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Improved high-energy bound on the logarithmic derivative of scattering amplitudes

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It is proved that the logarithmic derivative of the absorptive part of the scattering amplitude with respect to momentum transfer t has an improved rigorous upper bound $(7\ln s)/[2\sqrt{t}(2-\sqrt{t})]$ for a sequence of $s\to\infty$. It is also observed that the slope of the Regge trajectory is not only rigorously bounded from above by $7/[2\sqrt{t}(2-\sqrt{t})]$, but also by $1/[2\sqrt{t}(2-\sqrt{t})]$, if the total cross section increases in any fashion as $s\to\infty$. Here s is energy squared and $0<t<4$ (in units of pion mass).

I. INTRODUCTION

The logarithmic derivative of the absorptive part $A(s,t)$ of a scattering amplitude with respect to the momentum transfer t, $(d/dt)\ln A(s,t)$, has been studied in order to understand some features of high-energy scattering.[1] For example, in the forward direction, it describes the slope of the diffraction peak, provided that the high-energy scattering is dominated by the absorptive part. It is known[2,3] that this quantity for $t=0$ is bounded from above by $\ln^2 s$ in the high-energy limit, which suggests that the diffraction peak cannot grow faster than $\ln^2 s$. On the other hand, a lower bound on this quantity for $t=0$ has been obtained from unitarity in terms of cross sections,[4] which turns out to provide an elegant proof that the Regge trajectory is either a constant or has a positive slope at $t=0$.

It is shown by one of the present authors[1] that upper and lower bounds on this quantity in the unphysical region of t can also be set in the framework of axiomatic field theory. In particular, it is demonstrated that one can find such an upper bound as

$$\frac{d}{dt}\ln A(s,t)\le\frac{N(t_0)+M(t)+\frac{1}{2}}{2\sqrt{t}(\sqrt{t_0}-\sqrt{t})}\ln s\ . \tag{1.1}$$

Here $N(t_0)$ and $M(t)$ are given by the upper and lower bounds of the scattering amplitudes, namely,

$$A(s,t)\le\text{const}\times s^{N(t_0)}\ , \tag{1.2}$$

$$A(s,t)\ge\text{const}\times s^{-M(t)} \tag{1.3}$$

for $0<t\le t_0<4$ (in units of pion mass), as $s\to\infty$.

The upper bound (1.1) implies an upper bound on the slope of the Regge trajectory in high-energy scattering. In this sense it has already been observed[5] that the scattering amplitude must have a Regge behavior in the unphysical region. We will show in this paper that the coefficient of the upper bound (1.1) can be improved. Furthermore, it will also be shown that

$$N(t_0)\le 1+\sqrt{t_0/4} \tag{1.4}$$

for $0<t_0<4$ (in units of pion mass). Accordingly, their significance in the Regge behavior of the scattering amplitude will be examined once more.

For simplicity we deal with elastic scattering of spin-zero particles of equal mass. We restrict ourselves to the unphysical region of the momentum transfer. In Sec. II, the upper bound (1.4) is obtained, and in Sec. III, an improved coefficient of the upper bound (1.1) is derived. Section IV carries some comments on the results.

II. A RIGOROUS UPPER BOUND ON $A(s,t)$

The absorptive part $A(s,t)$ of the spinless, equal-mass elastic-scattering amplitude for $A+B\to A+B$ is, in the s channel, defined by

$$A(s,t)=\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l\left[1+\frac{2t}{s-4}\right]\ , \tag{2.1}$$

where s and t are the energy squared and momentum transfer, respectively, and $a_l(s)$ is the absorptive part of the partial waves, and $P_l(x)$ is the Legendre polynomials of the first kind. We use units where the (pion) mass is set equal to unity. Then, unitarity implies

$$0\le a_l(s)\le\left[\frac{s}{s-4}\right]^{1/2}\quad\text{for } s\ge 4\ . \tag{2.2}$$

First, we divide $A(s,t)$ into two parts:

$$A(s,t)=A_1(s,t)+A_2(s,t)\ , \tag{2.3}$$

where

$$A_1(s,t)=\sum_{l=0}^{L}(2l+1)a_l(s)P_l\left[1+\frac{2t}{s-4}\right]\ , \tag{2.4}$$

$$A_2(s,t)=\sum_{l=L+1}^{\infty}(2l+1)a_l(s)P_l\left[1+\frac{2t}{s-4}\right]\ . \tag{2.5}$$

Here $L=c\sqrt{s}\,\ln s$ and c is a positive integer which will be determined later.

