Classical radiation zeros in gauge-theory amplitudes

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The electromagnetic radiation from classical convection currents in relativistic n-particle collisions is shown to vanish in certain kinematical zones, due to complete destructive interference of the classical radiation patterns of the incoming and outgoing charged lines. We prove that quantum tree photon amplitudes vanish in the same zones, at arbitrary photon momenta including spin, seagull, and internal-line currents, provided only that the electromagnetic couplings and any other derivative couplings are as prescribed by renormalizable local gauge theory (spins ≤ 1). In particular, the existence of this new class of amplitude zeros requires the familiar gyromagnetic-ratio value $g=2$ for all particles. The location of the zeros is *spin independent*, depending only on the charges and momenta of the external particles. Such null zones are the relativistic generalization of the well-known absence of electric and magnetic dipole radiation for nonrelativistic collisions involving particles with the same charge-to-mass ratio and g factor. The origin of zeros in reactions such as $u\bar{d}\to W^+\gamma$ is thus explained and examples with more particles are discussed. Conditions for the null zones to lie in physical regions are established. A new radiation representation, with the zeros manifest and of practical utility independently of whether the null zones are in physical regions is derived for the complete single-photon amplitude in tree approximation, using a gauge-invariant vertex expansion stemming from new internal-radiation decomposition identities. The question of whether amplitudes with closed loops can vanish in null zones is addressed. A low-energy theorem for general quantum amplitudes (including closed loops) is found. Important relations between the photon couplings and Poincaré transformations are discovered. The null zone and these relations are discussed in terms of the Bargmann-Michel-Telegdi equation. The extension from photons to general massless gauge bosons is carried out.

I. INTRODUCTION

In this paper we describe a new general feature of theories that incorporate massless gauge fields: The existence of zones of null radiation independent of spin.[1] We present the details behind a theorem[1] for a new type of zero in tree-graph amplitudes for gauge-boson radiation/absorption involving any number of particles (spins ≤ 1) in collision. It is sufficient that any derivative couplings be of gauge-theory form.

The kinematic condition for the electromagnetic *null radiation zones* is simply that all external particles (charges Q_i and momenta p_i) have the same $Q_i/p_i\cdot q$ ratio, where q is the photon momentum. For definiteness, we refer to photons; the condition in non-Abelian gauge theories involves a generalized charge Q_g.

As a corollary to the theorem, each helicity amplitude can be written with the zeros displayed explicitly. This result is important since it defines a new canonical form (Sec. VI) for radiation amplitudes independent of whether the null zone lies in the physical region.

The physical basis of the theorem lies in a corresponding result for classical radiation patterns. For the same kinematic condition, we find that there is a complete destructive interference of the radiation from classical convection currents in relativistic n-particle collisions. In the nonrelativistic limit the null-zone condition reduces to the requirement that the charge/mass ratio Q_i/m_i is the same for all particles. Thus, the zeros are the relativistic version of the well-known absence of electric dipole radiation for nonrelativistic collisions involving particles with the same charge-to-mass ratio. The classical underpinnings are given in detail in Sec. III.

The null-zone condition directly applies to the simple quantum tree (single-photon) amplitude where all the other particles are spinless and scatter at a point,[2] and without restriction to low-energy photons. What is surprising about the theorem is that it continues to hold in more realistic amplitudes when we add contributions from spin currents, gauge-theoretic derivative couplings, and internal-line emission in tree approximation.

The restrictions on the derivatives specifically require that all photon couplings to the particles correspond to the same gyromagnetic ratio, $g=2$. In that case we find all spin currents can be described by the same first-order Lorentz transformation, a fact that is crucial to the theorem. This description and the null zones are destroyed by anomalous moments. The equivalence of spin- and Larmor-precession frequencies is thus intimately related to the null-zone phenomena.

Under such gauge-theoretic conditions only closed-loop graphs can undo the result. Quantum fluctuations in the sources of radiation, required by the uncertainty principle, spoil the exact cancellation; we need the long-range classical currents and perfect plane-wave states, such that the particle positions are completely unspecified, for null zones.

The reactions in which a weak boson and a photon are produced by the annihilation of quarks,[3] such as $u+\bar{d}\to W^{+}+\gamma$, which may be measurable in high-energy $p\bar{p}$ collisions and which may be important in the verification of the gauge properties of the W, offer striking examples of null-radiation-zone phenomena. Mikaelian, Samuel, and Sahdev (MSS) first pointed out that the lowest-order unpolarized cross sections vanish at an angle unrelated to any specific helicity state,[4] and only if $g_W=2$. Another example[4] is the reaction[3] $\bar{\nu}_e+e^{-}\to W^{-}+\gamma$.

The zeros in the W-production cross sections necessarily imply that each helicity amplitude calculated from the set of four-body tree graphs must have an overall factor $z=\cos\theta-\cos\theta_0$. The interesting algebra which shows this factorization has been developed by Goebel, Halzen, and Leveille (GHL).[5] Zeros and factorization in other four-body tree amplitudes have also been discussed in Ref. 5 and by Zhu.[6] Related work by Grose and Mikaelian concerns radiative W-decay channels.[7] These examples are restricted to the cases where no internal-line photon coupling occurs.

The motivation for our study stems from the fact that no explanation was known for such zeros. We can now recognize Mikaelian factorization, the MSS zero, and the GHL algebra as three-vertex examples of the general class of gauge-theoretic single-photon tree-amplitude zeros which are the relativistic generalization of the absence of electric and magnetic dipole radiation for nonrelativistic collisions of particles with the same charge-to-mass ratio and g factor.[1]

The plan of this paper is as follows: The theorem and its corollaries are presented in Sec. II, and the classical basis is developed in Sec. III. The conditions for physical null zones and examples are discussed in Sec. IV. The detailed proof of the theorem comprises Sec. V. The canonical representation is derived in Sec. VI. The special case where some of the particles are neutral is analyzed in Sec. VII. The union of the theorem and the standard low-energy theorem for general amplitudes including closed loops is considered in Sec. VIII. The fundamental role of Lorentz invariance (in the proof of the theorem) and the classical Bargmann-Michel-Telegdi (BMT) equations are investigated in Sec. IX. In Sec. X, our analysis is applied to other gauge groups. The last section is devoted to a summary and further remarks. There are two appendices where the details of physical null zones and a summary of rules for constructing radiation amplitudes are given.

II. THEOREM AND REPRESENTATION FOR RADIATION IN GAUGE THEORIES

This section contains the precise statement of the theorem, a brief outline of its proof, and corollaries. We need the following definitions:

(1) Gauge-theoretic vertices. These are interactions involving any number of fields with spin ≤ 1 but with no derivatives of Dirac fields and at most single derivatives of scalar and vector fields—all of which are aspects of local gauge theories. All vector derivative couplings must be of the Yang-Mills type. Products of single derivatives of distinct scalar fields as well as of the trilinear couplings are allowed. (The photon couplings must correspond to $g=2$.) This encompasses all renormalizable theories of current interest and an infinite class of nonrenormalizable theories. *(See note added in Sec. XI.)*

(2) Source graph. This is any Feynman diagram that serves as a source for photons.

