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Spin fluid in Einstein-Cartan theory: A variational principle and an extension of the velocity potential representation

R. de Ritis
Istituto di Fisica Teorica Universitá di Napoli, Mostra d'Oltremare Pad. 19, Napoli, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Napoli, Italy

M. Lavorgna
Scuola di Perfezionamento in Fisica Teorica e Nucleare, Napoli, Italy

G. Platania
Istituto di Fisica Sperimentale, Via Tari, 3 Napoli, Italy

C. Stornaiolo
Scuola di Perfezionamento in Fisica Teorica e Nucleare, Napoli, Italy
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We propose a variational principle describing a spin fluid in Einstein-Cartan theory. We also give a potential representation of the Taub current vector. We show that the dynamical description of the spin fluid obtained via this representation is equivalent to the standard equations, and that the material Lagrangian is the pressure.

I. INTRODUCTION

The problem of spin fluids in relativistic theories has long been of interest to physicists (see Refs. 1 and 2 and references therein[3–7]).

In special relativity spin fluids were exhaustively dealt with by Halbwachs.[1] He described a Weyssenhoff fluid starting with a variational principle, and proved that the Lagrangian, if the fluid is governed by a state equation, is just the pressure. In 1968 Seliger and Whitham[8] introduced a velocity potential representation (already discovered by Khalatnikov[9] in 1949) allowing them to describe the velocity field by means of potentials, each of which has an evolution equation. They derived their equations from a variational principle whose Lagrangian is still the pressure. A few years later Schutz[10] generalized their principle to include the effects of a general-relativistic gravitational field, and in 1978 Francaviglia and Khalatnikov[11] gave a Hamiltonian formulation of the problem. On the other hand, since the early 1960's, it has been shown (see Ref. 2 and references therein) that spin may have a dynamical role if we use, as the gravitational theory, the Einstein-Cartan (EC) theory instead of general relativity; in fact in the EC theory spin is coupled with the torsion of space-time. Our aim is to go further along this line because we think it is the most natural for a complete description of spin fluids. In this work we propose a variational principle for the description of the spin perfect fluid in the Einstein-Cartan theory.

Our Lagrangian becomes the Lagrangian used by Halbwachs[1] in the special relativistic limit. In so doing, we also deal with the potential representation of the velocity field of the spin fluid in the Einstein-Cartan theory. Although it is not possible to give an easily manageable potential representation of the four-velocity U_k, we have found the potential representation of the vector:

$$\hat{V}_k=(\bar{\epsilon}g_{kj}+\dot{\hat{S}}_{jk})U^j\ , \tag{1.1}$$

which is an extension of the current vector $\hat{V}_k=\bar{\epsilon}U_k$ defined by Taub.[12]

Ray and Smalley[6] have given a Lagrangian similar to the one we propose; the differences between the Ray-Smalley approach and our approach will be discussed in Sec. IV.

In our approach we show that the dynamical description of the spin fluid obtained via this representation is equivalent to the standard one (see, also, the discussion in Schutz[10]) and that, in this case, too, the Lagrangian is the pressure. In Sec. III we give the Lagrangian for spin fluids, and in Sec. IV we compute the equations of motion.

In Secs. V and VI we give the potential representation for the current vector and demonstrate the equivalence between the two descriptions.

II. THE EINSTEIN-CARTAN THEORY

The EC theory [also called the Einstein-Cartan-Sciama-Kibble (ECSK) theory; for details see Hehl[2]] is a generalization of general relativity (GR) since it introduces the asymmetric connection $\Gamma_{ij}{}^k$ whose antisymmetrical part

$$S_{ij}{}^k\equiv\tfrac{1}{2}(\Gamma_{ij}{}^k-\Gamma_{ji}{}^k)\equiv\Gamma_{[ij]}{}^k \tag{2.1}$$

is called the torsion tensor or, simply, torsion.

We will see that in this theory torsion is connected to the presence of the spin. Introducing a metric tensor g_{ij} of signature $(-,+,+,+)$ and assuming the metricity hypothesis ($\nabla_k g_{ij}=0$) we can show that the connection has the form

$$\Gamma_{ij}{}^k=\left\{{k \atop ij}\right\}-K_{ij}{}^k\ , \tag{2.2}$$

where $\{^k_{ij}\}$ are the Christoffel symbols, i.e., the symmetric connection of GR,

$$\{^k_{ij}\}=\tfrac{1}{2}g^{km}(g_{im,j}+g_{mj,i}-g_{ij,m})$$

and

$$K_{ij}{}^k=-S_{ij}{}^k+S_j{}^k{}_i-S^k{}_{ij}$$

is called the contortion tensor.

