, length scale transitions are taking place all the time. Even an ultra-low-frequency variation of $\sim 10^{-17}$ Hz is enough to cause transitions of the order of $\Delta n \sim 1$. Since no energy distribution is stable to this order, we have to conclude that definite lengths cease to have any meaning at around the Planck length.

This structure of “spacetime foam”²⁶ can also be arrived at in a different way. One can consider the conformal fluctuations around the flat-space background, using the action

$$J = -\frac{3}{8\pi} \int d^4x (\Omega_i \Omega^i). \quad (4.22)$$

By standard analysis,^{17,20,27} one can construct the ground-state wave functional that will give the probability distribution for finding various conformal factors in the flat vacuum; which turns out to be

$$\psi[\Omega(\vec{x})] = N \exp \left[-\frac{3}{8\pi^3} \frac{1}{L_p^2} \times \int \int d^3\vec{x} d^3\vec{y} \frac{(\vec{\nabla}_{\vec{x}} \Omega) \cdot (\vec{\nabla}_{\vec{y}} \Omega)}{|\vec{X} - \vec{Y}|^2} \right], \quad (4.23)$$

As one can see, rapid variations are possible at the Planck length scales.

Before concluding the section we would like to make a comment regarding the concept of stationary states in other models of the universe. In particular, one can talk of stationary states for degrees of freedom other than the conformal degree of freedom (though it is against the spirit of the present discussion, it is helpful to understand the mathematical structure of the theory). For example, the homogeneous Bianchi universes can be described classically by the action²⁷

$$J = \frac{1}{16\pi} \int_{t_1}^{t_2} L dt, \quad (4.24)$$

$$L = -e^{3\lambda} \left[6\dot{\lambda} - \frac{3}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2) \right] + e^{3\lambda} R^* + L_m. \quad (4.25)$$

Here the metric is given in terms of the variables $(\beta_1, \beta_2, \lambda)$ by

$$ds^2 = dt^2 - g_{ik}(t) \sigma^i \sigma^k, \quad (4.26)$$

$$g_{ik}(t) = e^{2\lambda} (e^{-2\beta})_{ik}.$$

The one-forms σ^i satisfy the commutation rules

$$[\sigma_i, \sigma_k] = c^j{}_{ik} \sigma_j \quad (4.27)$$

with $c^j{}_{ik}$ being the structure constants of the isotropy group. One can study the wave functions of the stationary states $\psi(\beta_1, \beta_2, \lambda)$ in exactly the same way as before.²⁸ It turns out that these wave functions vanish at the classical singularity thereby leading to zero probability for its existence. The same result is true for the simplest case of an interacting field as well.²⁸

This concept of stationary states can also be arrived at from a superspace analysis. The superspace is the space of three-geometries modulo coordinate transformations. One can introduce a metric in the space of three-geometries and write an action

$$J = \int G_{AB} \frac{dg^A}{d\lambda} \frac{dg^B}{d\lambda} + \int \underline{R} d\lambda. \quad (4.28)$$

Here A, B stand for a pair of indices, and

$$G_{AB} = G_{(ij)(lm)} = \frac{1}{2} (g_{il}g_{jm} + g_{im}g_{jl} - 2g_{ij}g_{lm}), \quad (4.29)$$

$$\underline{R} = g({}^3R), \quad (4.30)$$

where 3R is the three-curvature of the surfaces of homogeneity. By considering the same action to govern the quantum gravity, one can formulate a path integral in the superspace.^{28,29} This analysis leads to the same results as before.

To summarize, we have replaced the description

of spacetime geometry in terms of a classical metric by a description in terms of quantum stationary geometries. This should allow us to discuss the behavior arbitrarily close to a singularity. However, one task remains: even in this section we have assumed a particular form for the "background." It remains to be shown that this form is consistent with our "back reaction" equation.

V. STAGE THREE—SELF-CONSISTENT SOLUTION

We have now obtained the stationary states for the conformal factor. But two features of this solution point to an incompleteness. (i) In Eq. (4.3) we have assumed a form of g_{ik} which is again "God-given." (ii) In Eq. (4.5) we have obtained the classical limit without any reference to the source strength. In fact, the source is completely eliminated from discussion because of conformal invariance. In order to justify the choice of the background metric, we have to solve the complete set of equations, and prove the consistency of the formalism.