It follows from unitarity (2.2) as well as the formula

$$\sum_{l=0}^{L}(2l+1)P_l(x)=P'_{L+1}(x)+P'_L(x) \tag{2.6}$$

that, as $s\to\infty$,

$$A_1(s,t)\le\sum_{l=0}^{L}(2l+1)P_l\left[1+\frac{2t}{s-4}\right]$$
$$=P'_{L+1}\left[1+\frac{2t}{s-4}\right]+P'_L\left[1+\frac{2t}{s-4}\right]\ .$$

Here the prime implies that the function is differentiated with respect to its argument. On the other hand, we have the inequality[1]

$$P_l'(\cosh\alpha) \le \frac{l\tanh(l+\frac{1}{2})\alpha}{\sinh\alpha} P_l(\cosh\alpha) , \tag{2.7}$$

and the asymptotic form[6]

$$P_l\left[1+\frac{2t}{s-4}\right]=\left[\frac{s}{t}\right]^{1/4}\frac{\exp(2l\sqrt{t/s})}{2\sqrt{\pi}\sqrt{l}}\left[1+O\left[\frac{\sqrt{s}}{l}\right]\right], \tag{2.8}$$

for $t>0$, $s\gg 1$, $l/\sqrt{s}\gg 1$. Using (2.7) and (2.8), it is not difficult to obtain that

$$A_1(s,t)\le\frac{1}{2}\left[\frac{c}{\pi}\right]^{1/2}\frac{1}{t^{3/4}}s^{1+2c\sqrt{t}}\ln^{1/2}s . \tag{2.9}$$

Next we turn to $A_2(s,t)$. Squaring both sides of (2.5) and using the Schwartz inequality, we have

$$[A_2(s,t)]^2\le\left[\sum_{l=L+1}^{\infty}(2l+1)a_l(s)\frac{P_l^{\,2}\left[1+\frac{2t}{s-4}\right]}{P_l\left[1+\frac{2t_1}{s-4}\right]}\right]\left[\sum_{l=L+1}^{\infty}(2l+1)a_l(s)P_l\left[1+\frac{2t_1}{s-4}\right]\right].$$

If we square both sides of the above inequality and use the Schwartz inequality once more, we find that

$$[A_2(s,t)]^{2^2}\le\left[\sum_{l=L+1}^{\infty}(2l+1)a_l(s)\frac{P_l^{\,2^2}\left[1+\frac{2t}{s-4}\right]}{P_l^{\,3}\left[1+\frac{2t_1}{s-4}\right]}\right]\left[\sum_{l=L+1}^{\infty}(2l+1)a_l(s)P_l\left[1+\frac{2t_1}{s-4}\right]\right]^3 .$$

Repeating it n times in this manner, we obtain the following:

$$[A_2(s,t)]^{2^n}\le\left[\sum_{l=L+1}^{\infty}(2l+1)a_l(s)\frac{P_l^{\,2^n}\left[1+\frac{2t}{s-4}\right]}{P_l^{\,2^n-1}\left[1+\frac{2t_1}{s-4}\right]}\right]\left[\sum_{l=L+1}^{\infty}(2l+1)a_l(s)P_l\left[1+\frac{2t_1}{s-4}\right]\right]^{2^n-1} .$$

At this stage we make use of (1.2) as well as the fact that

$$(2l+1)a_l(s)\le\text{const}\frac{s^{N(t_1)}}{P_l\left[1+\frac{2t_1}{s-4}\right]} , \tag{2.10}$$

which can also be derived from (1.2). We are then led to the inequality

$$[A_2(s,t)]^{2^n} \le[\text{const}\, s^{N(t_1)}]^{2^n}\sum_{l=L+1}^{\infty}\left[\frac{P_l\left[1+\frac{2t}{s-4}\right]}{P_l\left[1+\frac{2t_1}{s-4}\right]}\right]^{2^n} . \tag{2.11}$$