A space-time with such a connection $\Gamma_{ij}{}^k$ is called U_4 space-time. As in GR we can introduce in U_4 a Riemann tensor:

$$R_{ijk}{}^l=2\partial_{[i}\Gamma_{j]k}{}^l+2\Gamma^l_{[i\,|\,m\,|}\Gamma_{j]k}{}^m$$

from which we get the Ricci tensor,

$$R_{ij}=R_{mij}{}^m ,$$

while the scalar curvature is defined by

$$R=g^{ij}R_{ij} .$$

The field equations are obtained by a variational principle by considering the variations of the action integral

$$I=\int(R+2kL)\sqrt{-g}\,d^4x ,$$

where k is the relativistic gravitational constant ($k=8\pi Gc^{-4}=2\times10^{-48}$ dyn) and L is the material Lagrangian.

Using the Palatini formalism[2] we have

$$G_{ij}=k\Sigma_{ij} , \quad (2.3)$$

$$T_{ij}{}^k=k\tau_{ij}{}^k , \quad (2.4)$$

where G_{ij} is the Einstein tensor defined by

$$G_{ij}=R_{ij}-\tfrac{1}{2}g_{ij}R ,$$

$T_{ij}{}^k$ is the modified torsion tensor,

$$T_{ij}{}^k=S_{ij}{}^k+2\delta^k_{[i}S_{j]l}{}^l ,$$

and Σ_{ij} and $\tau_{ij}{}^k$ are the dynamical asymmetric energy-momentum tensor and spin tensor, respectively, which are equivalent to the canonical tensors (see Hehl[13]):

$$\sqrt{-g}\,\Sigma_i{}^j\equiv\frac{\partial L}{\partial\partial_j\Psi}\nabla_i\Psi-\delta_i{}^jL ,$$

$$\sqrt{-g}\,\tau_{ij}{}^k\equiv\frac{\partial L}{\partial\partial_k\Psi}f_{[ji]}\Psi ,$$

where Ψ are the material fields and f_{ij} are the representation matrices of the infinitesimal coordinate transformations of Ψ.

Since G_{ij} and $T_{ij}{}^k$ satisfy the differential identities derived from the Bianchi identities in U_4, using Eqs. (2.3) and (2.4), we get that the energy-momentum tensor and the spin tensor must satisfy the same differential equations. These identities are considered the conservation laws in a U_4 space-time. This point of view is completely justified because, according to Hehl,[13] one can derive the same laws by applying the Noether theorem.

Introducing the symbols

$$\nabla^*_k=\nabla_k+2S_{kj}{}^j ,$$

$$\nabla^+_j\Psi_i=\nabla^*_j\Psi_i+2S_{ji}{}^k\Psi_k ,$$

the conservation laws are

$$\nabla^+_j\Sigma_i{}^j=\tau_{lm}{}^jR_{ij}{}^{lm} , \quad (2.5)$$

$$\nabla^*_k\tau_{ij}{}^k=\Sigma_{[ij]} . \quad (2.6)$$

We can use Eqs. (2.5) and (2.6) to get the equations which describe the motion of a spin fluid.

III. LAGRANGIAN FOR A PERFECT FLUID WITH SPIN

We consider a fluid constituted by spin particles described by Halbwachs[1] and which obey the Weyssenhoff condition $S_{ij}U^j=0$, where $S_{ij}=\epsilon_{ijkl}U^ks^l$, s^l being the spin vector associated to the particle and U_k the four-velocity. All the physical properties of the fluid are obtained by a volume average process, as in Halbwachs; so we can define the densities $F=\rho\bar{f}$ where ρ is the material density, i.e., the number of particles for the unit volume, and $\bar{f}$ is the average value for each quantity f (we always use the symbol $\hat{F}$ for the quantity $F/\rho=\bar{f}$). The Lagrangian we use is

$$L=\mu(\rho,s,h_0)+\rho U^i\partial_i\phi+\rho\theta U^i\partial_iS$$
$$+\rho AU^i\partial_iB+\rho h_0a^kU^i\nabla_ib_k+$$
$$+\lambda_{ij}(a^ia^j+b^ib^j+\hat{\sigma}^i\hat{\sigma}^j-U^iU^j-g^{ij}) , \quad (3.1)$$

where $\mu(\rho,S,h_0)$ is the total energy density of the fluid, S is the entropy, and B is one of the Lagrangian coordinates (note that the term $\rho AU^i\partial_iB$ imposes the conservation of the particle identity[14]; the use of just one coordinate is enough for this purpose, as discussed in Refs. 8 and 10). Moreover, ϕ, θ, and A are Lagrangian multipliers used to impose, respectively, the conservation of the number of the particles, of the entropy, and of the identity of the particles.

