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Strong-coupling quantum gravity. II. Solution without gauge fixing

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The quantization of a strong-coupling limit of general relativity begun in a previous paper is continued. The theory is quantized without fixing a gauge since in this form the problem of properly taking into account the gauge degrees of freedom can be directly attacked in the perturbation theory. The formalism of the scattering theory on $R\times SL(3,R)/SO(3)$ is developed with strong use made of analysis on that space.

I. INTRODUCTION

In a previous paper[1] (to be referred to as I), a strong-coupling limit of gravity was quantized in a fixed gauge. The chosen gauge simplified the mathematics in the quantization, and clarified the connection between the strong-coupling limit and earlier work on quantum cosmology. In addition, the gauge-restricted theory has an appealing geometrical interpretation. However, this gauge is not convenient for use in the further development of the strong-coupling theory; in particular, the proper contribution (if any) of gauge-fixing ghosts cannot be reliably computed yet. In the present paper a strong-coupling formalism is developed in which all components of the metric are quantized. There are gauge degrees of freedom included among the g_{ij}, but the proper way of taking this into account will not be discussed and the content is largely formal.

The strong-coupling limit of general relativity considered here is obtained by taking the Hamiltonian generator

$$\mathscr{H}_\perp=\kappa g^{-1/2}G_{ijkl}\pi^{ij}\pi^{kl}-\kappa^{-1}g^{1/2}R\ ,$$
$$G_{ijkl}=\tfrac{1}{2}(g_{ik}g_{jl}+g_{il}g_{jk}-g_{ij}g_{kl})$$

($g=\det g_{ij}$, $R=$scalar curvature of g_{ij}, $\pi^{ij}=$conjugate to g_{ij}, and $\kappa=16\pi G/c^3$), and replacing it by[1,2–4]

$$\mathscr{H}_0=g^{-1/2}G_{ijkl}\pi^{ij}\pi^{kl}\ . \tag{1.1}$$

The idea of looking at this limit as a basis for quantization (in addition to much of the point of view adopted in the present paper) seems to have been first put forward in Ref. 2.

This strong-coupling limit is characterized by the ultralocality of the dynamics, i.e., in this limit the terms in the full Hamiltonian containing the spatial derivatives that couple the dynamics at different spatial points have been eliminated. The light cones have closed up to become lines, and the causal structure of the theory is trivial. The quantization of (1.1) is to be used as the starting point for a perturbation theory with perturbation potential $g^{1/2}R$.

In a strong-coupling quantization the metric operator must be quantized as a whole without being divided into background plus perturbation. The positive definiteness of the metric operator g_{ij} leads to the consideration of the variables $\pi^i_j=\frac{1}{2}(g_{jl}\pi^{il}+\pi^{il}g_{jl})$.[1] These variables generate the action of GL(3,R) on the space of g_{ij}'s. There is an invariant metric on this space, namely G_{ijkl}, and this metric has signature $(-+++++)$.[5] The signature of the metric implies that there is one "timelike" coordinate among the g_{ij}'s which is found to be

$$\tau=\tfrac{1}{3}\ln g\ , \tag{1.2}$$

and its canonical conjugate (a timelike, hypersurface orthogonal Killing vector) is

$$\pi=\pi^i_i\ . \tag{1.3}$$

Factoring these timelike variables out of the g_{ij}, π^k_l leads to the consideration of the symmetric space $S=SL(3,R)/SO(3)$ parametrized by

$$\tilde g_{ij}=e^{-\tau}g_{ij},\ \ \det\tilde g_{ij}=1 \tag{1.4}$$

with

$$P^i_j=\pi^i_j-\tfrac{1}{3}\delta^i_j\pi,\ \ P^i_i=0\ , \tag{1.5}$$

satisfying

$$\{g_{ij}(x),P^k_l(x')\}=\tfrac{1}{2}(g_{li}P^k_j+g_{lj}P^k_i-\tfrac{2}{3}g_{ij}P^k_l)\delta(x,x')\ , \tag{1.6a}$$

$$\{P^i_j(x),P^k_l(x')\}=\tfrac{1}{2}(P^i_l\delta^k_j-P^k_j\delta^i_l)\delta(x,x')\ . \tag{1.6b}$$

The variables (1.5) generate the action or SL(3,R) on S with (1.6b) being the bracket relations of those generators.

In terms of these variables the generator (1.1) takes the form

$$\mathscr{H}_0 = P^i_j P^j_i - \tfrac{1}{6}\pi^2 = 0 \ . \tag{1.7}$$

Notice that $\{P^k_l, P^i_j P^j_i\} = 0$, i.e., $P^i_j P^j_i$ corresponds to a Casimir invariant on sl(3,R) [the Lie algebra of SL(3,R)], and $P^i_j P^j_i$ is directly related to the Laplace-Beltrami operator on S.[1] The Poisson brackets (1.6) imply that (1.7) is not the Hamiltonian for a linear field theory, and, in fact, it is only as a result of its ultralocality that a quantization of (1.7) is possible. The quantization of nongauge, ultralocal field theories has been accomplished by Klauder,[6] and we apply his methods to general relativity.

The full metric contains gauge information as well as the dynamical degrees of freedom. The problem for ultralocal quantization is to find the proper method of eliminating the contributions of the gauge modes. For weak-coupling quantum field theory the most convenient way of doing this is to introduce ("probability eating") ghosts. These ghosts are needed to guarantee the unitarity of the S matrix, and the form of the ghost interactions can be derived from this requirement.[7–10] The space-time S matrix is not relevant in strong-coupling quantum gravity, and the proper method of eliminating the effects of the gauge modes in ultralocal gauge theories needs to be discovered. The most direct method of doing this (i.e., the method requiring the fewest prejudices) is to quantize the theory ignoring the need for ghosts, develop the perturbation theory, and then see how it needs to be modified in order to give physically acceptable results; that is, we want to follow the spirit of Feynman's[7] original discovery of the need for ghosts in weak-coupling gauge theory. The present paper provides the preperturbation quantization of the strong-coupling theory with all components of the field present. From this first step the discovery of the strong-coupling analog of gauge-fixing ghosts may begin in the perturbation theory. The need for quantization without explicit gauge fixing is reinforced by the difficulty in expressing the perturbation potential $g^{1/2}R$ in terms of the dynamical degrees of freedom for the gauge used in I.

The paper is organized as follows. In Sec. II the Hilbert space and basic field operators are introduced. Section III gives the Hamiltonian operator, and the scattering formulation of the perturbation theory is discussed. The paper ends with a discussion of the results. In addition to the main body of the paper are two appendices, the first giving a representation of the generator of coordinate transformations on the Hilbert space given in Sec. II, and the second deriving the form of plane-wave states on R$\times$SL(3,R)/SO(3). Throughout the paper the coordinate label $\vec{x}$ represents a point in a coordinate patch on a compact, three-dimensional manifold.

II. FIELD REPRESENTATION

As discussed in the Introduction, and, more extensively, in I, the classical fields that are to be represented as quantum operators are the three-dimensional metric g_{ij} (signature $+ + +$) and π^i_j, the generators of GL(3,R). These variables satisfy the Poisson brackets

$$\{g_{ij}, \pi^k_l\} = \tfrac{1}{2}(g_{li}\delta^k_j + g_{lj}\delta^k_i) \ , \tag{2.1}$$

$$\{\pi^i_j, \pi^k_l\} = \tfrac{1}{2}(\pi^i_l\delta^k_j - \pi^k_j\delta^i_l) \ , \tag{2.2}$$

where factors of $\delta(x,x')$ have been suppressed, and will continue to be suppressed. The choice of variables π^i_j is made in order to be consistent with the positive definiteness of g_{ij}.

We mentioned in the Introduction that it is convenient to decompose these variables into

$$\tau = \tfrac{1}{3}\ln g \ , \tag{2.3}$$

$$\tilde{g}_{ij} = e^{-\tau} g_{ij} \ , \tag{2.4}$$

$$\pi = \pi^i_i \ , \tag{2.5}$$

$$P^i_j = \pi^i_j - \tfrac{1}{3}\pi\delta^i_j \ , \tag{2.6}$$

which satisfy

$$\{\tau, \pi\} = 1 \ , \tag{2.7}$$

$$\{\tau, P^i_j\} = \{\pi, P^i_j\} = \{\pi, \tilde{g}_{ij}\} = 0 \ , \tag{2.8}$$

$$\{\tilde{g}_{ij}, P^k_l\} = \tfrac{1}{2}(\tilde{g}_{li}\delta^k_j + \tilde{g}_{lj}\delta^k_i - \tfrac{2}{3}\tilde{g}_{ij}\delta^k_l) \ , \tag{2.9}$$

$$\{P^i_j, P^k_l\} = \tfrac{1}{2}(P^i_l\delta^k_j - P^k_j\delta^i_l) \ . \tag{2.10}$$

It is this set of fields (2.3)—(2.6) that we will represent as quantum operators satisfying the commutation relations associated with (2.7)—(2.10).

We define the Hilbert space[11] H by starting with Fock creation and annihilation operators $A^\dagger(\vec{x},\Omega,\gamma_{ij})$ and $A(\vec{x},\Omega,\gamma_{ij})$ and a fiducial state $|0\rangle$ of unit norm satisfying

$$[A(\vec{x},\Omega,\gamma_{ij}), A^\dagger(\vec{x}\,',\Omega',\gamma'_{ij})] = \delta(\vec{x},\vec{x}\,')\delta(\Omega,\Omega')\delta(\gamma_{ij},\gamma'_{ij}) \ ,$$

$$[A,A] = [A^\dagger, A^\dagger] = 0 \ , \tag{2.11}$$

$$A(\vec{x},\Omega,\gamma_{ij})\,|0\rangle = 0 \ , \tag{2.12}$$

where $\Omega \in$ R and γ_{ij} is a symmetric, positive-definite matrix of unit determinant, i.e., a point of the symmetric space SL(3,R)/SO(3). The Hilbert space H is the closure of the span of the set given by linear combinations of arbitrary powers of $A^\dagger$ acting on $|0\rangle$.