(3) Radiation graph. This is a graph generated by the attachment of a single photon onto a specific line or, in the case of derivative couplings, onto a vertex (seagulls) of a source graph.

(4) Radiation amplitude. This is the sum of all the radiation graphs generated from a given source graph(s).

We next state the main result:

Theorem. If $M_\gamma(T_G)$ is the radiation amplitude generated by the *tree* source graph T_G with gauge-theoretic vertices, then

$$M_\gamma(T_G)=0 \tag{2.1}$$

provided all ratios $Q_i/p_i\cdot q$ are equal.

Proof outline. In the special case where T_G is a single vertex V_G the corresponding radiation amplitude is

$$M_\gamma(V_G)=\sum\frac{Q_iJ_i}{p_i\cdot q}\ , \tag{2.2}$$

where J_i is the product of the current for photon emission by the ith leg and the remaining vertex factors. The theorem follows for (2.2) if

$$\sum J_i=0\ . \tag{2.3}$$

The proof for tree graphs with internal lines follows from a novel decomposition of the radiation amplitude into a sum over the source vertices of gauge-invariant terms,

$$M_\gamma(T_G)=\sum M_\gamma(V_G)R(V_G)\ , \tag{2.4}$$

where $M_\gamma(V_G)$ now includes internal legs [with (2.2) and (2.3) still valid] and $R(V_G)$ denotes the propagators and the other vertices of the source graph.

There are several results ancillary to the theorem:

(1) Complementary theorem: Equation (2.1) also holds if the ratios $\delta_iJ_i/p_i\cdot q$ are all equal.

This follows from (2.2) and charge conservation,[8]

$$\sum\delta_iQ_i=0\ , \tag{2.5}$$

where

$$\delta_i\equiv\begin{cases}+1 & \text{outgoing,}\\ -1 & \text{incoming.}\end{cases} \tag{2.6}$$

(2) Radiation representation. For a vertex source graph the zeros of the theorem and its complement imply

$$M_\gamma(V_G)=\sum\delta_ip_i\cdot q\left[\frac{Q_i}{p_i\cdot q}-\frac{Q_1}{p_1\cdot q}\right]\left[\delta_i\frac{J_i}{p_i\cdot q}-\delta_n\frac{J_n}{p_n\cdot q}\right]. \tag{2.7}$$

The off-shell $M_\gamma(V_G)$ in (2.4) can be expressed in a similar manner.

(3) Low-energy theorem. If $\mathscr{M}_\gamma(S_G)$ is the radiation amplitude corresponding to a general source graph S_G, which includes closed loops and arbitrary interactions, and if spinning external particles have $g=2$, then

$$\mathscr{M}_\gamma(S_G)=M_\gamma(S_G)+O(q)\ , \tag{2.8}$$

where $M_\gamma(S_G)$ satisfies the interference theorem.

III. CLASSICAL PRELUDE

In this section we look for completely destructive interference in classical radiation amplitudes. This provides the relativistic generalization of the well-known result[9,10] that classical electric dipole radiation vanishes for nonrelativistic collisions of particles with the same charge-to-mass ratio.

To review this result, the electric dipole moment is

$$\vec{d}=\sum Q_i\vec{r}_i \tag{3.1}$$

for particle positions $\vec{r}_i(t)$. If the charge-to-mass ratios have the same value for all n particles,

$$\frac{Q_i}{m_i}=\frac{Q_1}{m_1},\ \text{all}\ i\ , \tag{3.2}$$

and if there are no external forces, we find $\ddot{\vec{d}}=0$. There is completely destructive interference at all angles, a combined result of translational invariance and the constraint (3.2) on the constituent particles.

The inclusion of spin currents has a counterpart: Magnetic dipole radiation vanishes for nonrelativistic collisions at a point, when the particles have the same charge/mass ratio and the same gyromagnetic factor.[11] The magnetic dipole moment is

$$\begin{aligned}\vec{\mu}&=\sum\vec{\mu}_i\ ,\\ \vec{\mu}_i&=g_i\frac{Q_i}{2m_i}\vec{S}_i\ ,\end{aligned} \tag{3.3}$$

with spin $\vec{S}_i$ for each particle. If all g factors are the same and if there are no external torques, then (3.2) implies that $\dot{\vec{\mu}}=0$. Thus the magnetic dipole radiation field vanishes identically with rotational symmetry as a key ingredient.

The relativistic amplitude for radiation during collisions is found using a classical current[12] corresponding to k initial particles scattering into $n-k$ final particles with uniform velocities $\vec{v}_i=\dot{\vec{r}}_i$ before or after the localized collision. (Spin currents are ignored for the time being.) The classical infrared amplitude (frequency $\omega\to 0$) is[10,12,13]

$$A_{\rm IR}(k,n)=-\sum_1^n\delta_i\frac{Q_i}{\omega(1-\hat{n}\cdot\vec{v}_i)}\vec{v}_i\cdot\vec{\epsilon} \tag{3.4}$$

which reduces to the nonrelativistic electric dipole amplitude $A_{\rm IR}^{\rm NR}$. For (3.2), we see that

$$A_{\rm IR}^{\rm NR}(k,n)=-\frac{Q_1}{\omega m_1}\vec{\epsilon}\cdot\sum_1^n\delta_i m_i\vec{v}_i=0\ , \tag{3.5}$$

verifying the conclusion reached earlier.

Let us rewrite (3.4) in terms of the particle (four-) momenta, the photon (four-) polarization, and the photon (four-) momentum $q=\omega(1,\hat{n})$:

$$A_{\rm IR}(k,n)=\sum_1^n\frac{Q_i}{p_i\cdot q}\delta_i p_i\cdot\epsilon\ . \tag{3.6}$$

For common $Q/p\cdot q$ ratios we find $A_{\rm IR}(k,n)=0$ by momentum conservation and transversality $q\cdot\epsilon=0$. This is the relativistic generalization for arbitrary photon momenta of the cancellation of electric dipole radiation. Because the fields get folded forward, the general cancellation occurs only for the set of charges and momenta that give the same $Q/p\cdot q$, ranges for which are discussed in Sec. IV.

The classical treatment of the radiation generated by a system of moving intrinsic magnetic moments is relatively complicated except in the low-frequency, nonrelativistic limit, where the individual magnetic moments can be represented by their intrinsic (rest frame) values (3.3) and the radiation amplitude is[10]

$$A_m=i\sum_1^n\delta_i(\vec{\mu}_i\times\hat{n})\cdot\vec{\epsilon}\ , \tag{3.7}$$

noting the absence of ω^{-1} in comparison with (3.4). The expression (3.7) does indeed vanish under (3.2), if the g factors are all the same and if the total intrinsic spin is conserved. Note that orbital angular momentum, through its associated magnetic moment, contributes terms at the ω^0 level as well.