The next to last term is the kinetic spin energy K defined by

$$K=\tfrac{1}{2}S_{ij}\Omega^{ij} ,$$

where Ω_{ij} is the angular velocity associated with the spin of the particle, and the last one constrains the fields a^i, b^i, $\hat{\sigma}^i$, and U^i to form a vierbein at each point of the space-time (λ_{ij} is another Lagrangian multiplier). In fact, we have considered the field of vierbeins chosen in this way:

(i) The four-velocity U^i is the timelike vector ($U^iU_i=-1$).

(ii) A vector $\hat{\sigma}^i$ proportional to σ^i (spin-density vector) such that

$$\hat{\sigma}^i\hat{\sigma}_i=\sigma^i\sigma_i/(\rho h_0)^2,$$

where h_0 is a standard spin module function (we assume that $\sigma^iU_i=0$, so σ^i is a spacelike vector).

(iii) a^i and b^i are two spacelike vectors orthonormal to σ^i. Thus we are able to describe the spin tensor in two different ways:

$$S_{ij}=\epsilon_{ijkl}U^k\sigma^l=\rho h_0(a_ib_j-a_jb_i)$$

according to the properties of vierbein; in this way we get the Weyssenhoff hypothesis $S_{ij}U^j=0$, and also the form of the kinetic spin-energy density $K=\rho h_0a^kU^i\nabla_ib_k$ where Ω_{ij} is written in terms of the vectors $e_k^{(m)}$ of the vierbein:

$$\Omega_{ij}=\tfrac{1}{2}\sum_m(\dot{e}_i^{(m)}e_j^{(m)}-e_i^{(m)}\dot{e}_j^{(m)}) .$$

It is important to note that the general-relativistic limit of our Lagrangian is the one used by Schutz[10] and that the special-relativistic one is the Lagrangian proposed by Halbwachs.[1] Here we stress that in the Lagrangian (3.1) there is coupling between spin and torsion.

Before deriving the motion equations, we consider how the spin energy influences the thermodynamic behavior of the fluid described by the state equation $\mu(\rho,S,h_0)$. Following Landau and Lifshitz[15] and taking into account the contribution of the spin kinetic energy, in our case the first thermodynamical principle is written as

$$d\bar{\epsilon}=T\,dS+\rho^{-1}dp+\Lambda\,dh_0 , \tag{3.2}$$

where $\epsilon=(\mu+p)/\rho_0$ is the enthalpy, $\bar{\epsilon}=m\epsilon$, $\rho_0=\rho m$, m is the rest mass of each particle of the fluid, p is the pressure, and $\Lambda=a^k\dot{b}_k$.

The expression (3.2) of the first principle is a little different from the one given by Ray and Smalley,[6] but it is easy to show that using the motion equations (4.9) and (4.10) both the above quoted formulations are the same.

IV. EULER-LAGRANGE EQUATIONS AND MOTION EQUATIONS

Varying the independent fields of our Lagrangian, we get

$$\delta\phi:\ \dot{\rho}=0 , \tag{4.1}$$

$$\delta\theta:\ \dot{S}=0 , \tag{4.2}$$

$$\delta A:\ \dot{B}=0 , \tag{4.3}$$

$$\delta\lambda_{ij}:\ a^ia^j+b^ib^j+\hat{\sigma}^i\hat{\sigma}^j-U^iU^j=g^{ij} , \tag{4.4}$$

$$\delta S:\ \dot{\theta}=T . \tag{4.5}$$

Equation (4.5) is obtained from the first thermodynamic principle:

$$\delta B:\ \dot{A}=0 , \tag{4.6}$$

$$\delta\rho:\ \dot{\phi}=-\frac{\partial\mu}{\partial\rho}-h_0a^k\dot{b}_k , \tag{4.7}$$

$$\delta h_0:\ \frac{\partial(\mu/\rho)}{\partial h_0}=\Lambda . \tag{4.8}$$

The dot means, for a density, $\dot{F}=\nabla_i(FU^i)$, and for a vector, $\dot{f}=U^i\nabla_i f$. Substituting Eqs. (4.1) to (4.7) in the Lagrangian, we obtain