Let us consider again the spacetime presented in the previous section,

$$ds^2 = \langle \Omega^2 \rangle \left[c^2 dt^2 - \frac{dr^2}{1-r^2/a^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (5.1)$$

Suppose the spacetime is in the n th stationary state. Since the Hamiltonian corresponds to that of a harmonic oscillator, we have the results

$$\langle \Omega^2 \rangle = \left[\frac{\hbar}{M\omega} \right] \left(n + \frac{1}{2} \right), \quad (5.2)$$

$$\langle \dot{\Omega}^2 \rangle = \left[\frac{\hbar\omega}{M} \right] \left(n + \frac{1}{2} \right). \quad (5.3)$$

The metric g_{ik} has to satisfy the equations, see Eq. (2.8),

$$\langle \Omega^2 \rangle (R^i{}_k - \frac{1}{2} \delta^i{}_k R) + 6t^i{}_k = -8\pi G T^i{}_k + (\delta^j{}_k \square - \nabla^i \nabla_k) \langle \Omega^2 \rangle. \quad (5.4)$$

Since $\langle \Omega^2 \rangle$ is independent of time in a stationary state, the second term on the right-hand side vanishes. For a homogeneous, isotropic spacetime there are only two independent equations in the set (5.4). These may be taken to be the trace equation and $\langle \Omega^2 \rangle$ component equation. The trace equation leads to

$$\langle \Omega^2 \rangle \omega^2 = \langle \dot{\Omega}^2 \rangle, \quad (5.5)$$

which is identically satisfied in the stationary states.

The $\langle \Omega^0 \rangle$ component equation, however, is nontrivial, giving

$$\langle \Omega^2 \rangle \left[\frac{3}{2} \frac{1}{a^2} - \frac{3}{a^2} \right] - 3 \langle \dot{\Omega}^2 \rangle + 8\pi G \epsilon = 0, \quad (5.6)$$

$$\epsilon = \frac{9}{16\pi G} \omega^2 \langle \Omega^2 \rangle = \frac{3}{8\pi^2} \left[\frac{\hbar c}{a^4} \right] \left(n + \frac{1}{2} \right). \quad (5.7)$$

We get the interesting result that energy density is quantized if the quantum gravity equations are to be consistent. Notice that this result is obtained without using any form of quantization on the matter variable. Classical gravity can lead to the classical dynamics of the source in a natural fashion; can quantum gravity lead to at least some features involving the quantum dynamics of the source? Such an attractive possibility is suggested by Eq. (5.7).

In this particular example, the conformal invariance of the source simplifies matters considerably. Hence it is important to see whether the results are valid for other types of sources. The analysis can be repeated for other types of sources. If we take the source to consist of dust with the energy-momentum tensor

$$T^i_k = \rho(1, 0, 0, 0), \quad (5.8)$$

we still obtain a quantum condition

$$\rho = \frac{c^4}{\pi^2 G} \frac{L_p}{a^3} \left(n + \frac{1}{2} \right)^{1/2}, \quad (5.9)$$

which shows that conformal invariance is not an essential ingredient of this feature.²⁰

Thus we have produced a static, self-consistent solution to our coupled quantum-gravity equations. This completes the logical structure of the formalism. In order to explore further, one has to consider the solutions to Eqs. (2.5), (2.6), and (2.8) under various circumstances. The complete solution, in the case of a maximally symmetric cosmological model, is given in the following paper.¹³ Here we shall consider some general features of the equations.

The equations differ from Einstein's equations by the extra terms which involve the derivatives of Ω . Since Ω and $\dot{\Omega}$ are going to become canonically related variables, one expects nontrivial values for $\langle \Omega^2 \rangle$ and $\langle \Omega_i \Omega_k \rangle$ in any state. (By choosing stationary states, one can avoid the derivatives of $\langle \Omega^2 \rangle$.) We see that t_{ik} , which is based on the expectation values of the form $\langle \Omega_i \Omega_k \rangle$, has the structure of the energy-momentum tensor for a *negative-energy* scalar field. This leads, in a qualitative way, to two possibilities. First, the solutions of these

equations could be nonsingular because of the predominance of the negative-energy field over matter near the singularity. Second, notice that the energy-momentum tensor of matter T^i_k is no longer conserved. Only the combination of t_{ik} and T_{ik} together is conserved. This allows for the possibility of matter creation at the expense of gravitational energy. Since the result is important only near singularities, macroscopic energy-momentum conservation is not violated, within observable limits. [Similar ideas for creating the matter have been proposed before, (see Ref. 30), but the formalisms are entirely different.]