Substituting (2.8) into (2.11), we finally end up with the inequality

$$A_2(s,t)\le\text{const}\left[\frac{t_1}{t}\right]^{1/4}\frac{s^{N(t_1)+1/2^n-2c(\sqrt{t_1}-\sqrt{t})}}{(2^n)^{1/2^n}[2(\sqrt{t_1}-\sqrt{t})]^{1/2^n}} ,$$

which reduces to

$$A_2(s,t)\le\text{const}\, s^{N(t_1)-2c(\sqrt{t_1}-\sqrt{t})} \tag{2.12}$$

as $n\to\infty$.

Comparing (2.12) with (2.9), we can determine c such that

$$c=\frac{N(t_1)-1}{2\sqrt{t_1}} , \tag{2.13}$$

which makes $A_2(s,t)$ much smaller than $A_1(s,t)$, as $s\to\infty$. We now substitute (2.13) into (2.9) and neglect $A_2(s,t)$ to find such an upper bound as

$$A(s,t)\le\frac{1}{2}\left[\frac{N(t_1)-1}{2\pi\sqrt{t_1}}\right]^{1/2}\frac{1}{t^{3/4}}s^{1+[N(t_1)-1](t/t_1)^{1/2}}\ln^{1/2}s \tag{2.14}$$

for $0<t<t_1$, as $s\to\infty$.

Now, from rigorous results alone we already know[7,8] the fact that

$$N(4)<2,\quad M(t)\le 5 \tag{2.15}$$

for a sequence of $s\to\infty$. Therefore, we obtain from (2.14) an upper bound on $N(t_0)$ itself:

$$N(t_0)\le 1+\left[\frac{t_0}{4}\right]^{1/2} \tag{2.16}$$

for $0<t_0<4$. The bound (2.16) corresponds to Theorem 2(B) of Ref. 3, but enlarges the validity domain of the same result given in Ref. 9.

III. A RIGOROUS UPPER BOUND ON $(d/dt)\ln A(s,t)$

In order to derive an upper bound on $(d/dt)\ln A(s,t)$, we again divide $A(s,t)$ into two parts, namely, $A_1(s,t)$ and $A_2(s,t)$ as given in the previous section by (2.4) and (2.5), respectively. However, the positive constant c in $L=c\sqrt{s}\,\ln s$ must be newly determined so that it gives the most desirable upper bound for our purpose.

From unitarity (2.2) and the inequality (2.7), we have an upper bound, as shown in Ref. 1:

$$\frac{d}{dt}A_1(s,t)\le\frac{c}{\sqrt{t}}(\ln s)A_1(s,t)\ , \tag{3.1}$$

for $0<t<4$, as $s\to\infty$.

The improvement over (1.1) mentioned in Sec. I does not come from (3.1), but from $A_2(s,t)$. Using (2.7) again we have an inequality,

$$\sqrt{st}\,\frac{dA_2(s,t)}{dt}\le\sum_{l=L+1}^{\infty}l(2l+1)a_l(s)P_l\left[1+\frac{2t}{s-4}\right]\ , \tag{3.2}$$

as $s\to\infty$. As we had done in the previous section, we square both sides of (3.2) n times and make use of the Schwartz inequality at each stage. We then find, from (3.2) and (2.10), such an inequality as

$$\left[\sqrt{st}\,\frac{dA_2}{dt}\right]^{2^n}$$

$$\le[\text{const}\ s^{N(t_0)}]^{2^n}\sum_{l=L+1}^{\infty}l^{2^n}\left[\frac{P_l\left[1+\dfrac{2t}{s-4}\right]}{P_l\left[1+\dfrac{2t_0}{s-4}\right]}\right]^{2^n}\ . \tag{3.3}$$