$$L=\mu(\rho,S,h_0)-\rho\frac{\partial\mu}{\partial\rho} .$$

From (3.2) we get $L=-p$ which shows that in the U_4 space-time, also, the fluid Lagrangian is the pressure, as in SR (Ref. 1) and GR (Ref. 10). The other variations give

$$\delta a_k:\ \rho h_0U^i\nabla_i b_k+2\lambda_{kj}a^j=0 , \tag{4.9}$$

$$\delta b_k:\ -\rho\dot{h}_0Q_k-\rho h_0\dot{Q}_k+2\lambda_{kj}b^j=0 , \tag{4.10}$$

$$\delta\hat{\sigma}_k:\ 2\lambda_{kj}\hat{\sigma}^j=0 , \tag{4.11}$$

$$\delta U_k:\ \rho(\partial_k\phi+\theta\partial_kS+A\partial_kB+h_0a^i\nabla_kb_j)-2\lambda_{kj}U^j=0 . \tag{4.12}$$

From Eqs. (4.9) to (4.12) we obtain

$$U^k(\rho\bar{\epsilon}g_{ki}+\dot{S}_{ki})=\rho(\partial_i\phi+\theta\partial_iS+A\partial_iB+h_0a^k\nabla_ib_k) \tag{4.13}$$

and $\dot{\sigma}_0=0$ with $\sigma_0{}^2=\sigma^i\sigma_i$, which implies

$$\dot{h}_0=0 , \tag{4.14}$$

i.e., the module of the spin is constant along the flow lines.

From (4.12) we can calculate the Lagrange multiplier λ_{ij}; we have

$$2\lambda_{ij}U^j=\rho(\partial_i\phi+\theta\partial_iS+A\partial_iB+h_0a^k\nabla_ib_k) .$$

By (4.13) we can also write

$$\lambda_{ij}=\tfrac{1}{2}(\rho\bar{\epsilon}q_{ij}+\dot{S}_{ji})+H_{ij} ,$$

where $H_{ij}U^j=0$. For simplicity we set $H_{ij}=0$; so we get

$$\lambda_{ij}=\tfrac{1}{2}(\rho\bar{\epsilon}g_{ij}+\dot{S}_{ji}) . \tag{4.15}$$

The energy-momentum and spin canonical tensors can be obtained by Eqs. (4.9) to (4.12):

$$\Sigma_{ij}=\rho U_j(\partial_i\phi+\theta\partial_iS+A\partial_iB+h_0a^k\nabla_ib_k)+g_{ij}p .$$

By (4.13) we can write it as

$$\Sigma_{ij}=(\rho\bar{\epsilon}g_{ki}+\dot{S}_{ki})U^kU_j+pg_{ij} \tag{4.16}$$

while the spin tensor is

$$\tau_{ij}{}^k=\tfrac{1}{2}\rho h_0U^k(a_ib_j-a_jb_i)=\tfrac{1}{2}S_{ij}U^k . \tag{4.17}$$

The tensor Σ_{ij} obtained here is different from that obtained by Ray and Smalley[6]; this happens because the Lagrangian (3.1) we have used differs from the Lagrangian used by Ray and Smalley with respect to the way the constraints are expressed.

By the conservation laws and expressions (4.16) and (4.17), the motion equations follow:

$$\nabla_j(\rho\bar{\epsilon}U_iU^j+S_{ik}\dot{U}^kU^j+p\delta_i{}^j)=\tfrac{1}{2}S_{jk}{}^lR_{il}{}^{jk} , \tag{4.18}$$

where we have used (2.4), (2.5), and the Weyssenhoff condition ($S_{ij}U^j=0$). Contracting (4.18) with the projector

$$P_i{}^j=(\delta_i{}^j+U_iU^j)$$

we obtain the Euler-generalized equation for spin fluids in a U_4 space-time:

$$P_k{}^i\partial_ip=f_k-\rho\bar{\epsilon}\dot{U}_k-S_{jk}\ddot{U}^j , \tag{4.19}$$

where $f_k=\tau_{ij}{}^lR_{kl}{}^{ij}$ is the Mathisson force density. Contracting (4.18) with U^i and remembering (3.2) and (4.14) we obtain the entropy conservation law

$$T\frac{dS}{d\tau}=0 ,$$

where τ is the proper time.