Rather than proceeding further in a semiclassical manner, we turn to quantum amplitudes, for which we have already found the infrared factors exactly. The infrared term of the full radiation amplitude $\mathscr{M}$, shown in Fig. 1(a), is derived from the graphs of Fig. 1(b). If the scattering amplitude for k particles $\to n-k$ particles is denoted by $T(p_1,\ldots,p_n)$, then the ω^{-1} term is given by[12]

$$\mathscr{M}_{\rm IR}=A_{\rm IR}(k,n)T(p_1,\ldots,p_n)\ . \tag{3.8}$$

Clearly, the radiation theorem always holds for the infrared part of *any* amplitude. The zeros in the infrared factor have apparently gone unnoticed until now.[1] (See Sec. VIII.)

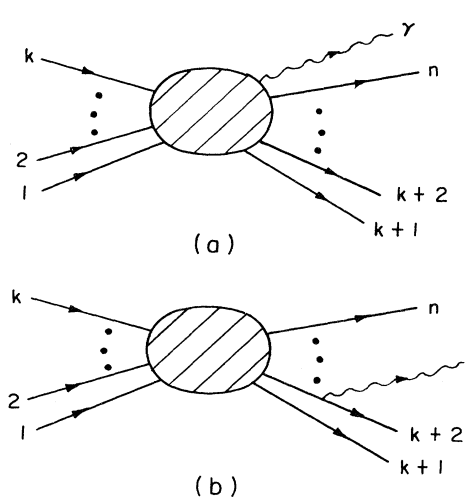

FIG. 1. (a) The general amplitude for photon emission in the interactions of n particles, $k\to n-k+\gamma$. (b) A contribution with an infrared divergence.

IV. $Q/p\cdot q$ FACTORS AND PHYSICAL NULL ZONES

In this section, we investigate the kinematics corresponding to equal $Q/p\cdot q$ ratios and the possible overlap with physical phase space.

A. Preliminaries

Definition: The null radiation zone is the momentum-space region where all the $Q/p\cdot q$ factors[14] are equal, corresponding to the $n-2$ equations,

$$\frac{Q_i}{p_i\cdot q}=\frac{Q_j}{p_j\cdot q},\quad \text{all } i\neq j,l\ , \tag{4.1}$$

for fixed distinct pairs j,l. Charge and momentum conservation are assumed,

$$\sum_i \delta_i Q_i=0\ , \tag{4.2a}$$

$$\sum_i \delta_i p_i=-q\ , \tag{4.2b}$$

as are the mass-shell conditions.

The $n-1$ possible equations reduce to $n-2$ because of (4.2), $q^2=0$, and the simple fact that, if $a/A=b/B=c/C=\dots$, then composite ratios such as $(a+b+c+\cdots)/(A+B+C+\cdots)$ are also the same. In general, one $Q/p\cdot q$ is determined by the rest through the identity

$$\frac{a+b+c+\cdots}{A+B+C+\cdots}-\frac{a}{A}=\left[\left[\frac{b}{B}-\frac{a}{A}\right]B+\left[\frac{c}{C}-\frac{a}{A}\right]C+\cdots\right]\frac{1}{A+B+C+\cdots}\ . \tag{4.3}$$

Care must be exercised in the use of an arbitrary set of $n-2$ equalities in place of (4.1), since they may not always be independent. For example, in the electron-electron reaction,

$$e^-(p_1)+e^-(p_2)\rightarrow e^-(p_3)+e^-(p_4)+\gamma(q)\ , \tag{4.4}$$

$p_1\cdot q=p_3\cdot q$ is equivalent to $p_2\cdot q=p_4\cdot q$ by momentum conservation, and therefore, they are not independent equations. This problem does not arise if the prescription in (4.1) is followed.

In the nonrelativistic limit for all n particles, (4.1) reduces to (3.2). Mass conservation replaces (4.2b) in going from $n-1$ to $n-2$ equations.

Since $p_i\cdot q\geq 0$, (4.1) implies that *all nonzero charges in both the initial and final states must have the same sign,*

$$\frac{Q_i}{Q_j}\geq 0,\quad \text{all } i,j\ . \tag{4.5}$$

Neutral particles are required by the null-zone condition to have zero mass and to travel in the same direction as the photon. (Neutral particles are addressed in more detail in Sec. VII.) For a given total charge, the more particles there are, the smaller their charges, and consequently fractional charges can play a special role.[15]

B. Null zone: $n\leq 3$

Given (4.5) the next step is to find the constraints on the energies and angles due to (4.1).

(1) $n=1$. For completeness, we include this "mixing" transition which is realized only off-shell for well-defined particle states and has a tadpole source graph. The radiation representation is trivial since $Q_1=0$.

(2) $n=2$. An example is $\mu\rightarrow e\gamma$ lepton-number-violating radiative decays. The momentum and charge conservation equations automatically satisfy $Q_1/p_1\cdot q=Q_2/p_2\cdot q$, in accord with the fact that there is no independent equation. The radiation representation is identically zero and, indeed, the most general $\mu\rightarrow e\gamma$ amplitude[16] $\Psi(a+b\gamma_5)\sigma_{\mu\nu}\psi q^\mu\epsilon^\nu$ is $O(q)$, with contributions from derivative couplings or closed loops. (See Sec. VIII.)

(3) $n=3$ *decay.* Equations (4.2) read $p_1=p_2+p_3+q$ and $Q_1=Q_2+Q_3$. In terms of the energies in the rest frame of the parent, we take the two decay variables to be

$$\begin{aligned} x&\equiv\frac{2p_3\cdot q}{m_1{}^2}=1-\frac{2E_2}{m_1}+\mu_2{}^2-\mu_3{}^2\ ,\\ y&\equiv\frac{2p_2\cdot q}{m_1{}^2}=1-\frac{2E_3}{m_1}+\mu_3{}^2-\mu_2{}^2\ , \end{aligned} \tag{4.6}$$

where

$$\mu_i\equiv m_i/m_1\ , \tag{4.7}$$

which coincide with those of Ref. 7 in the limit $m_2=m_3=0$. The single null-zone equation may be written

$$y=\frac{Q_2}{Q_3}x\ , \tag{4.8}$$

and the question is whether this straight line intersects the physical domain in x-y space.

In Appendix A, we find the physical x range,

$$0\leq x\leq(1-\mu_2)^2-\mu_3{}^2\ , \tag{4.9}$$

and, for a given x, the y range,

$$\begin{aligned} &y_-\leq y\leq y_+\ ,\\ &y_\pm\equiv\frac{x}{2A}[B\pm(B^2-4\mu_2{}^2A)^{1/2}]\ , \end{aligned} \tag{4.10}$$

with $A\equiv x+\mu_3{}^2$ and $B\equiv 1-\mu_2{}^2-\mu_3{}^2-x$. The roles of x and y can be reversed by relabeling $2\leftrightarrow 3$.