An overcomplete set of states (i.e., a set such that the closure of its span is H, but not all elements of the set are linearly independent) for H is determined by

$$|f\rangle \equiv \exp\left[-\tfrac{1}{2}\int |f|^2 d\vec{x}\, d\Omega\, d\gamma\right] \exp\left[\int d\vec{x}\, d\Omega\, d\gamma\, f(\vec{x},\Omega,\gamma_{ij}) A^\dagger(\vec{x},\Omega,\gamma_{ij})\right] |0\rangle \ . \tag{2.13}$$

The c-number function $f(\vec{x},\Omega,\gamma_{ij})$ is taken to be square integrable in the measure $d\vec{x}\, d\Omega\, d\gamma$ with $d\gamma$ being the measure on SL(3,R)/SO(3) induced by the Haar measure on SL(3,R) (see Appendix B). The first exponential in (2.13) guarantees that $|f\rangle$ has unit norm. Denote by l the space of functions f. This space is itself a Hilbert space and (2.13) provides a mapping from l to an overcomplete subset of H. The states $|f\rangle$ are eigenstates of the operator A,

$$A(\vec{x},\Omega,\gamma_{ij})|f\rangle = f(\vec{x},\Omega,\gamma_{ij})|f\rangle \ , \tag{2.14}$$

and the inner product of any two of these states is given by

$$\langle f | f'\rangle = \exp[-\tfrac{1}{2}||f||^2 - \tfrac{1}{2}||f'||^2 + (f,f')] \ , \tag{2.15}$$

where

$$(f,f') = \int d\vec{x}\, d\Omega\, d\gamma\, f^* f' \ ,$$

and $|| \ ||$ is the associated norm. H is the space on which the field operators act, but when discussing the dynamics of the theory it is more useful to work with l.

The representation on the space H of the field operators is[1,6,12]

$$\tau(\vec{x}) = \int d\Omega\, d\gamma\, B^\dagger(\vec{x},\Omega,\gamma_{kl})\Omega B(\vec{x},\Omega,\gamma_{kl}) \ , \tag{2.16}$$

$$\pi(\vec{x}) = -i\int d\Omega\, d\gamma\, B^\dagger \frac{\partial}{\partial\Omega} B \ , \tag{2.17}$$

$$\tilde{g}_{ij}(\vec{x}) = \int d\Omega\, d\gamma\, B^\dagger \gamma_{ij} B \ , \tag{2.18}$$

$$P^i_j(\vec{x}) = i\int d\Omega\, d\alpha\, B^\dagger(\rho^{i\dagger}_j - \rho^i_j)B \tag{2.19}$$

with

$$\rho_j = \gamma_{jl}\frac{\partial}{\partial\gamma_{il}} \tag{2.20}$$

and

$$B(\vec{x},\Omega,\gamma_{ij}) = A(\vec{x},\Omega,\gamma_{ij}) + C(\Omega,\gamma_{ij}) \ . \tag{2.21}$$

The c-number function $C(\Omega,\gamma_{ij})$ in (2.21) is taken to be real, and it satisfies

$$\int C^2(\Omega,\gamma_{ij}) d\Omega\, d\gamma = \infty \ . \tag{2.22}$$

It is easy to check that (2.16)—(2.19) satisfy the commutation relations assocated with (2.7)—(2.10) [use needs to be made of the identity

$$\frac{\partial}{\partial\gamma_{ij}}\gamma_{kl} = \tfrac{1}{2}(\delta^i_k\delta^j_l + \delta^i_l\delta^i_k - \tfrac{2}{3}\gamma_{kl}\gamma^{ij})] \ .$$

The translation (2.21) of the operator A by the non-square-integrable function C implies that the operators B are unitarily inequivalent to A.[6,11,12] This translation is necessary to ensure the uniqueness of the state $|0\rangle$, the irreducibility of the commutation relations, and that the spectrum of the field operators is continuous.[6,12] If two different operator representations (2.16)—(2.19) are formed by translating A by two different functions C and C', then the two representations are inequivalent if $C - C'$ is not square integrable[6,12] (if three-space is not compact then the representation for different C's are always inequivalent). The function C thus determines the representation, and, as we shall see, determines a potential term in the Hamiltonian. Note that the relation

$$B\,|0\rangle = C\,|0\rangle \tag{2.23}$$

establishes a close relation between $|0\rangle$ and C through the action of the operators (2.16)—(2.19).

There are several ways to restrict the choice of C. The most important restriction comes from the fact that the functional form of C determines a contribution to the Hamiltonian operator (this results from the requirement $\mathscr{H}_0|0\rangle = 0$, see below). If $\mathscr{H}_0$ is to transform under coordinate changes as a scalar density, then this C-dependent part of $\mathscr{H}_0$ must correspond to a cosmological constant. This implies that C must be an eigenfunction of the operator $\Delta_S - \partial^2/\partial\Omega^2$, where Δ_S is the Laplace-Beltrami operator on S. A result of this requirement is that the operator $\mathscr{H}_0$ commutes with π which means that the dynamics generated by $\mathscr{H}_0$ are conformally invariant since π generates conformal rescalings of the metric g_{ij}

$$[e^{-i\int \lambda(x')\pi(x')dx'} g_{ij}(x) e^{i\int \lambda(x')\pi(x')dx'} = e^{-3\lambda(x)} g_{ij}(x)] \;.$$

In addition to the restriction resulting from $\mathscr{H}_0$ being a scalar density, one can impose other, less well-motivated, conditions on C. For instance, (2.23) establishes the connection between C and the "ground state" $|0\rangle$, and it might be natural to assume that C is Ω-time independent. A different restriction is that C be such that $\mathscr{H}_i|0\rangle=0$. This is discussed in Appendix A where it is found that this implies $C=e^{-9\Omega/2}$. A final possible restriction is to force $\mathscr{H}_0$ to be the conformally covariant wave operator on $\mathsf{R}\times S$. The scalar curvature on $\mathsf{R}\times S$ is a constant so this can be done consistently with the restriction that $\mathscr{H}_0$ be a scalar density. None of these possible restrictions on C is compelling; so below we make the simplest possible choice, $C=1$.

The choice of C should ideally be such that the field operators are well defined. For example, if C is too singular, then it is easy to see that

$$\langle 0|\tilde{g}_{ij}|0\rangle=\int d\Omega\, d\gamma\, \gamma_{ij} C^2(\Omega,\gamma_{ij})$$

is not finite. The requirement that C be an eigenstate of $\Delta_S-\partial^2/\partial\Omega^2$ has the embarrassing consequence that the field operators are not well defined, and a method of regularizing the attendant infinities needs to be found. Any choice of C that makes the field operators finite will of necessity result in a potential contribution to $\mathscr{H}_0$ that does not transform as a scalar density. Since we are dealing with a gauge theory, not all of the components of the field need be properly defined, self-adjoint operators; only the dynamical degrees of freedom need have this property. A possibility that suggests itself is that the gauge invariance can be broken and simultaneously the dynamical fields made well defined with a single choice of C. Whether this can be done consistently is not yet known.

The simplicity of ultralocal theories is to a large extent the result of their reducing to the study of the Hilbert space l. The states $|f\rangle$ are convenient for taking expectation values, for instance

$$\langle f|P^i_j(\vec{x})|f'\rangle=i\int d\Omega\, d\gamma\, f^*(\vec{x},\Omega,\gamma_{kl})\left[\frac{\overleftarrow{\partial}}{\partial\gamma_{im}}\gamma_{jm}-\gamma_{jm}\frac{\overrightarrow{\partial}}{\partial\gamma_{im}}\right]f'(\vec{x},\Omega,\gamma_{kl}) \;. \tag{2.24}$$

The expectation value in H reduces to one in l when using the overcomplete set of $|f\rangle$. Notice that the differential operators in the right-hand side of (2.24) do not act on the x dependence of f and f'. The same will be true of the Hamiltonian operator (3.1) which is what one would expect of an ultralocal theory.