The equations also lead to another interesting result, which goes to confirm the above view. A flat vacuum metric is a perfectly valid (though trivial) solution to the standard classical Einstein equations. However, stationary-state solutions with flat ($g_{ik} = \eta_{ik}$), vacuum ($T_{ik} = 0$) conditions do not exist for our equations (2.8). In some sense, quantum conformal fluctuations lead to the creation of matter, which may be interpreted as the "creation of the universe."

Such an interpretation of big bang leads to a more concrete prediction. One can consider the probability amplitude for the transition from flat space to a maximally symmetric universe. Since the maximal universes are all conformally flat, this question can be easily analyzed in our formalism. It turns out that this probability is a maximum for the flat Friedmann model, with zero curvature for spacelike hypersurfaces. This could be a purely quantum gravitational solution to the flatness problem.^{31,32}

VI. CONCLUSION AND OUTLOOK

There are two aspects of the general formalism which we would like to point out. Notice that our basic equation (2.8) is conformally invariant. However, we have conformal invariance at a much "better" level than what is usual. In standard conformally invariant theories of gravity involving a scalar field and a metric (ϕ and g_{ik} , say) the equations will be invariant under the transformation

$$g_{ik} \rightarrow f^2 g_{ik}, \quad \phi \rightarrow \phi f^{-1}. \quad (6.1)$$

However, ϕ has no direct role to play (in standard theories) in the spacetime geometry. Thus, from the very definition of the conformal transformation ($g_{ik} \rightarrow f^2 g_{ik}$), the spacetime geometry is *not* invariant under the transformation. (After all one can go from flat space to a closed Friedmann model by a conformal transformation.) Thus in conventional approaches "conformal invariance" is restricted to the form of the equations alone (as an example, see Ref. 33). However, in our model under the transfor-

mations

$$g_{ik} \rightarrow f^2 g_{ik}, \quad \Omega \rightarrow f^{-1} \Omega, \quad (6.2)$$

not only the equations of motion but also the space-time interval remains invariant, i.e.,

$$ds^2 = \langle \Omega^2 \rangle g_{ik} dx^i dx^k = \langle \Omega'^2 \rangle g'_{ik} dx^i dx^k. \quad (6.3)$$

Thus the theory, at the quantum level, is truly conformally invariant. In this sense the similarity of the equations with those in the conformally invariant theory of Hoyle and Narlikar³³ is noteworthy. Nevertheless, the significant difference pointed out above (that Ω is part of spacetime geometry) must be kept in mind.

This conformal invariance is broken when the choice is made for the quantum state of the universe. As we have remarked earlier, the "measurement made by the observer on the universe" (whatever that means) causes this breaking of symmetry. In a given "conformal frame" so chosen, we will get Einstein's equations in the classical limit, when $\langle \Omega^i \Omega_k \rangle$ can be neglected. This term cannot be neglected near the singularity and hence we get the nontrivial quantum aspects of the theory. In this way, the formalism is similar in structure to attempts that treat gravity as a low-energy effective Lagrangian theory.¹²

As regards the creation of matter from the negative-energy term, the theory is reminiscent of the steady-state models for the universe.³⁴ However,

there is one major difference. The Ω field arises from quantum theory and does not require the introduction of an *ad hoc* "coupling constant" to matter. Moreover, a natural dynamical equation for the conformal factor is available in the form of the quantization prescription. (This also helps one to have the right number of equations.) Above all the negative-energy field is not an extra structure but an integral part of the geometry.

It will be noticed that the transformation from Ω to a ϕ by the substitution $\Omega^2 = G\phi^2$ will eliminate the Newtonian constant of gravity, G , from the equations. In the stationary state Ω^2 always appears with a (constant) scaling freedom. It is not clear whether G can be predicted by the theory and, if so, whether it evolves in time.

We have presented here an approach to quantum gravity which seems to show promise. It is rather unfortunate that the formalism is very different from conventional ones, making it difficult to compare the results with other approaches. Evidently three tasks remain to be achieved. (i) Connect up this approach with other formalisms of quantum gravity so as to compare and contrast the results. (ii) Consider the solutions to quantum gravitational equations under more general conditions. (iii) Produce a formalism in which the matter variables can also be treated (more realistically) as quantized. All these aspects are under investigation.

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