The intersection of (4.8) with (4.9) and (4.10) depends on the masses m_2 and m_3, and is analyzed in Appendix A. One particularly interesting result is that there is a physical null zone for all masses and charges such that $Q_2/m_2=Q_3/m_3, m_2+m_3\leq m_1$. This is consistent with the soft-photon, nonrelativistic limit, where $m_2+m_3=m_1$ and all Q_i/m_i are equal.

In the massless limit $m_2=m_3=0$, the inequalities reduce to the case already discussed in Ref. 7: $0\leq x\leq 1$, $0\leq y\leq 1-x$. Thus, there is always a line of intersection in $x-y$ space as long as the three charges have the same sign.

(4) $n=3$ *scattering.* With $p_1+p_2=p_3+q$ and $Q_1+Q_2=Q_3$, the single null-zone equation yields

$$P\cos\theta=\frac{Q_2E_1-Q_1E_2}{Q_2+Q_1}\ , \tag{4.11}$$

in terms of the c.m. energies E_i and angle θ (between $\vec{p}_1$ and $\vec{q}$) with $P\equiv|\vec{p}_1|=|\vec{p}_2|$. The physical null zone, θ of (4.11) for which $|\cos\theta|\leq 1$, is discussed for general masses and charges in Appendix A. In the ultrarelativistic limit,

$$\cos\theta=\frac{Q_2-Q_1}{Q_2+Q_1}\ , \tag{4.12}$$

so that all positive Q_2/Q_1 values produce physical null points. It is seen that (4.12) checks[4] with $q\bar{q}'\to W\gamma$. The nonrelativistic limit is consistent with total interference at all angles. Appendix A contains a demonstration that, if $Q_1/m_1=Q_2/m_2$, a physical null zone exists whatever the energies.

C. Null zone: $n=4$ example

The $n>3$ results can be built up from the preceding analysis. For $n=4$, consider $p_1+p_2=p_3+p_4+q$ and $Q_1+Q_2=Q_3+Q_4$ as equivalent to a three-body decay of a system with mass $E=E_1+E_2$ (the total c.m. energy). The photon angle is still given by (4.11). The second null-zone equation is expressed in terms of variables analogous to (4.6),

$$y=\frac{Q_3}{Q_4}x\ , \tag{4.13}$$

where

$$x\equiv\frac{2p_4\cdot q}{E^2}=1-\frac{2E_3}{E}+\frac{m_3{}^2-m_4{}^2}{E^2}\ ,$$
$$y\equiv\frac{2p_3\cdot q}{E^2}=1-\frac{2E_4}{E}+\frac{m_4{}^2-m_3{}^2}{E^2}\ . \tag{4.14}$$

The two null-zone equations do not follow the prescription of (4.1) but still are independent.

We count the dimensions of the null zone by recalling that the photon polar angle is fixed and noting that its azimuth can be arbitrarily chosen. The energy of particle 4 is determined by (4.13), and, after imposing momentum conservation, the last two free dimensions may be taken to be the energy x of particle 3 and the azimuth of the plane of particles 3 and 4 (and γ) relative to the photon axis. These constitute a two-dimensional null zone.

We may use the decay equations (4.8)—(4.10) and in Appendix A, *mutatis mutandis,* to determine whether the null zone is in the physical region. Again, if the ratios Q_i/m_i are all identical, there is a physical null zone for *any* c.m. energy. This suggests a striking example.

Bremsstrahlung in electron scattering, (4.4), satisfies the radiation theorem in lowest order and, in addition, the Q_i/m_i ratios are identical for all charges. Thus, we discover amplitude zeros in a textbook reaction that have gone unnoticed up to now and that occur somewhere for all energies ($E\geq 2m$, $m_i=m$). *Having two (or more) source graphs is immaterial.* The physical null zone is the two-dimensional region described above and in Appendix A:

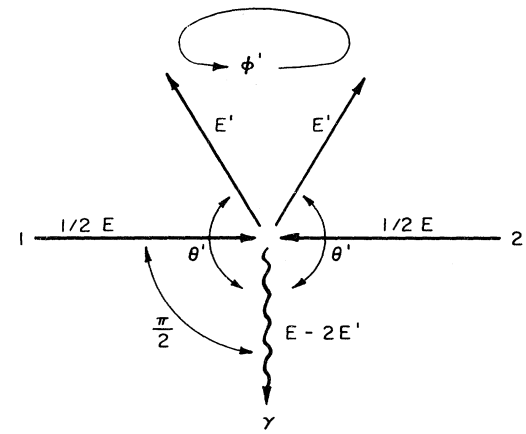

FIG. 2. The amplitude zero in $e^-e^-\to e^-e^-\gamma$ occurs when both the photon is at right angles to the c.m. beams and the final electrons have equal energies. This is a two-dimensional null zone: E',ϕ' or θ',ϕ' at fixed $\theta=\pi/2$.

$$\begin{aligned} &E'(1-v'\cos\theta')=E/2\ ,\\ &\pi/2\leq\theta'\leq\pi\ ,\\ &0\leq\phi'\leq 2\pi\ ,\\ &\theta=\pi/2\ , \end{aligned} \tag{4.15}$$

in which $E_3=E_4\equiv E'$, the final velocities $v_3=v_4\equiv v'$, and $\theta_3=\theta_4\equiv\theta'$. The final-state plane of the two electrons and the photon has an azimuthal angle ϕ' about the photon axis, pictured in Fig. 2.

In contrast to identical scalar bosons, the null zone in Fig. 2 is not forbidden by angular momentum conservation for identical spin-$\frac{1}{2}$ fermions. It is radiation interference, and not the exclusion principle, that forces every tree helicity amplitude for reaction (4.4) to vanish in (4.15).

D. Null zone: Theorem

It is possible to give a general criterion for the existence of physical null zones:

Physical null-radiation-zone theorem. There is a null radiation zone for *any* c.m. energy in the physical region of the reaction, k particles $\to n-k$ particles + photon, if the initial particles have an identical charge/mass ratio and the final particles share another common charge/mass ratio, not necessarily the same as the initial ratio.

Corollary. As a special limit of this theorem, one can require instead that subsets of the initial and/or the final particles be massless.

In short, we can always find physical regions where all $Q/p\cdot q$ are equal, provided that the Q/m are equal or that particles are massless, conditions which can be restricted separately to the initial or final states. In decay processes, obviously, the parent must not be massless, and *in all cases the nonzero charges must have the same sign.* We note also that, alternatively, the photon may be in the initial state. The proof and further remarks are given in Appendix A.

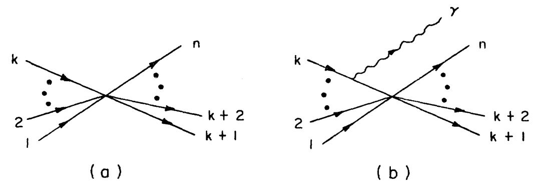

FIG. 3. (a) The n-vertex souce graph. (b) A photon attachment to an external leg.

V. PROOF OF THE THEOREM

A. Spin-zero fields and constant couplings

We first examine scalar/pseudoscalar particles whose couplings to each other may involve an arbitrary number of fields but no derivatives other than the standard convective photon coupling.