The simplicity found in working in l can be increased by a judicious choice of complete set of states in l such that the image of this set in H is overcomplete. One convenient set of states is derived from the generalized coherent states in H,

$$e^{-i\int dx\, V(x)\pi(x)}e^{-i\int\omega^{rs}(x)\tilde{g}_{rs}(x)}e^{i\int T(x)\pi(x)}e^{i\int\theta^i_j(x)P^j_i(x)}|0\rangle\equiv|V,\omega^{rs},T,\theta^i_j\rangle \;, \tag{2.25}$$

where $\theta^i_i=0$ and ω^{rs} have five independent components corresponding to the five components of $\tilde{g}_{rs}$. The element of l corresponding to (2.25) can be found by computing $f_{V,\omega^{rs},T,\theta^i_j}\in l$, where

$$A\,|V,\omega^{rs},T,\theta^i_j\rangle=f_{v,\omega^{rs},T,\theta^i_j}|V,\omega^{rs},T,\theta^i_j\rangle \;. \tag{2.26}$$

The result of this computation is

$$f=e^{-iV(\vec{x})\Omega}e^{-i\omega^{rs}(\vec{x})\gamma_{rs}}C(\Omega+T(\vec{x}),e^{\theta^i_r(\vec{x})/2}\gamma_{ij}e^{\theta^j_s(\vec{x})/2})-C(\Omega,\gamma_{rs}) \;. \tag{2.27}$$

Not all of the states (2.25) are particularly useful for the remainder of this paper. It is most useful to take the plane waves $e_{\omega,\lambda,b}(\Omega,\gamma)$ (discussed in Appendix B where now ω,λ,b are functions of $\vec{x}$) as a complete set of states in l. The corresponding states in H are

$$|\omega(\vec{x}),\lambda(\vec{x}),b(\vec{x})\rangle=\exp\left[\int d\vec{x}\, d\Omega\, d\gamma[e_{\omega,\lambda,b}(\Omega,\gamma_{ij})-C]A^{\dagger}(\vec{x},\Omega,\gamma_{ij})\right]|0\rangle \tag{2.28}$$

satisfying

$$B\,|\omega,\lambda,b\rangle=e_{\omega,\lambda,b}\,|\omega,\lambda,b\rangle \;. \tag{2.29}$$

The plane waves are not square integrable, but this can, as usual, be overcome by forming wave packets.

We finally mention the proper definition of products of operators in this type of representation.[6,12,13] Naively, given a functional of the matrix $F(\tau,\tilde{g}_{ij})$, the corresponding quantum operator is

$$\hat{F}(\tau,\tilde{g}_{ij})=\int d\Omega\, d\gamma\, B^{\dagger}F(\Omega,\gamma_{ij})B \;. \tag{2.30}$$

As discussed in Appendix A, the quantum operator

$\hat{F}$ will not necessarily transform under changes of coordinates in the same way as the classical F. This can often be corrected by multiplying $F(\Omega,\gamma_{ij})$ in (2.30) by $e^{k\Omega}$ for some appropriate constant k. The extension of this discussion to more general combinations of fields and momenta is straightforward.

III. THE HAMILTONIAN AND SCATTERING THEORY

The representation on the Hilbert space H of the Hamiltonian operator $\mathscr{H}_0$ in (1.1) is

$$\mathscr{H}_0(x)=\int d\Omega\, d\gamma B^{\dagger}(x,\Omega,\gamma_{ij})hB(x,\Omega,\gamma_{ij}) , \tag{3.1}$$

$$h=\frac{\partial^2}{\partial\Omega^2}-\Delta_S+V(\Omega,\gamma_{ij}) , \tag{3.2}$$

where Δ_S is the Laplacian on SL(3,R)/SO(3) (see Appendix B), and $V(\Omega,\gamma_{ij})$ is a regularization term (an analog of the $\frac{1}{2}$ in the harmonic-oscillator Hamiltonian after normal ordering) determined by

$$\mathscr{H}_0\,|\,0\rangle=0 . \tag{3.3}$$

This implies that

$$V(\Omega,\gamma_{ij})=C^{-1}(\Omega,\gamma_{ij})\left[\Delta_S-\frac{\partial^2}{\partial\Omega^2}\right]C(\Omega,\gamma_{ij}) . \tag{3.4}$$

The form (3.1) represents a specific choice of factor ordering in (1.1). Other choices are possible, but this is the simplest.

If the commutator of (3.1) with $\mathscr{H}_i$ of Appendix A is to be that of a scalar density, then $V(\Omega,\gamma_{ij})$ can only be a constant. This restricts the form of $C(\Omega,\gamma_{ij})$ to be an eigenfunction of $\Delta_S-\partial^2/\partial\Omega^2$. For the sake of simplicity we will choose $C=1$ which implies $V=0$, but, as any choice $V=$constant merely represents a shift in the spectrum of h, this choice is formally not as restrictive as it might seem.

For the choice $C=1$ the field operators (2.16)—(2.19) are not well defined; so some method of regulation will eventually need to be adopted. As mentioned previously, not all components of the metric operator need be well defined, since not all components are physical. To regulate the theory one can imagine taking

$$C=C_\epsilon(\Omega,\gamma_{ij}) \tag{3.5}$$

with $\lim_{\epsilon\to 0}C_\epsilon=1$, computing physical expressions, and then taking $\epsilon\to 0$. Physical quantities would hopefully be finite as the regulator is taken away. A choice of C_ϵ that also represents the possibility of gauge fixing is

$$C_\epsilon(\Omega,\gamma_{ij})=e^{-\epsilon(r_+{}^2+r_-{}^2)} , \tag{3.6}$$

where r_+ and r_- are as in Appendix B for some fixed direction b. This choice will regulate those components of g_{ij} corresponding to r_+,r_- but those components corresponding to the coordinates on the orbits of bNb^{-1} are left unregulated. This choice of C_ϵ also leads through (3.4) to gauge-breaking terms in $\mathscr{H}_0$. This inspires the conjecture that choosing (3.6) is equivalent to the choice of gauge in I, but, in its current state, ultralocal gauge field theory is far from being able to confirm this conjecture. For the remainder of this paper we will simply use $C=1$, and leave the problems and complications of regularization and gauge fixing for the future.

The operator h in (3.2) is not multiplied by the $e^{-3\Omega/2}$ that one might expect from the classical scalar density $g^{-1/2}G_{ijkl}\pi^{ii}\pi^{kl}$. This fact is formally equivalent to similar choices made for very different reasons by Misner[14] and Teitelboim.[4] It is (3.1) with the choice (3.2) that behaves under coordinate transformations as a scalar density as is discussed in Appendix A.

The treatment of the variable τ associated with the intrinsic time requires further discussion. This variable is taken to be nondynamical, and this might indicate that it should be eliminated by imposing a gauge condition of the form $\tau=\phi$ (other canonical variables). This is known to be incorrect.[4] Any such attempt to eliminate the intrinsic time by a canonical gauge condition will eliminate not only unphysical modes, but physical ones as well. The basic point is that the Hamiltonian $\mathscr{H}_0$ is quadratic in the momentum conjugate to τ; so eliminating τ and π by fixing a gauge and solving $\mathscr{H}_0=0$ for π will force a choice of sign when solving $\pi^2=6P_j^iP_i^j$ for π. The result of this choice will be to eliminate either all of the positive or negative ω 's (2.28) from the theory, i.e., either the expanding or the contracting geometries are eliminated. This is physically unacceptable, so Ω, $\partial/\partial\Omega$, and $\int d\Omega$ must be included in the field operators (2.16)—(2.19) and (3.2).

The Hamiltonian operator has particularly simple matrix elements with the states $|\,\omega,\lambda,b\,\rangle$. These are

$$\begin{aligned}\langle\omega',\lambda',b'\,|\,\mathscr{H}_0\,|\,\omega,\lambda,b\,\rangle&=\langle\omega',\lambda',b'\,|\,\omega,\lambda,b\,\rangle\int d\Omega\, d\gamma\, e^*_{\omega',\lambda',b'}he_{\omega,\lambda,b}\\&=\langle\omega',\lambda',b'\,|\,\omega,\lambda,b\,\rangle[-\omega^2(x)+\lambda_+{}^2(x)+\lambda_-{}^2(x)+\tfrac{1}{2}]\int d\Omega\, d\gamma\, e^*_{\omega',\lambda',b'}e_{\omega,\lambda,b}\\&=\langle\omega,\lambda,b\,|\,\omega,\lambda,b\,\rangle[-\omega^2(x)+\lambda_+{}^2(x)+\lambda_-{}^2+\tfrac{1}{2}]\delta(\omega',\lambda',b';\omega,\lambda b) .\end{aligned} \tag{3.7}$$

The quantity $\langle \omega,\lambda,b \mid \omega,\lambda,b \rangle$ would be equal to one for normalized states, but here is roughly $\exp[\delta(0)-\delta(0)]$. We call the states $|\omega,\lambda,b\rangle$ satisfying

$$\omega^2(x)=\lambda_+{}^2(x)+\lambda_-{}^2(x)+\tfrac{1}{2} \tag{3.8}$$

physical states since they satisfy the constraint

$$\mathscr{H}_0\,|\,\omega,\lambda,b\rangle=0\;. \tag{3.9}$$

The best way of extracting physical information from perturbation theory applied to the above formalism is as yet unknown, and all that we can do is make a guess. The close connection between the ultralocal quantizations for the full gravitational field and earlier work on quantum cosmology was pointed out in I. A main result of quantum cosmology[14] was the treatment of cosmology as a scattering problem in superspace. The asymptotic states in this scattering theory were usually quantum Kasner universes, and the scattering potential was determined by the scalar-curvature contribution to $\mathscr{H}_\perp$. It is known that the most general metric solution to the classical, ultralocal equations of motion has a form that can be interpreted as an independent Kasner universe at each spatial point[15–17] ($G_{ijkl}\pi^{ij}\pi^{kl}=0$ defines the light cone in superspace; Kasner universes are solutions traveling on the light cone). Using this fact and the analogy with quantum cosmology, we guess that an S-matrix theory on superspace (for us the space of g_{ij}) is the proper way of formulating the perturbation theory. The Hilbert space of asymptotic states will be that spanned by the physical states (3.8) of the ultralocal theory, and the perturbation potential R will induce transitions between the asymptotic states and thereby define an S matrix.[4] In the remainder of this section an outline of this S-matrix theory will be given following the lines set out in Ref. 18. The problem of actually computing finite S-matrix elements based on the perturbation R will not be discussed here. This problem is currently under investigation, and the results will be presented in a future publication.