It follows from (3.3) and (2.8), if we take n large enough, that

$$\frac{d}{dt}A_2(s,t)\le\text{const}\frac{1}{\sqrt{t}}\left[\frac{t_0}{t}\right]^{1/4}s^{N(t_0)-2c(\sqrt{t_0}-\sqrt{t}\,)}\ln s\ . \tag{3.4}$$

We therefore have, from (3.1) and (3.4), an upper bound such as

$$\frac{1}{A}\frac{dA}{dt}\le\frac{c}{\sqrt{t}}\ln s+\text{const}\frac{s^{N(t_0)-2c(\sqrt{t_0}-\sqrt{t}\,)}\ln s}{A(s,t)}\ . \tag{3.5}$$

Substituting (1.3) into (3.5) and taking c such that the second term becomes smaller than the first in (3.5), we finally end up with such an upper bound as

$$\frac{d}{dt}\ln A(s,t)\le\frac{N(t_0)+M(t)}{2\sqrt{t}\,(\sqrt{t_0}-\sqrt{t}\,)}\ln s\ . \tag{3.6}$$

Therefore, the best upper bound from the rigorous results (2.15) and (2.16) is

$$\frac{d}{dt}\ln A(s,t)\le\frac{12+\sqrt{t_0}}{4\sqrt{t}\,(\sqrt{t_0}-\sqrt{t}\,)}\ln s \tag{3.7}$$

for $0<t<t_0<4$, and for a sequence of $s\to\infty$. This is an improvement over the upper bound derived in Ref. 1.

IV. REMARKS

We would like to make a few comments on the results derivable from our upper bounds (3.6) and (3.7). First of all, the upper bound (3.6) is an improvement over the bound given in Ref. 1.

If we assume that the total cross section $\sigma_{tot}(s)$ increases in any fashion as $s\to\infty$, then we can take $M(t)=-1$ and we have an upper bound like

$$\frac{d}{dt}\ln A(s,t)\le\frac{\sqrt{t_0}}{4\sqrt{t}\,(\sqrt{t_0}-\sqrt{t}\,)}\ln s\ .$$

This in turn gives us the bound

$$\frac{d}{dt}\ln A(s,t)\le\frac{1}{2\sqrt{t}\,(2-\sqrt{t}\,)}\ln s \tag{4.1}$$

for $0<t<4$, as $s\to\infty$.

If we assume on the other hand the Regge-pole behavior of the high-energy scattering amplitude, namely, $A(s,t)=\beta(t)s^{\alpha(t)}$, then we see that this amplitude gives the same energy dependence for $(d/dt)\ln A(s,t)$ as the upper bound (3.6) in the unphysical region. This can be taken as another way of confirming the Regge behavior.[5] We find from (2.16), as shown in Ref. 3, such an upper bound as

$$\alpha(t)\le1+\left[\frac{t}{4}\right]^{1/2}\ . \tag{4.2}$$

From the upper bound (4.1) we can now set an upper bound to the slope of the Regge trajectory, namely,

$$\frac{d\alpha(t)}{dt}\le\frac{1}{2\sqrt{t}\,(2-\sqrt{t}\,)}\ , \tag{4.3}$$

whereas the rigorous bound from (3.7) is

$$\frac{d\alpha(t)}{dt}\le\frac{7}{2\sqrt{t}\,(2-\sqrt{t}\,)}\ . \tag{4.4}$$

If we further assume that the Regge trajectory is linear,[10] as it is usually taken for granted, then we can put $\alpha(t)=at+b$, a and b being constants. If we set, for instance, $b=1$ for Pomeron, and $M(t)=-at-b=-at-1$, then, from (3.6), we obtain

$$a\le\tfrac{1}{4}\approx13\ \text{GeV}^{-2}\ . \tag{4.5}$$

Although (4.5) is far larger than the phenomenological value of $a=1$ GeV^{-2} for Regge amplitudes, it is an impressive improvement over previous results.[1] Of course, we can derive the same result as our (4.5) also from both (4.2) and the assumption that the trajectory is linear. Our (4.3) holds without the linearity assumption, though.

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