A vertex source graph $V_G(n)$, is defined to have n external lines coupled through a single vertex [Fig. 3(a)] and, in the absence of derivative couplings, only external-line photon attachments [Fig. 3(b)] are present in the corresponding radiation amplitude. For photon emission by an external scalar leg with charge Q flowing along momentum p, we have the following factors[13]:

$$\text{outgoing particle: } \frac{Q}{p\cdot q}p\cdot\epsilon\ , \tag{5.1a}$$

$$\text{incoming particle: } (-p\cdot\epsilon)\frac{Q}{p\cdot q}\ . \tag{5.1b}$$

If λ_n denotes the constant vertex in $V_G(n)$,

$$M_\gamma[V_G(n)]=\lambda_n A_{\rm IR}(k,n)\ , \tag{5.2}$$

in terms of (3.7). The theorem is obvious in this case.

The $n=3$ vertex leads to the spinless version of $q\bar{q}'\to W\gamma$ which is already known[5] to have the same amplitude zero. The new aspects of the preceding results for vertex source graphs are the demonstration that amplitude zeros also exist for $n>3$ together with the identification of the conditions (4.1) for their location, and the understanding of the physical basis for their occurrence.

Remarkably, the same zeros survive in arbitrary scalar tree graphs, where we encounter photon attachments to internal lines. (See Fig. 4.) A crucial step in handling such contributions involves the use of an identity for real photon emission from a scalar internal line ($p'\equiv p-q$):

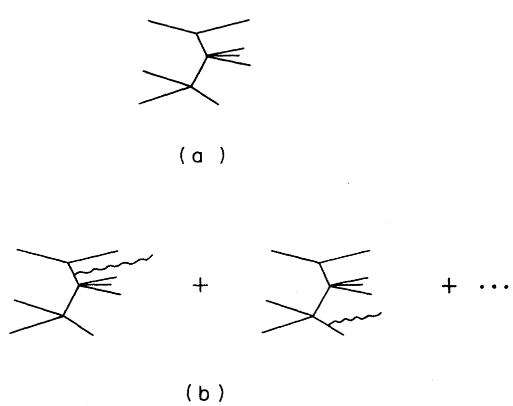

FIG. 4. (a) A sample tree source graph and (b) its associated radiation amplitude, as defined in Sec. II.

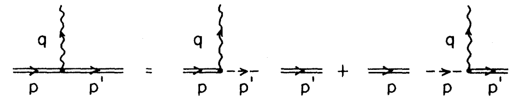

FIG. 5. The radiation decomposition identity for the coupling of an external photon to an internal particle line. A double line represents a propagator. A dashed line is quasiexternal in that the calculation of each current on the right-hand side is carried out as if the dashed line were real. See Eqs. (5.3), (5.18), and (5.26). Additional contributions to the left-hand side due to seagull graphs where the photon is attached to either end are easily incorporated into the respective quasiexternal factor on the right-hand side. See Appendix B.

$$i\frac{1}{p'^2-m^2}Q(p'+p)\cdot\epsilon\frac{1}{p^2-m^2}$$

$$=\frac{i}{p'^2-m^2}\frac{Q}{p'\cdot q}p'\cdot\epsilon+(-p\cdot\epsilon)\frac{Q}{p\cdot q}\frac{i}{p^2-m^2}\ , \tag{5.3}$$

using $q\cdot\epsilon=q^2=0$.

We refer to (5.3) and similar relations in the sequel as *radiation decomposition identities,* representing a manifestly gauge-invariant split of the internal vertex into two terms (Fig. 5) each of which is a product of a propagator and a quasiexternal-leg emission factor.

In the scalar case (5.3) holds to all orders. The invariant-amplitude expansion for the scalar-photon-scalar vertex function,

$$\Gamma^\mu=(p'-p)^\mu f(p'^2,p^2)+(p'+p)^\mu g(p'^2,p^2)\ ,$$

implies that

$$\Gamma\cdot\epsilon=(p'+p)\cdot\epsilon\, g\ .$$

Alternatively, the Ward-Takahashi identity can be used to show that

$$\Delta'(p'^2)^{-1}-\Delta'(p^2)^{-1}=-2p\cdot q\ g\ ,$$

where Δ' is the full scalar propagator. Thus (5.3) is valid with $(p'+p)\cdot\epsilon$ replaced by $\Gamma\cdot\epsilon$ and the free propagators replaced by Δ'.[17]

Let us illustrate (5.3) with an $n=4$ example depicted in Fig. 6. The radiation amplitude can be expressed in the form

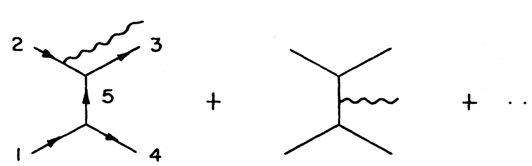

FIG. 6. The radiation amplitude for an $n=4$ tree source graph.

$$M_\gamma(\text{Fig.}6)=\frac{i\lambda_3^2}{(p_3-p_2)^2-m_5^2}\left[\frac{Q_4}{p_4\cdot q}p_4\cdot\epsilon-\frac{Q_1}{p_1\cdot q}p_1\cdot\epsilon+\frac{Q_1-Q_4}{(p_1-p_4)\cdot q}(p_1-p_4)\cdot\epsilon\right]$$
$$+\frac{i\lambda_3^2}{(p_1-p_4)^2-m_5^2}\left[\frac{Q_3}{p_3\cdot q}p_3\cdot\epsilon-\frac{Q_2}{p_2\cdot q}p_2\cdot\epsilon+\frac{Q_2-Q_3}{(p_2-p_3)\cdot q}(p_2-p_3)\cdot\epsilon\right]. \quad (5.4)$$

The two square brackets in (5.4) are the separately gauge-invariant classical $A_{\rm IR}$ amplitudes (3.6), each associated with one of the ($n=3$) source vertices. Both are multiplied by the original source graph amplitude, the momentum assignment in each source amplitude determined by momentum conservation at the other vertex. These features are quite general.

From a general scalar tree graph T_G and (5.3), we obtain a *radiation vertex expansion* [cf. (2.4)]:

$$M_\gamma(T_G)=\sum_v \lambda_v A_{\rm IR}(v)R(v)\ , \quad (5.5)$$

summing over the vertices v of T_G. $A_{\rm IR}(v)$ is the *gauge-invariant* off-shell version of the classical amplitude (3.6) for radiation by the legs of vertex v where the sums are over all external *and* internal lines into and out of the vertex. $R(v)$, comprised of the remaining factors in T_G including all propagators, is simply T_G/λ_v in the scalar case, but with momentum unconserved at the vertex v.

The validity of (5.5) follows from the fact that (5.3) partitions each internal-line photon attachment into two quasiexternal-line attachments which are respectively and unambiguously assigned to the two vertices joined by the internal line. For every vertex v, we are left with a complete set of photon emission factors, one factor for each attached line and each factor with the *same* coefficient $R(v)$. The momentum of each propagator on the right-hand side of (5.3) is consistent with q leaving the vertex to which its quasiexternal factor is ultimately associated, giving the same R that the external-leg radiation does.