Another conceptually important fact about the classical ultralocal theory is that it provides a description of the behavior of the gravitational field near a large class of singularities.[15,16] This is what one might expect of a strong-coupling limit. The space of states $|\omega,\lambda,b\rangle$ may be considered as a collection of states that either expand out of (positive ω) or collapse into (negative ω) a singularity. This follows from the identification of Ω with $\frac{1}{3}\ln g$ as the intrinsic time. As $\Omega\to-\infty$, the geometry becomes singular.

The Hamiltonian operator (3.1) is a Klein-Gordon operator, and normally the quantization of this sort of operator proceeds through second quantization. In the present case that would mean converting the elements ϕ of l into field operators. This would represent a third quantization of the above theory, and the Hilbert space on which the new field operators act would be interpreted as multiuniverse states (there would be operators to create and annihilate universes). It would take a brave physicist indeed to fully face the physical and theological implications of such a formalism. Luckily for this author it is not necessary to follow this path. A fully consistent quantum scattering theory can be based on the above formalism without modification.

Historically, fully consistent first quantizations of Klein-Gordon operators have depended on studying the Schrödinger equation

$$h\psi=i\frac{\partial}{\partial\theta}\psi\;, \tag{3.10}$$

where θ is an auxiliary time parameter identified with the proper time.[18–21] The theory is formulated in such a way that it is causal in the time parameter θ, but, since $\partial/\partial\Omega$ occurs in h as $\partial^2/\partial\Omega^2$, the intrinsic time parameter Ω can both increase and decrease (expansion and contraction in our case, particle and antiparticle for the Klein-Gordon equation). The theory so constructed is a single-particle theory and only "single line" diagrams (Fig. 1) can be described (the arrows on the lines point in the direction of increasing θ, the vertical direction on the graphs represents the intrinsic time Ω). When formulating these theories as scattering theories, all reference to the time θ disappears. The most complete such formulation is found in Ref. 18, and the remainder of this section is largely lifted from that paper. Other relevant references are those of Stückelberg,[19] Feynman,[20] and Nambu.[21]

The S-matrix theory will be formulated directly on the Hilbert space l. By using (2.13) and (2.15) this scattering theory can be immediately taken over to the field Hilbert space H (e.g., the unitary scattering operator defined explicitly on l below implicitly defines a unitary scattering operator on H). In the following $(\;|\;)$ will represent the inner product coming from integration in the measure $d\Omega\,d\gamma$. In general the result of such an integration will be a function of x.

The plane-wave states are normalized by

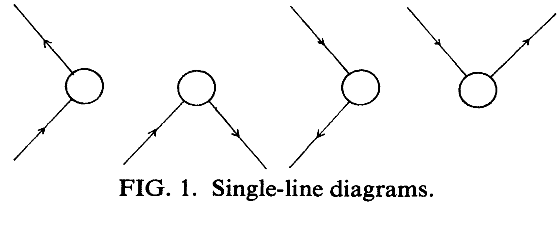

FIG. 1. Single-line diagrams.

$$(e_{\omega'(x'),\lambda'(x'),b'(x')} \mid e_{\omega(x),\lambda(x),b(x)}) = \delta(\omega(x),\omega'(x))\delta^{(2)}(\lambda(x),\lambda'(x))\delta^{(3)}(b(x),b'(x))\delta^{(3)}(\vec{x},\vec{x}') \ . \quad (3.11)$$

The nonobvious $\delta^{(3)}(x,x')$ contribution in (3.11) comes from the fact that if wave-packet states

$$\int g(\omega(x),\lambda(x),b(x))e_{\omega(x),\lambda(x),b(x)} \times d\omega\, \mu^{-1}(\lambda)d\lambda\, db$$

[and similarly with $g'(\omega'(x'),\lambda'(x'),b'(x'))$] are used in (2.15) the resultant inner product is

$$\int g^*(\omega(x),\lambda(x),b(x))g'(\omega(x),\lambda(x),b(x)) \times dx\, d\omega\, \mu^{-1}(\lambda)d\lambda\, db$$

Equation (3.11) is defined such that the appropriate inner product is obtained when multiplying (3.11) by g^* and g' and integrating over all primed and unprimed variables, x and x' included. The S matrix defined below will thus depend on x and x'. Let $\omega'(x')$, $\lambda'(x')$, $b'(x')$ be denoted as a group by $i(x')$ and ω,λ,b by $f(x)$. The scattering amplitude to go from wave packet g' to g is given by

$$\int S_{f(x)i(x')}g^*(f(x))g'[i(x')]dx\, dx'di\, df \quad (3.12)$$

with di and df being given by $d\omega\, \mu^{-1}(\lambda)d\lambda\, db$, as defined in Appendix B. From now on we will drop the $\delta^{(3)}(x,x')$ from (3.11) with it being understood that the appropriate expressions depend on x and x'.

The plane waves are eigenstates of h satisfying

$$he_{\omega,\lambda,b} = E_{\omega,\lambda}e_{\omega,\lambda,b} \ , \quad (3.13)$$

where

$$E_{\omega,\lambda} = -\omega^2 + \lambda_+{}^2 + \lambda_-{}^2 + \tfrac{1}{2} \quad (3.14)$$

(the θ dependence of such states is $e^{-iE_{\omega,\lambda}\theta}$). The physical states are those with $E=0$, (3.8).

If instead of ω,λ,b we parametrize the states by E, $\epsilon=\text{sign}(\omega),\lambda,b$, we find that[18]

$$(e_{\omega',\lambda',b'} \mid e_{\omega,\lambda,b}) = 2\pi\delta(E-E')[e_{\omega',\lambda',b'} \mid e_{\omega,\lambda,b}] \ , \quad (3.15)$$

where [|] is given by

$$[e_{\omega,\lambda,b} \mid e_{\omega',\lambda',b'}] = \frac{\omega}{\pi}\delta_{\epsilon\epsilon'}\delta^{(2)}(\lambda,\lambda')\delta^{(3)}(b,b') \quad (3.16)$$

and ω in (3.16) is equal to $(\lambda_+{}^2+\lambda_-{}^2+\frac{1}{2}-E)^{1/2}$. The [|] inner product is just the absolute value of the Klein-Gordon inner product, i.e.,

$$[\phi \mid \psi] = \left| i\int d\gamma \left[\phi^* \frac{\partial\psi}{\partial\Omega} - \frac{\partial\phi^*}{\partial\Omega}\psi \right] \right| \ . \quad (3.17)$$

This is simply the inner product determined by the complex structure on the space of solutions to the Klein-Gordon equation.[22] Notice that (3.17) is independent of the intrinsic time without having to fix a gauge to eliminate it.

If the interaction potential V is independent of the time (as it is for gravity), then the $E_{\omega,\lambda}$ eigenvalues for the initial and final states are the same. In particular, physical states ($E=0$) scatter only into other physical states (although internal lines can go off-shell).

The scattering matrix elements are given by[4,18]

$$S_{fi} = (\psi_{(f)} \mid \psi_{(i)}) - 2\pi i\delta(E_f-E_i)[(\psi_{(f)} \mid V \mid \psi_{(i)}) - (\psi_{(f)} \mid VP(E_i)V \mid \psi_{(i)}) + \cdots] \ , \quad (3.18)$$

where

$$P(E_i) = \int_0^\infty e^{-\theta(h-E_i)}d\theta \ , \quad (3.19)$$

and i and f label initial and final states of definite energy E_i and E_f. $P(E_i=0)$ is just the propagator derived in Appendix B. Using (3.14) we find

$$S_{fi} = (\psi_{(f)} \mid S \mid \psi_{(i)}) = 2\pi\delta(E_i-E_f)[\psi_{(f)} \mid S \mid \psi_{(i)}] \equiv 2\pi\delta(E_i-E_f)\tilde{S}_{fi} \ . \quad (3.20)$$

From (3.20) it immediately follows that unitarity of S_{fi} in the inner product (|) is equivalent to the unitarity of $\tilde{S}_{fi}$ in the inner product [|]. The perturbative form of $\tilde{S}_{fi}$ is

$$\tilde{S}_{fi} = [\psi_{(f)} \mid \psi_{(i)}] - i[(\psi_{(f)} \mid V \mid \psi_{(i)}) - (\psi_{(f)} \mid VP(E_i)V \mid \psi_{(i)}) + \cdots] \ , \quad (3.21)$$

as follows from (3.18). For a scalar field (3.21) gives, for a restricted set of processes, the same answers for scattering probabilities as quantum field theory.[18] We conjecture that it is the proper object to compute in ultralocal gravity. If the potential V contains no spatial derivatives, then $\widetilde{S}_{f(x)i(x')}=\widetilde{S}_{f(x)i(x)}\delta(x,x')$, but for perturbations like $g^{1/2}R$ there can be contributions to $\widetilde{S}_{fi}$ that are formally proportional to things like $\delta_{,ii}(x,x')$. We have not yet been successful in making sense of this type of perturbation theory, but work in this direction is currently in progress.

It is helpful to review some of the steps involved in developing the above formalism. The canonical treatment of gravity leads one to consider the constraint

$$\mathscr{H}_{\perp}=G_{ijkl}\pi^{ij}\pi^{kl}+V(g_{ij})=0 \ ,$$

and dropping the potential V leads to the ultralocal Hamiltonian $\mathscr{H}_0$. It has been found useful to think of $\mathscr{H}_0$ as a differential operator on the space of metrics.[5] The natural metric to put on this space is G_{ijkl} which as signature $(-+++++)$, and, as a result of this hyperbolic signature, $\mathscr{H}_0$ is identified with the d'Alembertian on the space of metrics. The hyperbolic nature of this operator is an essential attribute of a symmetric tensor field,[23] and the analogous operators for the scalar and Yang-Mills fields are elliptical. As a result, many of the above considerations (e.g., states developing forward or backward in an intrinsic time) are, in field theory, essentially unique to gravity (the relativistic free particle shares many of these properties with gravity[4]).