Equation (2.6) implies the spin-conservation law

$$\dot{S}_{ij}=-\dot{U}^k(S_{ki}U_j-S_{kj}U_i) . \tag{4.20}$$

Equation (4.19) differs from the generalized Euler equation in GR by the presence of two terms connected with the spin distribution. The first is $S_{ij}\ddot{U}^j$, that is, a third-order term with respect to the coordinates and is present also in the theory of spin fluids in SR (Ref. 1); the second is the Mathisson force density that describes the coupling between spin tensor and Riemann tensor, which is peculiar to the ECSK theory.

V. POTENTIAL REPRESENTATION

The Clebsch representation for the velocity field of a fluid

$$U_i=\partial_i\alpha+\beta\partial_i\gamma ,$$

in which the "velocity potentials" α, β, and γ are used to simplify the dynamics of the fluid, have been generalized by many authors in order to be more manageable both in Newtonian fluid dynamics[8,16] and in relativistic fluid dynamics.[9–11] In particular, Schutz has shown that the generalized Euler equation for a perfect fluid without spin in GR may be obtained also by the following potential representation of four-velocity:

$$U_i=(\bar{\epsilon})^{-1}(\partial_i\phi+\theta\partial_i S+A\partial_i B) \tag{5.1}$$

in which the symbols are the same of those defined in Sec. III. In fact, making the Lie derivative with respect to the four-velocity of (5.1), we obtain

$$\rho^{-1}\nabla_j T_i{}^j=L_U(\bar{\epsilon}U_i-\partial_i\phi-\theta\partial_i S)$$

because $L_U(A\partial_i B)=0$; here T_{ij} is the energy-momentum tensor for a perfect fluid without spin in GR:

$$T_{ij}=\rho\bar{\epsilon}U_iU_j+pg_{ij} .$$

Schutz has proved that $\nabla_j T_i{}^j=0$ and

$$L_U(\bar{\epsilon}U_i-\partial_i\phi-\theta\partial_i S)=0$$

are equivalent equations, in the sense that both give the Euler equation in GR. This means that there are two different approaches to obtain the same equations. We demonstrate that the same is also true in our theory.

If we define the vector

$$V_i=(\rho\bar{\epsilon}g_{ij}+\dot{S}_{ji})U^j , \tag{5.2}$$

we see from (4.13) that it has a potential decomposition:

$$V_i=\rho(\partial_i\phi+\theta\partial_i S+A\partial_i B+h_0a^k\nabla_i b_k) . \tag{5.3}$$

Note that V_i is the vector (1.1) which we have indicated as a generalization of Taub's current vector. Besides we have to note that we could also have a potential decomposition for U_i, but that would be much too involved and definitely useless because we can achieve our aim through V_i.

Comparing (5.1) and (5.3) we see that spin density introduces two potentials more than the GR case, i.e., a^k and b^k. Let us prove that the decomposition (5.3) leads to Eq. (4.18): From (4.16) and (5.2) we can write

$$\Sigma_{ij}=\rho\hat{V}_iU_j+g_{ij}p ,$$

which gives from (2.4) and (2.5) and the Weyssenhoff condition

$$\nabla_j(\rho\hat{V}_iU^j)+\partial_i p=f_i ,$$

i.e.,

$$\rho\dot{\hat{V}}_i+\partial_i p=f_i , \tag{5.4}$$

but

$$\dot{\hat{V}}_i=U^j\nabla_j(\partial_i\phi+\theta\partial_i S+A\partial_i B+h_0a^k\nabla_i b_k)$$
$$=\partial_i\dot{\phi}+T\partial_i S+\nabla_i(h_0a^k\dot{b}_k)+a^k\dot{b}_k\nabla_i h_0+\hat{f}_i .$$

From the first principle and (4.7) we have

$$\rho\dot{\hat{V}}_i=f_i-\partial_i p ,$$

that is,

$$\rho\dot{\hat{V}}_i+\partial_i p=f_i .$$

Q.E.D.

VI. ACTION FUNCTIONAL IN A U_4 SPACE-TIME

Since the Lagrangian we have introduced is the pressure, the integral action in a U_4 space-time reads

$$I=\int(R-2kp)\sqrt{-g}\,d^4x . \tag{6.1}$$

Our hypotheses are

$$U^iU_i=-1 , \tag{6.2}$$

$$V_i=(\rho\bar{\epsilon}g_{ij}+\dot{S}_{ji})U^j$$
$$=\rho(\partial_i\phi+\theta\partial_i S+A\partial_i B+h_0a^k\nabla_i b_k) , \tag{6.3}$$

$$S_{ij}=\rho h_0(a_ib_j-a_jb_i) , \tag{6.4}$$

$$dp=-\rho T\,dS+\rho\,d\bar{\epsilon}-\rho a^k\dot{b}_k dh_0 . \tag{6.5}$$

Equations (6.3) and (6.5) give the relations between the physical and the geometrical variables:

$$g^{ij}V_iV_j=-\rho^2\bar{\epsilon}^2+g^{ij}\dot{S}_{ki}\dot{S}_{lj}U^kU^l . \tag{6.6}$$