The proof is completed by noting that the internal $Q/p\cdot q$ factors are determined through (4.3) by the external ones. If (4.1) is satisfied, then

$$\frac{Q_I}{p_I\cdot q}=\frac{Q_j}{p_j\cdot q}\ \text{(null zone)} \quad (5.6)$$

for all fixed internal I. Therefore, each $A_{\rm IR}(v)$ (and consequently M_γ) vanishes in the null zone.

Evidently, the null-zone cancellation depends only on the external charges and momenta; *all* source graphs (with the prescribed couplings) generate tree radiation amplitudes that vanish at exactly the same places for a given set of external particles. Notice how the proof breaks down for closed-loop source graphs, since (5.6) does not follow unless the internal line is fixed by the external charges and momenta.

The theorem can be checked by the example in (5.4) which in particular demonstrates the interesting case of vanishing internal charges. If $Q_5=Q_1-Q_4=Q_3-Q_2=0$, one null-zone condition is $p_1\cdot q=p_4\cdot q$ (or $p_3\cdot q=p_2\cdot q$), the cancellation still goes through in (5.4) but now *between* the square brackets. This is not surprising since the original demonstration did not depend on the magnitudes of Q_i, and the limit $Q_5\rightarrow 0$ could be taken before or after demanding (4.1). In general, we may regard any two vertices connected by a neutral internal line as a single compound vertex in expansions like (5.5). Neutral external scalar lines conform to the theorem as well but in a more subtle fashion. Their inclusion is analyzed in Sec. VII.

B. Including spin-half particles

Now each tree source graph may involve any even number $2D$ of Dirac particles along with an arbitrary number $n-2D$ of scalars (but no derivative couplings).

A vertex source graph may be written

$$V_G(n,D)=\prod_{i=1}^{D}\bar w'_i\Gamma_i w_i\ , \quad (5.7)$$

where w,w' are chosen as needed from the familiar u,v spinors. The Γ_i are spin matrices, possibly contracted together, with the coupling constant and the presence of the $n-2D$ scalars understood.

The factors corresponding to (5.1) for photon emission by an external Dirac leg are computed from minimal (gauge-theoretic) coupling to be

$$\text{outgoing particle:}\ \frac{Q}{p\cdot q}\bar u(p)(p\cdot\epsilon+\tfrac14[\not\epsilon,\not q])\ , \quad (5.8a)$$

$$\text{incoming particle:}\ (-p\cdot\epsilon-\tfrac14[\not\epsilon,\not q])u(p)\frac{Q}{p\cdot q}\ , \quad (5.8b)$$

$$\text{outgoing antiparticle:}\ (p\cdot\epsilon-\tfrac14[\not\epsilon,\not q])v(p)\frac{Q}{p\cdot q}\ , \quad (5.8c)$$

$$\text{incoming antiparticle:}\ \frac{Q}{p\cdot q}\bar v(p)(-p\cdot\epsilon+\tfrac14[\not\epsilon,\not q])\ . \quad (5.8d)$$

Each is a sum of convection and spin currents, replacing the original spinor in the source graph.

The radiation amplitude for the vertex source graph (5.7) can be obtained directly from (5.1) and (5.8). With k initial particles,

$$M_\gamma[V_G(n,D)]=V_G(n,D)A_{\rm IR}(k,n)+\sum_{i=1}^{D}S_i\prod_{j\neq i}^{D}\bar w'_j\Gamma_j w_j\ , \quad (5.9)$$

where

$$S_i=\tfrac14\bar w'_i(p'_i)\left\{\frac{Q'_i}{p'_i\cdot q}[\not\epsilon,\not q]\Gamma_i-\Gamma_i[\not\epsilon,\not q]\frac{Q_i}{p_i\cdot q}\right\}w_i(p)\ . \quad (5.10)$$

The convection currents combine to give (3.6), as before, and clearly cancel in the null zone.

We can show that the Dirac spin currents also conspire to cancel in the null zone but by Lorentz, rather than translational, invariance. The spin currents in (5.8) are proportional to first-order wave-function corrections all of which can be associated with the *same* (called "universal" hereafter) first-order Lorentz transformation,

$$\Lambda_{\mu\nu}=g_{\mu\nu}+\lambda\omega_{\mu\nu}\ , \tag{5.11}$$

where

$$\omega_{\mu\nu}=q_\mu\epsilon_\nu-\epsilon_\mu q_\nu \tag{5.12}$$

and λ is an infinitesimal length. The spinor wave function ψ transforms as[18]

$$\psi'(x')=S(\Lambda)\psi(x)\ , \tag{5.13}$$

where $x'=\Lambda x$ and

$$S(\Lambda)=1-\frac{i}{4}\lambda\sigma_{\mu\nu}\omega^{\mu\nu}=1-\tfrac{1}{4}\lambda[\not\epsilon,\not q]\ . \tag{5.14}$$

Comparison of (5.8) and (5.14) establishes the relationship.

When the $Q/p\cdot q$ factors are equal, (5.9) reduces to

$$M_\gamma[V_G(n,D)]=\frac{Q_1}{p_1\cdot q}\sum_{i=1}^{D}\overline{w}'_i\Delta\Gamma_i w_i\prod_{j\neq i}^{D}\overline{w}'_j\Gamma_j w_j \quad \text{(null zone)}\ , \tag{5.15}$$

with

$$\Delta\Gamma_i=\tfrac{1}{4}[\not\epsilon,\not q]\Gamma_i-\Gamma_i\tfrac{1}{4}[\not\epsilon,\not q] = \frac{i}{4}[\sigma_{\mu\nu}\omega^{\mu\nu},\Gamma_i]\ . \tag{5.16}$$

We see that (5.16) is proportional to the complete first-order change[19] in (5.8). By the Lorentz invariance of V_G, we conclude that $M_\gamma[V_G(n,D)]=0$ in the null zone.

To extend the proof to internal lines, we need an identity analogous to (5.3). The alternative expressions

$$(\not p'+m)\not\epsilon(\not p+m)=2(\not p'+m)(p'\cdot\epsilon+\tfrac{1}{4}[\not\epsilon,\not q])-(p'^2-m^2)\not\epsilon \tag{5.17a}$$

$$=2(p\cdot\epsilon+\tfrac{1}{4}[\not\epsilon,\not q])(\not p+m)-\not\epsilon(p^2-m^2)\ , \tag{5.17b}$$

lead to

$$i\frac{1}{\not p'-m}Q\not\epsilon\frac{1}{\not p-m}=\frac{i}{\not p'-m}\frac{Q}{p'\cdot q}(p'\cdot\epsilon+\tfrac{1}{4}[\not\epsilon,\not q])+(-p\cdot\epsilon-\tfrac{1}{4}[\not\epsilon,\not q])\frac{Q}{p\cdot q}\frac{i}{\not p-m}\ . \tag{5.18}$$

This also follows the schematic of Fig. 5, offering an immediate demonstration of the associated Ward-Takahashi identity.