The dynamical generator $\mathscr{H}_0$ contains the timelike momentum squared, and as a result $\mathscr{H}_0$ cannot be eliminated as a constraint by fixing a canonical gauge[4] (this also applies to the constraint $P^{\mu}P_{\mu}+m^2=0$ for the free particle). Such a gauge would fix an intrinsic time; thus for gravity there can be no attempt to eliminate an intrinsic time as a canonical variable. This means in the quantization of the relativistic free particle that x^0 must be a quantum operator along with x^i.[18] For ultralocal quantum gravity we must include the variable Ω (identified with the intrinsic time) in the quantization, i.e., there is a field operator associated with it, and the relevant inner product includes an integration over $d\Omega$. The operator $\mathscr{H}_0$ is identified with the Klein-Gordon operator

$$h=\frac{\partial^2}{\partial\Omega^2}-\Delta_S \ . \tag{3.2}$$

In the development of the scattering theory the relation between the full inner product with its integration over Ω and the Klein-Gordon inner product is made clear [(3.15)–(3.17)].

Solutions to the classical ultralocal theory are such that for a fixed $\vec{x}$ the metric is in the form of a Kasner metric.[17] In this sense one speaks very loosely of an independent (ultralocality means no correlations) Kasner universe at each spatial point. Quantum mechanically the physical plane-wave states [i.e., those satisfying (3.8)] can be thought of as independent quantum Kasner universes[14] at each spatial point. The scattering theory will describe the scattering between these states (when unambiguous calculation becomes possible). At a fixed point $\vec{x}$ there are four basic processes, i.e., an expanding "universe" bounces off a potential into another expanding "universe", the potential scatters an expanding "universe" into a contracting one, and the Ω time reverses of these processes. For a fixed $\vec{x}$ there are no "multiuniverse" processes. The general scattering event will of necessity be very complicated since "universes" at neighboring points will couple through the spatial derivatives in the perturbing potential. There will be a full scattering formalism in $\mathrm{R}\times\mathrm{SL}(3,\mathrm{R})/\mathrm{SO}(3)$ at each point $\vec{x}$ with the scattering at $\vec{x}$ being influenced by what is happening at $\vec{x}+\delta\vec{x}$.

The word universe has been thrown around pretty freely above, so it is appropriate to mention that, in its present form, the above formalism has nothing to say about the macrocosmic universe. Strong-coupling gravity is applicable to the short-distance behavior of the gravitational field, and therefore might prove to be important for understanding the extremely early universe. We are now far from being able to say anything about this.

IV. DISCUSSION

There are two major questions raised in the above for which there are at the moment no definitive answers. The first has to do with the type of representation chosen. For ultralocal theories there exist straightforward generalizations of the standard Fock representation in addition to the type of representation that we have chosen, but these Fock representations are not applicable to interacting ultralocal theories[24] (gravity is self-interacting). The affine variables π^i_j fit naturally into the representation that we have chosen, but not into the standard Fock representation.[12] There are also more complicated ultralocal representations of the affine commutation relations,[25] but as the one that we have chosen is the simplest, it seems natural to devote our energy initially to it.

The other major question concerns the choice of $C(\Omega,\gamma_{ij})$. The range of possible $C(\Omega,\gamma_{ij})$ give a wide range of possible theories although the requirement that $\mathscr{H}_0$ be a scalar density severely restricts the possible C's. As a consequence of this restriction the field operators are undefined (infinite). If the infinities are to be regulated the coordinate transformation behavior of $\mathscr{H}_0$ must be changed. It is hoped that the tasks of regulating the infinities and fixing a gauge may be done simultaneously, but at the moment more work on this is required. The choice $C=1$ drastically simplies the formalism, but more general choices cause no problems in principle.

In its current state strong-coupling gravity is very formal. There are purely technical barriers to our being able to do concrete calculations. These include not knowing how to fix gauges and determine the associated analog of Faddeev-Popov ghosts, and regulating the ultralocal limit. The major problem is to develop the ultralocal perturbation theory, and definitive answers to the problems of gauge fixing and regularization await this development.

In addition to these technical details the theory requires an injection of physics. We are not yet completely confident of what it is that we want to calculate. For Fock-based field theories the classical concept of particle plays a large role in keeping the formalism from being sterile. The corresponding concept for ultralocal gravity comes from homogeneous cosmology. We have some understanding of the classical behavior of gravity in the ultralocal limit with the important ideas coming from homogeneous cosmology. In the next paper[26] we will exploit this classical understanding in a semiclassical approximation to the quantum field theory.

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APPENDIX A: COORDINATE TRANSFORMATION GENERATOR

The main part of this paper concentrates on the dynamics of ultralocal gravity, but coordinate invariance is an important part of the theory and it is useful to have an explicit representation on the Hilbert space of this theory of the generator of coordinate transformations $\mathscr{H}_i$. This we do below as well as discuss the coordinate transformation properties of some important operators and the form of a coordinate-invariant "ground" state $|0\rangle$.

We will assume that Ω,γ_{ij} are not functions of x. This simplifies the formulas, but it is not a necessary assumption.[27] To derive the results of this appendix (under this assumption), it is necessary to use the following relations:

$$\left[\int d\Omega\, d\gamma\, B^\dagger(x)aB(x), \int d\Omega' d\gamma' [B^\dagger_{,i}(x')bB(x')+B^\dagger(x')bB_{,i}(x')]\right]$$
$$= -\int d\gamma\, d\Omega\, B^\dagger(x,\Omega,\gamma)[a,b]B(x,\Omega,\gamma)\delta_{,i}(x,x') , \quad \text{(A1)}$$

$$\left[\int d\gamma\, d\Omega\, B^\dagger(x)aB(x), \int d\gamma' d\Omega' [B^\dagger_{,i}(x')B(x')-B^\dagger(x')B_{,i}(x')]\right]$$
$$= -\int d\gamma\, d\Omega [2B^\dagger(x)aB(x)\delta_{,i}(x,x')+2(B^\dagger aB_{,i}+B^\dagger_{,i}aB)\delta(x,x')] , \quad \text{(A2)}$$

where a and b are arbitrary operators formed from Ω, γ_{ij}, $\partial/\partial\Omega$, $\partial/\partial\gamma_{ij}$.

The operator $\mathscr{H}_i$ is determined by the requirement that it gives the proper coordinate transformation laws for the metric g_{mn} and the affine momenta π^i_j where

$$g_{mn}=\int d\gamma\, d\Omega B^\dagger k_{mn} B , \quad \text{(A3)}$$

$$\pi^i_j=\frac{i}{2}\int d\gamma\, d\Omega\, B^\dagger(\bar{\rho}^{i\dagger}_j-\bar{\rho}^i_j)B , \quad \text{(A4)}$$

$$k_{mn}=e^{\Omega}\gamma_{mn} , \quad \text{(A5a)}$$

$$\bar{\rho}^i_j=\gamma_{jl}\frac{\partial}{\partial\gamma_{il}}+\tfrac{1}{3}\delta^i_j\frac{\partial}{\partial\Omega}=k_{jl}\frac{\partial}{\partial k_{il}} . \quad \text{(A5b)}$$

The generator $\mathscr{H}_i$ can be written in the form

$$\mathscr{H}_i(x)=-2\pi^j_{i,j}(x)+\pi^k_{k,i}(x)-\frac{i}{2}\int d\gamma\, d\Omega(B^\dagger_{,i}B-B^\dagger B_{,i}) , \quad \text{(A6)}$$

where

$$\pi^j_{i,j}=\frac{i}{2}\int d\gamma\, d\Omega[B^\dagger_{,j}(\bar\rho^{j\dagger}_i-\bar\rho^j_i)B+B^\dagger(\bar\rho^{j\dagger}_i-\bar\rho^{ij}_i)B_{,j}]\ , \tag{A7}$$

and similarly for $\pi^k_{k,i}$. It is easy to check [using (A1) and (A2)] that

$$\left[g_{mm}(x),\int dx'\xi^i(x')\mathcal{H}_i(x')\right]=i(g_{mn,i}\xi^i+g_{im}\xi^i_{,n}+g_{in}\xi^i_{,m})\ , \tag{A8}$$

$$\left[\pi^m_n(x),\int dx'\xi^i(x')\mathcal{H}_i(x')\right]=i(\pi^m_{n,i}\xi^i+\pi^m_i\xi^i_m-\pi^i_n\xi^m_{,i}+\pi^m_n\xi^i_{,i})\ , \tag{A9}$$

$$[\mathcal{H}_i(x),\mathcal{H}_j(x')]=i[\mathcal{H}_i(x')\delta_{,j}(x,x')-\mathcal{H}_j(x)\delta_{,i}(x',x)]\ , \tag{A10}$$

as one requires of a generator of coordinate transformations. The $(B^\dagger_{,i}B-B^\dagger B_{,i})$ term in (A6) is just the translation generator for a flat-space ultralocal theory[12]; so its appearance is not surprising even though nothing similar is in the classical expression for $\mathcal{H}_i$.