Varying (6.1) with respect to g_{ij} we have

$$G^{ij}-\nabla^*_k(T^{ijk}+T^{kji}+T^{kij})=k(V^{(i}U^{j)}+g^{ij}p) , \tag{6.7}$$

where we have used the representation for the pressure

$$-p=\mu+g^{ij}V_iU_j \tag{6.8}$$

obtained from (3.1).

From (6.8) we are able to have also the variations of (6.1) with respect to $K_{ij}{}^k$:

$$T_{ij}{}^k=\tfrac{1}{2}kS_{ij}U^k . \tag{6.9}$$

Combining (6.7) and (6.9) and using Eqs. (6.2) and (6.3) we have

$$G^{ij}=k[\tfrac{1}{2}\nabla^*_k(S^{ij}U^k)+\rho\bar{\epsilon}U^iU^j+\dot{S}^{k(i}U^{j)}U_k+pg^{ij}]$$
$$+\frac{k}{2}\nabla^*_k(S^{ki}U^j+S^{kj}U^i) .$$

From the equivalence between the dynamical and canonical tensors[13] and from (2.6) we have

$$G^{ij}=k(\rho\bar{\epsilon}U^iU^j+\dot{S}^{ki}U^jU_k+pg^{ij})$$
$$+\frac{k}{2}\nabla_k^*(S^{ki}U^j+S^{kj}U^i)\ .$$

Now, using the Weyssenhoff condition, the last term of the above expression can be eliminated by adding in (6.1) the null term

$$\int\frac{k}{2}\sqrt{-g}\,g_{ij}\nabla_k^*(S^{ki}U^j+S^{kj}U^i)d^4x\ .$$

This term does not affect any variations, so we have

$$G^{ij}=k\Sigma^{ij}\ . \tag{6.10}$$

The other variations give the equations:

$$\delta\phi:\ \dot{\rho}=0\ , \tag{6.11}$$

$$\delta\theta:\ \dot{S}=0\ , \tag{6.12}$$

$$\delta A:\ \dot{B}=0\ , \tag{6.13}$$

$$\delta S:\ \dot{\theta}=T\ , \tag{6.14}$$

$$\delta B:\ \dot{A}=0\ , \tag{6.15}$$

$$\delta a_l:\ \rho h_0\dot{b}_l+(\rho\bar{\epsilon}g_{lk}+\dot{S}_{kl})a^k=0\ , \tag{6.16}$$

$$\delta b_l:\ -\rho h_0\dot{a}_l+(\rho\bar{\epsilon}g_{lk}+\dot{S}_{kl})b^k=0\ , \tag{6.17}$$

$$\delta h_0:\ \left.\frac{\partial p}{\partial h_0}\right|_{S,\bar{\epsilon}}=-\Lambda\ . \tag{6.18}$$

Equations (6.9)—(6.18) are the complete set of the equations of motion that we can get by the action integral (6.1). Equations (6.9) and (6.10) are the Einstein equations in U_4; (6.11), (6.12), and (6.13) are the conservation of energy, entropy, and identity of the particles; (6.14) is the evolution equation for what von Danzig[10] defined as thermasy; (6.15) is the evolution equations of the Lagrange multipliers A. Equations (6.16), (6.17), and (6.18) express how the spin influences the pressure.

VII. CONCLUSIONS

We have shown a full theory of spin fluids in a U_4 space-time by means of a variational principle, and the Eulerian description of the fluid. The variational principle shows that it is not possible to get a straightforward potential decomposition of the four-velocity, but we have proved that the potential decomposition of the vector V_i in (1.1) gives the same results. This fact suggests that it is the current vector that in general must be decomposed in potentials, as may be seen also by Eq. (2.24) in Ref. 10. We have noted that in a spin fluid the vectors, V_i and U_i are not parallel because of the presence of the spin density; this circumstance prevents us from obtaining a direct potential decomposition of four-velocity by the decomposition of the current vector.

At last we want to stress that all our equations give the usual equations in the GR and in the SR limit.

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