Equation (5.18) provides the correct internal incoming and outgoing convection and spin currents of a given vertex in a source graph, for the null-zone cancellations. We obtain the radiation vertex expansion (2.4),

$$M_\gamma(T_G)=\sum_v M_\gamma[V_G(v)]R(v)\ , \tag{5.19}$$

where $M_\gamma[V_G(v)]$ is the (separately gauge-invariant) radiation vertex amplitude including internal legs. For internal legs, we replace the corresponding spinors in (5.9) by spin indices that are tied to the remaining factor $R(v)$, which contains all propagators. $R(v)$ is T_G less the vertex v, with momentum assignments consistent with photon emission from v.

In the null zone, the conservation of momentum (modulo q), the rank-zero nature[19,20] of the string of Γ_i's at each vertex v, and (5.6) lead to $M_\gamma[V_G]=0$ for all v in (5.19). The theorem is thus proven for scalar-spinor tree source graphs with constant couplings.

Since any deviation from minimal coupling for Dirac particles ruins the $n=3$ factorization,[5] it is expected to undermine the radiation interference theorem. An anomalous-magnetic-moment coupling[21] leads to the modified vertex

$$\not\epsilon\to\not\epsilon+\frac{a}{4m}[\not\epsilon,\not q]\ , \tag{5.20}$$

where the magnetic moment and gyromagnetic ratio are $\mu=e/2m$, $g=2(1+a)$. The external current (5.8a), for example, is then changed to

$$\frac{Q}{p\cdot q}\bar u(p)\left[p\cdot\epsilon+\tfrac{1}{4}[\not\epsilon,\not q](1+a)+\frac{a}{2m}\omega_{\mu\nu}p^\mu\gamma^\nu\right]\ . \tag{5.21}$$

The argument for $a=0$ depends on the relationship between the spin currents and a universal Lorentz transformation, but the p dependence $\omega_{\mu\nu}p^\mu\gamma^\nu$ destroys this.

Therefore, we require the minimal Dirac electromagnetic coupling for the radiation theorem. This shows that electromagnetic gauge invariance is not sufficient. The Pauli terms, for example, are gauge invariant, but lead to $g\neq 2$, nonrenormalizability, and a violation of the radiation theorem, all of which appear to be intimately related to one another.

C. Including spin-one particles

We now add an arbitrary number N of vectors to the $2D$ Dirac particles and $n-2D-N$ scalars in the tree source graph, but still with no derivative couplings beyond the scalar and vector electromagnetic currents. The photon-vector-vector coupling has the form[22] of the locally gauge-invariant Yang-Mills trilinear (Fig. 7) and corresponds to $\kappa=1$ for the magnetic moment parameter of the vector particle ($g=2$). The quadrilinear vector couplings in which the photon participates are regarded as seagull terms in Sec. V D. The incorporation of neutral vector

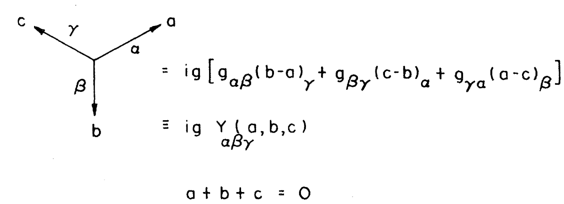

FIG. 7. The Feynman rule for a Yang-Mills locally gauge-invariant three-vertex for vector fields, with four-momenta a,b,c and polarization indices α,β,γ. The coupling constant g would be augmented by a matrix representation for the general internal-symmetry gauge group. In the U(1) case where a vector boson with charge Q emits a photon, we have $g=Q$. See Ref. 22.

particles into the proof is considered in Sec. VII.

The vertex source graph is generalized from (5.7) to

$$V_G(n,D,N)=\prod_{l=1}^{N}\eta_l^{\mu_l}\left[\prod_{i=1}^{D}\bar{w}_i\Gamma_i w_i\right]_{\mu_1\mu_2\cdots\mu_N}, \quad (5.22)$$

in terms of the vector polarization factors[13] η. We may include possible $g_{\mu\nu},\epsilon_{\mu\nu\sigma\rho}$ tensors along with the Dirac matrices in the definition of the Γ_i, making up the (constant) rank-N Lorentz tensor into which the η_l are contracted.

The photon-emission factors for vector legs with polarization $\eta(p)$ ($\eta\cdot p=0$) are calculated by contracting the vector propagator, $iP_{\mu\nu}(p)/(p^2-m^2)$, where

$$P_{\mu\nu}(p)\equiv -g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}, \quad (5.23)$$

with the photon vertex inferred from Fig. 7. We find

$$\text{outgoing particle: } \frac{Q}{p\cdot q}(p\cdot\epsilon\eta_\mu+\omega_{\mu\nu}\eta^\nu), \quad (5.24a)$$

$$\text{incoming particle: } (-p\cdot\epsilon\eta_\mu+\omega_{\mu\nu}\eta^\nu)\frac{Q}{p\cdot q}. \quad (5.24b)$$

These replace η_μ in the original source graph.

We learn from (5.24) that the relationship between spin currents and the universal Lorentz transformation is not just an accidental aspect of Dirac particles, since the vector spin currents are also proportional (for $g=2$) to the first-order change under (5.11) in their associated wave functions. This relationship is again the key to the null-zone cancellation.

From (5.1), (5.8), and (5.24) it follows that (5.9) generalizes to

$$M_\gamma[V_G(n,D,N)]=V_G(n,D,N)A_{\rm IR}(k,n)+\prod_{l=1}^{N}\eta_l^{\mu_l}\left[\sum_{i=1}^{D}S_i\prod_{j\neq i}^{D}\bar{w}'_j\Gamma_j w_j\right]_{\mu_1\cdots\mu_N}$$
$$+\sum_{l=1}^{N}\frac{Q_l}{p_l\cdot q}\omega_\nu^{\mu_l}\eta_l^\nu\prod_{r\neq l}\eta_r^{\mu_r}\left[\prod_{i=1}^{D}\bar{w}'_i\Gamma_i w_i\right]_{\mu_1\cdots\mu_N}, \quad (5.25)$$

with (3.6) as the familiar repository of the complete set of convection currents. In the null zone, the Dirac spin currents in the second term of (5.25) are proportional to the first-order universal transformation of the rank-N spinor product according to the remarks in Sec. V B and are therefore canceled by the third term which is similarly related to the first-order transformation of the rank-N vector polarization product. The total first-order change of the rank-zero V_G vanishes under (4.1) by its Lorentz invariance. Hence (5.25) satisfies the theorem.