Now that we have an expression for $\mathcal{H}_i$ the coordinate transformation behavior of operators other than the basic ones can be investigated. For instance,

$$V(x)=\int d\gamma\, d\Omega\, B^\dagger(x,\Omega,\gamma_{ij})B(x,\Omega,\gamma_{ij}) \tag{A11}$$

is, strictly speaking, undefined, but formally it transforms as a scalar density, i.e.,

$$[V(x),\int dx'\xi^i(x')\mathcal{H}_i(x')]=i(V_{,i}\xi^i+V\xi^i_{,i})\ . \tag{A12}$$

This is an unexpected result since the obvious representation for the classical scalar density $g^{1/2}$ is $\int B^\dagger k^{1/2}B$ which does not transform as a scalar density, instead (A11) does. In fact, this sort of naive identification between classical field and quantum does not in general preserve behavior under coordinate transformations. The most important example of this is $\mathcal{H}_0=g^{-1/2}(\pi^i_j\pi^j_i-\frac{1}{2}\pi^2)$. We have taken the quantum realization of this to be

$$\mathcal{H}_0=\int d\gamma\, d\Omega\, B^\dagger\left[\frac{\partial^2}{\partial\Omega^2}-\Delta_S\right]B\ , \tag{A13}$$

without a $k^{-1/2}$, since it is (A13) that satisfies

$$\left[\mathcal{H}_0(x),\int dx'\xi^i(x')\mathcal{H}_i(x')\right]=i(\mathcal{H}_{0,i}\xi^i+\mathcal{H}_0\xi^i_{,i})\ , \tag{A14}$$

i.e., transforms as a scalar density. Notice that as a result of (A11) and (A12) a cosmological constant term $\lambda g^{1/2}$ in the classical Hamiltonian results in replacing

$$\frac{\partial^2}{\partial\Omega^2}-\Delta_s \text{ by } \frac{\partial^2}{\partial\Omega^2}-\Delta_S+\lambda\ ,$$

i.e., in translating the spectrum of the differential operator by a constant.

The expression (A6) can be used to find $C(k,\gamma_{ij})$ such that

$$\mathcal{H}_i\,|\,0\rangle=0\ . \tag{A15}$$

Equation (A15) implies that

$$\int d\Omega\, d\gamma\langle\phi\,|\,B^\dagger_{,j}\,|\,0\rangle\left[-2\bar\rho^j_i+\delta^j_i\bar\rho^k_k-\frac{i}{2}\delta^j_i\right]\times C(k,\gamma_{ij})=0$$

for all $\phi\in l$; so

$$\left[-2\bar\rho^j_i+\delta^j_i\bar\rho^k_k-\frac{i}{2}\delta^j_i\right]C(k,\gamma_{ij})=0\ . \tag{A16}$$

Assuming that $C(k,\gamma_{ij})$ is of the form

$$C(k,\gamma_{ij})=k^{-3/2}=e^{-(9/2)\Omega}\ , \tag{A17}$$

(A16) implies

$$P=-\tfrac{3}{2}\ , \tag{A18}$$

$$\gamma_{jl}\frac{\partial}{\partial\gamma_{il}}f=0\ , \tag{A19}$$

i.e., f is an SL(3,R)-invariant function of γ_{ij}, so f is constant. If $C(k,\gamma_{ij})$ is of the form

$$C(k,\gamma_{ij})=k^{-3/2}=e^{-(9/2)\Omega}\ , \tag{A20}$$

then (A15) is satisfied. The steps involved in deriving (A20) depended on k,γ_{ij} being independent, but the result is the same even when this restriction is lifted.[27]

The function C in (A20) is very reminiscent of similar functions used for ultralocal scalar fields.[6,12,24] Here it is very much more singular than there, because the integration measure is $d\Omega\propto k^{-1}dk$ and because there is no dependence on γ_{ij} in (A20).

APPENDIX B: PROPAGATORS ON SL(3,R)/SO(3)

In this appendix the discussion of SL(3,R)/SO(3) begun in I is extended. In particular, the propagator for the wave operator $\Delta_S-\partial^2/\partial\Omega^2$ [Δ_S is the Laplace-Beltrami operator for SL(3,R)/SO(3)] is computed in momentum space (following the discussion of Ref. 28), and the configuration space form is discussed (following Ref. 29). Most of the presentation of the material here was developed in discussion with the authors of Ref. 28. None of this material is original with this paper, but a very detailed discussion is given, since extracting the relevant information from the mathematical literature is difficult and the content of this appendix is so important for the problem at hand (as important here as Fourier analysis on R^3 is for ordinary quantum mechanics).

For finding a complete set of eigenfunctions for the operator Δ_S, it is convenient to work in a special coordinate system on S, the five-dimensional manifold of 3×3, symmetric, positive-definite matrices of unit determinant (this choice of coordinates will be an analog of choosing Cartesian coordinates for R^n). We make use of the fact that an arbitrary element γ of S can be uniquely written in the form[30]

$$\gamma=naa^tn^t\ , \tag{B1}$$

where t denotes transposition, and a and n are matrices of the form

$$a=\begin{bmatrix} e^{r_1} & 0 & 0\\ 0 & e^{r_2} & 0\\ 0 & 0 & e^{-(r_1+r_2)}\end{bmatrix}\ , \tag{B2}$$

$$n=\begin{bmatrix} 1 & n_1 & n_2\\ 0 & 1 & n_3\\ 0 & 0 & 1\end{bmatrix}\ . \tag{B3}$$

The matrices a form a maximal Abelian subgroup A of SL(3,R), and the matrices n form a unipotent subgroup N. The coordinate system that we choose for S is given by r_1,r_2,n_1,n_2,n_3.

The appropriate SL(3,R)-invariant metric for S (Refs. 5 and 31) is

$$G(d\gamma,d\gamma)=\mathrm{tr}(\gamma^{-1}d\gamma\,\gamma^{-1}d\gamma)\ , \tag{B4a}$$

or, expressed in terms of γ,

$$G^{ijkl}=\tfrac{1}{2}(\gamma^{ik}\gamma^{jl}+\gamma^{il}\gamma^{jk}-\tfrac{2}{3}\gamma^{ij}\gamma^{kl})\ . \tag{B4b}$$

Expressions (B1)—(B4) imply that the line element in the r_1,r_2,n_1,n_2,n_3 coordinate system is given by

$$G(d\gamma,d\gamma)=8[(dr_1)^2+(dr_2)^2+dr_1dr_2]+2e^{-2(r_1-r_2)}(dn_1)^2+2e^{-2(2r_1+r_2)}(dn_2)^2-4e^{-2(2r_1+r_2)}n_1dn_2dn_3$$
$$+2(e^{-2(2r_2+r_1)}+n_1{}^2e^{-2(2r_1+r_2)})(dn_3)^2\ . \tag{B5}$$

It is convenient to define the coordinates

$$r_+=\sqrt{6}(r_1+r_2),\quad r_-\sqrt{2}(r_1-r_2) \tag{B6}$$

in terms of which the line element is

$$G(d\gamma,d\gamma)=(dr_+)^2+(dr_-)^2+e^{-\sqrt{2}r_-}(dn_1)^2+2e^{-(\sqrt{3}r_++r_-)/\sqrt{2}}(dn_2)^2-4e^{-(\sqrt{3}r_++r_-)/\sqrt{2}}n_1dn_2dn_3$$
$$+2(e^{-(\sqrt{3}r_+-r_-)/\sqrt{2}}+n_1{}^2e^{-(\sqrt{3}r_++r_-)/\sqrt{2}})(dn_3)^2\ . \tag{B7}$$

The variables r_+ and r_- correspond to an orthonormal basis in the Lie algebra of A.

The Laplace-Beltrami operator is given by

$$\Delta_S=G^{-1/2}\frac{\partial}{\partial\gamma_A}G^{1/2}G_{AB}\frac{\partial}{\partial\gamma_B}$$
$$=\frac{\partial^2}{\partial r_+{}^2}-(\tfrac{3}{2})^{1/2}\frac{\partial}{\partial r_+}+\frac{\partial^2}{\partial r_-{}^2}-(\tfrac{1}{2})^{1/2}\frac{\partial}{\partial r_-}+\tfrac{1}{2}e^{\sqrt{2}r_-}\frac{\partial^2}{\partial n_1{}^2}$$
$$+\tfrac{1}{2}(e^{(\sqrt{3}r_++r_-)/\sqrt{2}}+n_1{}^2e^{(\sqrt{3}r_+-r_-)/2})\frac{\partial^2}{\partial n_2{}^2}$$
$$+e^{(\sqrt{3}r_+-r_-)/2}n_1\frac{\partial^2}{\partial n_2\partial n_3}+\tfrac{1}{2}e^{(\sqrt{3}r_+r_-)/2}\frac{\partial^2}{\partial n_3{}^2}\ , \tag{B8}$$

where G_{AB} is determined by (B7) and $G=\det G_{AB}=8e^{-\sqrt{2}(\sqrt{3}r_++r_-)}$. The Laplace-Beltrami operator is SL(3,R) invariant.

In order to find a complete set of eigenfunctions to (B8) it is useful to begin with those that are independent of n_1, n_2, and n_3. Finding eigenfunctions of (B8) that depend only on r_+ and r_- is made easy by the fact that derivatives with respect to r_+ and r_- occur in (B8) with constant coefficients. These eigenfunctions are called plane waves with "planes" (more correctly they are called horocycles) on which the functions are constant being the three-dimensional images of the action of N on diagonal matrices, i.e., the plane through the point a^2 of S is the set of points na^2n^+ for all $n\in N$. These plane waves are

$$\psi_{\lambda_+,\lambda_-}(r_+,r_-)=e^{i(\lambda_+r_++\lambda_-r_-)+\sqrt{3/8}r_++\sqrt{1/8}r_-}\ , \tag{B9}$$

which satisfy

$$\Delta_S\psi_{\lambda_+,\lambda_-}=-(\lambda_+{}^2+\lambda_-{}^2+\tfrac{1}{2})\psi_{\lambda_+,\lambda_-}\ , \tag{B10}$$

where λ_+ and λ_- represent the wave number of the plane wave [it is a property of SL(3,R)/SO(3) that the wave number is two dimensional rather than one dimensional as in R^n]. The planes that we have introduced so far are all parallel to each other, and the plane waves (B9) propagate in a single direction (the measure of "distance" along a given direction is two dimensional, labeled by r_+ and r_-). To get a complete set of eigenfunctions we must be able to generalize the definition of planes and plane waves to accommodate waves propagating in arbitrary directions.

To give a useful description of the directions in which plane waves propagate it is necessary to repeat a definition given in Appendix A of I for the boundary of S [notice that for R^2 the boundary SO(2) is the same as the set of directions in which plane waves propagate]. The boundary B of S is by definition[30] SO(3)/M, where

$$M=\{k\in \mathrm{SO}(3)\,|\,\mathrm{Ad}(k)H=H \text{ for all } H\in\mathscr{A}\text{, the Lie algebra of } A\}$$
$$=\left\{\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix},\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix},\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right\}. \tag{B11}$$

If one accepts that a Weyl chamber (see I or Ref. 30) in A is a generalization of a measure of radius [something that is made more plausible by the fact that for SL(2,R) a Weyl chamber can be identified with the positive numbers], then this definition can be motivated by the theorem that S is essentially diffeomorphic to $B\times\exp(\mathscr{A}^+)$ ($\mathscr{A}^+$ is a Weyl chamber). The diffeomorphism $B\times\exp(\mathscr{A}^+)\to S$ puts a polar coordinate system on S, and there are subsets of S of lower dimension, analogs of the origin of polar coordinates on R^2, that are not in the image of $B\times\exp(\mathscr{A}^+)$.

A set of parallel horocycles was defined above by the orbits of N on a point $a\cdot o$ ($g\cdot o$ means $g\mathbb{1}g^t$). Note that by Eq. (B1) any point s in S can be written as $na\cdot o$ for some n and a; so the orbit of N acting on s is the same as N acting on $a\cdot o$. The orbits of N acting on points of S are the same as the set of planes defined above. Generally a set of parallel horocycles is given by the orbits of gNg^{-1} for $g\in$ SL(3,R). Any element g can be written as $g=kan$, $k\in$ SO(3); so $gNg^{-1}=kNk^{-1}$ since $aNa^{-1}=N$ (see the definition of N given in I or Ref. 30). For $m\in M$ we have $mNm^{-1}=N$, as can be seen directly from the form (B11) of the four elements of M, but a more general proof is also straightforward. As a result the set of groups kNk^{-1} whose orbits provide the horocycles is in fact the same as the set of bNb^{-1}, $b\in B=$ SO(3)/M. In fact, one can show that any horocycle $gNa^{-1}h\cdot 0$ can be written uniquely as[32] $bNa\cdot 0$; b is called the direction of the horocycle and a is called the complex distance of the horocycle from the origin (complex because a is determined by two quantities). For fixed $b\in B$, all horocycles $bNa\cdot 0$ are parallel to each other. A similar formula can be applied to the x-y plane, where the analog of N is the group of translations parallel to the y axis, a is an x coordinate, and b is an element of SO(2), the boundary of R^2. Given a line (horocycle) in R^2, b is determined by the angle required to rotate the line about the origin until it is orthogonal to the x axis and a is the x coordinate of this rotated line (i.e., the orthogonal distance from the origin to the line).

SO(3) is parametrized by the Euler angles, with an arbitrary group element being given by

$$R=e^{\psi\kappa_3}e^{\theta\kappa_1}e^{\phi\kappa_3} \tag{B12}$$

with

$$\kappa_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \kappa_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

The Euler angles have the ranges $0 \le \phi, \psi \le 2\pi$ and $0 \le \theta \le \pi$. It is easy to check that the effect of taking the quotient by M is to restrict the ranges of the Euler angles to $0 \le \phi, \psi \le \pi$ and $0 \le \theta \le \pi$. The space of directions in which plane waves on SL(3,R)/SO(3) travel, i.e., SO(3)/M, is parametrized by the Euler angles restricted to these ranges.

The plane wave (B9) is a wave propagating in the $\theta = \phi = \psi = 0$ direction (we will call this direction I and b will be an arbitrary point of B). To obtain plane waves propagating in directions other than $b = I$, we act on the plane waves (B9) with an arbitrary element of B. The SL(3,R) invariance of the Laplacian will guarantee that the functions so obtained are eigenfunctions with the same eigenvalue (B10). The plane wave propagating in the direction b is given by

$$e_{\lambda,b}(\gamma) = e_{\lambda,I}(b^{-1}(\gamma)) = e_{\lambda,I}(b^t \gamma b) , \tag{B13}$$

where $\gamma \in S$ and

$$e_{\lambda,I}(b^t \gamma b) = \psi_{\lambda_+,\lambda_-}(r_+, r_-) \tag{B14}$$

with ψ being given by (B9) and r_+, r_- being determined by the element a of A in the Iwasawa decomposition

$$b^t \gamma b = n a a^t n . \tag{B15}$$

Just as functions (B9) are constant on the horocycles generated by N (i.e., they are independent of n_1, n_2, n_3) the functions (B13) are constant on the horocycles generated by the group bNb^{-1} [proof:

$$e_{\lambda,b}(bnb^{-1}\gamma b n^t b^{-1}) = e_{\lambda,I}(nb^{-1}\gamma b n^t) = e_{\lambda,I}(b^{-1}\gamma b) = e_{\lambda,b}(\gamma)] .$$

The plane waves (B13) form a complete set of functions on S. Any function $f(\gamma)$, $\gamma \in S$ can be expanded as a Fourier series,[30] i.e.,

$$f(\gamma) = \frac{1}{2\pi} \int_{\lambda_+ > \sqrt{3}\lambda_- > 0} \int_B \hat{f}(\lambda, b) e_{\lambda,b}(\gamma) \mu^{-1}(\lambda) d\lambda \, db , \tag{B16}$$

where $d\lambda = d\lambda_+ d\lambda_-$, $db = (2\pi^2)^{-1} \sin\theta \, d\phi \, d\theta \, d\psi$ [from the Haar measure on SO(3)],

$$\mu(\lambda) = \frac{1}{\sqrt{8}\pi} [\lambda_-(3\lambda_+^2 - \lambda_-^2)]^{-1} \coth(\pi\sqrt{2}\lambda_-) \coth\left[\frac{\pi}{\sqrt{2}}(\sqrt{3}\lambda_+ + \lambda_-)\right] \coth\left[\frac{\pi}{\sqrt{2}}(\sqrt{3}\lambda_+ - \lambda_-)\right] , \tag{B17}$$

and the Fourier components $\hat{f}(\lambda, b)$ are given by

$$\hat{f}(\lambda, b) = \int_S f(\gamma) e^*_{\lambda,b}(\gamma) d\gamma . \tag{B18}$$

The measure $d\gamma$ in (B18) is the one induced on S by the Haar measure on SL(3,R), e.g., in the r_+, r_-, n_1, n_2, n_3 coordinate system

$$d\gamma = \sqrt{8} e^{-(\sqrt{3}r_+ + r_-)/\sqrt{2}} dr_+ dr_- dn_1 dn_2 dn_3 . \tag{B19}$$

The expansion (B16) in terms of plane waves forms the basis of our further discussion of the wave equation.

The restriction of the range of λ in (B16) to $\lambda_+ > \sqrt{3}\lambda_- > 0$ requires further discussion. In spite of the fact that it has two components, λ is the analog of the wave number of plane waves on Euclidean space. The restriction on λ adopted here is the generalization of the restriction that the Euclidean wave number be positive. This restriction on the Euclidean wave number results from the invariance of plane-wave solutions under reflections. Including plane waves with all angles of propagation and both positive and negative wave numbers results in a double counting of the plane waves. For SL(3,R)/SO(3) the role of the reflections is played by the Weyl group,[30] a six-element group given by

$$W = \left[w_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; w_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; w_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; w_4 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \right.$$

$$\left. w_5 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \; w_6 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \right] . \tag{B20}$$

The plane waves (B13) are invariant under the action of W; its action on λ is given by

$$\begin{aligned} &w_1\lambda_+=\lambda_+,\;\; w_1\lambda_-=\lambda_- ,\\ &w_2\lambda_+=\lambda_+,\;\; w_2\lambda_-=-\lambda_- ,\\ &w_3\lambda_+=-\tfrac{1}{2}(\lambda_++\sqrt{3}\lambda_-),\;\; w_3\lambda_-=-\tfrac{1}{2}(\sqrt{3}\lambda_+-\lambda_-) ,\\ &w_4\lambda_+=-\tfrac{1}{2}(\lambda_+-\sqrt{3}\lambda_-),\;\; w_4\lambda_-=\tfrac{1}{2}(\sqrt{3}\lambda_++\lambda_-) ,\\ &w_5\lambda_+=-\tfrac{1}{2}(\lambda_+-\sqrt{3}\lambda_-),\;\; w_5\lambda_-=-\tfrac{1}{2}(\sqrt{3}\lambda_++\lambda_-) ,\\ &w_6\lambda_+=-\tfrac{1}{2}(\lambda_++\sqrt{3}\lambda_-),\;\; w_6\lambda_-=\tfrac{1}{2}(\sqrt{3}\lambda_+-\lambda_-) . \end{aligned} \tag{B21}$$

The action of W on r_+,r_- is similar. To eliminate the six-fold overcounting of the plane waves the restriction $\lambda_+>\sqrt{3}\lambda_->0$ is imposed. This restriction defines a region in the (λ_+,λ_-) plane. No two points of this region are connected by a transformation (B21), and the image of this region under these transformations is the whole plane.

Now that we have a complete set of eigenstates of Δ_S, computing the propagator in momentum space is straightforward.[28] Writing

$$e_{\omega,\lambda,b}\equiv|\,\omega,\lambda,b)\equiv(2\pi)^{-1}e^{-i\omega\Omega}e_{\lambda,b} , \tag{B22}$$

and using

$$\left[\Delta_S-\frac{\partial^2}{\partial\Omega^2}\right]|\,\omega,\lambda,b) =[\omega^2-(\lambda_+{}^2+\lambda_-{}^2+\tfrac{1}{2})]\,|\,\omega,\lambda,b) , \tag{B23}$$

the propagator in momentum space is easily found to be

$$\left[\omega_2,\lambda_2,b_2\left|\left[\Delta_S-\frac{\partial^2}{\partial\Omega^2}\right]^{-1}\right|\omega_1,\lambda_1,b_1\right] = \delta(\omega_2;\omega_1)\delta(\lambda_2,b_2;\lambda_1,b_1) \times[\omega^2-(\lambda_+{}^2+\lambda_-{}^2+\tfrac{1}{2})]^{-1} , \tag{B24}$$

where the use of Feynman boundary conditions is understood. The δ function

$$\delta(\lambda_2,b_2;\lambda_1,b_1)=\int_S d\gamma\, e^*_{\lambda_2,b_2}(\gamma)e_{\lambda_1,b_1}(\gamma)$$

satisfies

$$\int_{\lambda_+>\sqrt{3}\lambda_->0}\int_B d\lambda\, db\,\mu^{-1}(\lambda) \times\delta(\lambda',b';\lambda,b)f(\lambda,b) =f(\lambda',b'). \tag{B25}$$

The use of Feynman boundary conditions guarantees that positive-frequency states propagate forward in Ω and negative-frequency states backward.

One can easily do everything in the above discussion with no restriction on the range of λ. In that case expressions such as (B16) must be divided by a factor of 6 because of the six-fold over-counting and the δ function $\delta(\lambda_2,b_2;\lambda_1,b_1)$ is not only concentrated on points where $\lambda_2=\lambda_1$, but also points where $\lambda_2=w\lambda_1$ for any $w\in W$.

For completeness we now discuss the coordinate-space form of the propagator on $\mathrm{R}\times\mathrm{SL}(3,\mathrm{R})/\mathrm{SO}(3)$. We will not be able to get far in this discussion since an explicit form of the coordinate-space Green's function is unknown. The extent of what we can do will depend, as above, on exploiting the fact that $\partial/\partial r_+$ and $\partial/\partial r_-$ occur in Δ_S with constant coefficients.

The discussion will follow that given in Ref. 29. The method used there is to exploit the heat equation to find the heat kernel, i.e., the distribution F satisfies the heat equation

$$\frac{\partial}{\partial\theta}F(\theta;\Omega,\gamma;\Omega',\gamma')=\Box F(\theta;\Omega,\gamma;\Omega',\gamma') \tag{B26}$$

with boundary conditions

$$\lim_{\theta\to0}F(\theta;\Omega,\gamma;\Omega',\gamma')=\delta(\Omega,\Omega')\delta(\gamma,\gamma') . \tag{B27}$$

The symbol $\Box$ in (B26) stands for $\Delta_S+\partial^2/\partial\Omega^2$ (a Wick rotation on Ω has been made). The $\mathrm{GL}^+(3,\mathrm{R})$ invariance of $\Box$ and the fact that it is a symmetric operator in the measure $d\Omega\, d\gamma$ lead to the properties

$$F(\theta;\Omega;\gamma;\Omega',\gamma')=F(\theta;\Omega',\gamma';\Omega,\gamma) , \tag{B28}$$

$$F(\theta;\Omega+k,g\gamma g^t;\Omega'+k,g\gamma' g^t) =F(\theta;\Omega,\gamma;\Omega',\gamma') , \tag{B29}$$

where $k\in\mathrm{R}$ and $g\in\mathrm{SL}(3,\mathrm{R})$.

As mentioned above, we will exploit the parts of Δ_S with constant coefficients. To do this define

$$\bar{F}_{b=I}(\theta;\Omega,\gamma;\Omega',\gamma')=\bar{F}_{b=I}(\theta;\Omega,r_+,r_-;\Omega',r'_+,r'_-)$$
$$=\sqrt{8}e^{-(\sqrt{3}r_+ + r_-)/\sqrt{2}}\int F(\theta;\Omega,\gamma;\Omega',n\gamma' n^t)dn_1dn_2dn_3\ , \tag{B30}$$

where n_1,n_2,n_3 are as in (B3) ($dn_1dn_2dn_3$ is the Haar measure on N). From (B29) it follows that $\bar{F}_{b=I}$ depends only on r_+,r_- and not all of γ; i.e., $\bar{F}_{b=I}$ is constant on the horocycles generated by N. For general b define

$$\bar{F}_b(\theta;\Omega,\gamma;\Omega',\gamma')\equiv\bar{F}_{b=I}(\theta,\Omega,b^t\gamma b;\Omega',b^t\gamma' b)\ , \tag{B31}$$

which is constant on horocycles generated by bNb^t as in the discussion of planes waves. If r_+ and r_- are defined by the a in

$$b^t\gamma b = naa^t n\ ,$$

then $\bar{F}_b$ is a function of γ through r_+ and r_- only (similarly for γ'). If $f(\Omega,\gamma)$ is a function constant on the horocycles generated by bNb^t, then [from (B19) and (B30)]

$$\int F(\theta;\Omega,\gamma;\Omega',\gamma')f(\Omega',\gamma')d\Omega' d\gamma' = \int \bar{F}_b(\theta;\Omega,\gamma_j\Omega',\gamma')f(\Omega',\gamma')dr'_+dr'_-\ . \tag{B32}$$

So the Green's function derived from $\bar{F}$ will propagate functions constant on horocycles.

The heat kernel $\bar{F}$ satisfies the equation

$$\left[\frac{\partial^2}{\partial r_+{}^2}-\left[\frac{3}{2}\right]^{1/2}\frac{\partial}{\partial r_+}+\frac{\partial^2}{\partial r_-{}^2}-\frac{1}{\sqrt{2}}\frac{\partial}{\partial r_-}+\frac{\partial^2}{\partial\Omega^2}\right]\bar{F}=\frac{\partial}{\partial\theta}\bar{F} \tag{B33}$$

with the boundary condition

$$\lim_{\theta\to 0}\bar{F}(\theta;\Omega,r_+,r_-;\Omega',r'_+,r'_-)=\delta(\Omega-\Omega')\delta(r_+-r'_+)\delta(r_--r'_-)\ . \tag{B34}$$

As is well known, the solution to (B33) satisfying (B34) is

$$\bar{F}=\frac{1}{(4\pi\theta)^{3/2}}\exp\left[-\frac{1}{4\theta}\{[r_+-r'_+-(\tfrac{3}{2})^{1/2}\theta]^2+[r_--r'_--(\tfrac{1}{2})^{1/2}\theta]^2+(\Omega-\Omega')^2\}\right]\ . \tag{B35}$$

The existence and functional form of $\bar{F}$ may be used to prove that the full Green's function $G(\Omega,\gamma;\Omega',\gamma')$ exists,[29] but no explicit functional form is known. It might be possible to use the momentum-space form of the Green's function in a direct derivation of the coordinate-space form, but so far the complicated expression $\mu^{-1}(\lambda)d\lambda$ (B17) has prevented this.

For functions f constant on parallel horocycles the explicit form of the Green's function can be found. It is

$$\bar{G}_b(\Omega,r_+,r_-;\Omega',r'_+,r'_-)=\int_0^\infty d\theta\,\bar{F}_b(\theta;\Omega,r_+,r_-;\Omega',r'_+,r'_-)$$
$$=\left[\frac{1}{2\pi}\right]^{3/2}\left[\frac{1}{\sqrt{2}\sigma}\right]^{1/2}K_{1/2}\left[\frac{\sigma}{\sqrt{2}}\right]\exp\{\tfrac{1}{2}[(\tfrac{3}{2})^{1/2}(r_+-r'_+)+(\tfrac{1}{2})^{1/2}(r_--r'_-)]\}$$
$$=\frac{1}{4\pi}\exp\{\tfrac{1}{2}[(\tfrac{3}{2})^{1/2}(r_+-r'_+)+(\tfrac{1}{2})^{1/2}(r_--r'_-)]\}\frac{1}{\sigma}e^{-\sigma/\sqrt{2}}\ , \tag{B36}$$

where

$$\sigma=[(r_+-r'_+)^2+(r_--r'_-)^2+(\Omega-\Omega')^2]^{1/2}\ .$$

Knowing this is not particularly helpful since all one can do for general functions $f(\gamma)$ is apply (B36) to each of the plane waves in the plane-wave decomposition of f. This just results in the momentum-space propagator derived previously, since

$$\int \bar{G}_b(\Omega,\gamma;\Omega',\gamma')e_{\omega,\lambda b}(\Omega',\gamma')d\Omega' d\gamma' = -\frac{1}{\omega^2+(\lambda_+{}^2+\lambda_-{}^2+\frac{1}{2})}e_{\omega,\lambda,b}(\Omega,\gamma)\ .$$

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