The radiation decomposition identity for a real photon attached to an internal vector line is (Fig. 5)

$$\frac{-iQI_{\gamma\delta}}{(p'^2-m^2)(p^2-m^2)}=\frac{iP_{\gamma\beta}(p')}{p'^2-m^2}\frac{Q}{p'\cdot q}(p'\cdot\epsilon g_\delta^\beta+\omega_\delta{}^\beta)+(-p\cdot\epsilon g_\gamma^\alpha+\omega_\gamma{}^\alpha)\frac{Q}{p\cdot q}\frac{iP_{\alpha\delta}(p)}{p^2-m^2}, \quad (5.26)$$

where (Fig. 7)

$$I_{\gamma\delta}\equiv P_{\gamma\beta}(p')Y^{\beta\sigma\alpha}(p',q,-p)P_{\alpha\delta}(p)\epsilon_\sigma. \quad (5.27)$$

Equation (5.26) is derived using both of the alternate expressions for (5.27)

$$I_{\gamma\delta}=-2P_{\gamma\beta}(p')(p'\cdot\epsilon g_\delta^\beta+\omega_\delta{}^\beta)+\frac{1}{m^2}(p'^2-m^2)(\epsilon_\gamma p_\delta+p'_\gamma\epsilon_\delta) \quad (5.28a)$$

$$=2(-p\cdot\epsilon g_\gamma^\alpha+\omega_\gamma{}^\alpha)P_{\alpha\delta}(p)+\frac{1}{m^2}(\epsilon_\gamma p_\delta+p'_\gamma\epsilon_\delta)(p^2-m^2). \quad (5.28b)$$

The decomposition (5.26) leads to a radiation vertex expansion as before but now including internal and external vector particles. For every internal particle with spin attached to a given vertex v of T_G, the factor $V_G(v)$, defined as in (5.22), has a free index in place of the spinor or polarization vector. The off-shell radiation amplitude $M_\gamma[V_G(v)]$ is likewise multi-spinor-indexed and a Lorentz tensor.

We may regard V_G (and M_γ) as Lorentz invariants in a manner following the spinor description.[20] For each internal vector leg, index μ, we rewrite $(V_G)_\mu$ as $(V_G)_\delta\eta^\delta(\mu)$ for $\eta^\delta(\mu)=g_\mu^\delta$, defining an internal vector wave function. If all wave functions, vector and spinor, external and internal, are universally Lorentz transformed, the (first-order) terms cancel. Since (5.26) provides exactly these internal first-order changes, M_γ derived from general source tree graphs continues to satisfy the theorem.

A non-gauge-theoretic photon coupling to vector particles spoils the cancellation. For $\kappa\neq 1$, the vertex for $(p,\alpha)\rightarrow(p',\beta)+(q,\mu)$ is augmented by the term[22]

$$iQ(\kappa-1)(g_{\beta\mu}q_\alpha-q_\beta g_{\mu\alpha}). \quad (5.29)$$

The currents are changed by the addition of

$$\frac{Q}{p\cdot q}\frac{\kappa-1}{2}P_{\mu\nu}(l)\omega^{\nu\rho}\eta_\rho, \quad (5.30)$$

where $l=p+q$ ($p-q$) for the first (second) factor in (5.24). The p dependence of $P_{\mu\nu}$ in (5.30) ruins the universality of the spin currents. We need $\kappa=1$, or $g=2$, in the vector magnetic moment, $\mu=ge/2m$, $g=1+\kappa$ in order to maintain the relationship between the spin currents and the universal Lorentz transformation (5.11).

D. Including derivative couplings: seagulls

It remains to consider the possibility of derivative couplings in the interactions among the source particles. We show next that the current associated with the presence of

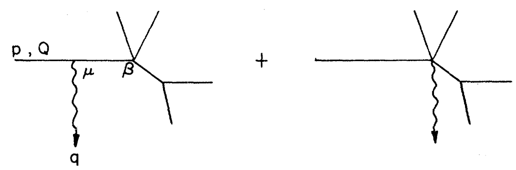

FIG. 8. An example of photon emission by an incoming or outgoing particle, with momentum p and charge Q, that is coupled through a derivative ∂_β of its own field to other particles. The seagull factor is $-Qg_{\beta\mu}$ for photon polarization ϵ^μ.

a derivative coupling is described by the same Lorentz transformation that characterizes spin currents and consequently the radiation theorem holds for the general class of gauge-theoretic interactions in the source graph.

We first examine single-derivative factors: Lagrangian interactions of the form $(\partial_\mu \Psi_i)(\Psi_j \Psi_k \cdots)^\mu$, or *products* thereof, $(\partial_\mu \Psi_i)(\partial_\nu \Psi_j)\cdots$, where each field Ψ, boson or fermion, has at most one derivative. Obviously, these include interactions that can be brought into single-derivative form through an integration by parts. Electromagnetic gauge invariance requires a direct photon attachment, adding a seagull current to the convective and spin currents, for the ensuing momentum-dependent source vertex.

Consider a vertex in which there is a derivative coupling, $(\partial^\beta \Psi)\cdots$, and the external *or* internal leg (particle of Ψ) connected to this vertex, as an isolated part of a source tree graph. In momentum space, the vertex may be denoted by $p^\beta r_\beta$, in terms of the momentum p of the leg and the remaining vertex factors r.

The contribution to the radiation vertex amplitude in (5.19) due to the particle Ψ from this isolated vertex-leg system (Fig. 8) is

$$M_\Psi = \left[\pm \frac{Q}{p\cdot q} p\cdot\epsilon (p\pm q)^\beta - Q\epsilon^\beta + \text{spin term} \right] r_\beta \quad (5.31)$$

for an outgoing(+)/incoming(−) particle. In the internal-leg case we include only the radiation-decomposition term relevant to this vertex. Aside from a possible external wave function, r resembles R in (5.19) in that it can be expressed entirely in terms of momenta other than p and q.

The seagull term in (5.31) comes from the vertex factor, $-Qg^{\mu\beta}$. We note that the spin currents are separately gauge invariant and that the convective current in (5.31), $\pm Qp\cdot\epsilon p\cdot r/p\cdot q$, is conjointly gauge invariant with the other convection currents in the radiative vertex amplitude.

The *seagull* and *momentum-shift* contributions to (5.31) can be rewritten in the suggestive form

$$\frac{Q}{p\cdot q}(p\cdot\epsilon q^\beta - p\cdot q\epsilon^\beta) r_\beta \,. \quad (5.32)$$

These terms go hand-in-hand for any single-derivative coupling in the source graph. (They also appear together in first-order q for higher derivatives.)

The significance of (5.32) is that it allows us to identify a *universal contact current*,

$$\frac{Q}{p\cdot q}\omega^{\mu\nu} \,, \quad (5.33)$$

for photon emission from a line coupled through a (linear) derivative coupling to a vertex, to be added to the convection and (any) spin currents. The rule[23] is that (5.33) replaces $g^{\mu\nu}$ in the *derivative* coupling, $p^\mu = g^{\mu\nu} p_\nu$. A summary of all photon emission factors is given in Appendix B.

The contact current is thus proportional to the first-order Lorentz transformation (5.11) of the rank-one derivative. Recalling that the spin currents transform the wave functions, Lorentz invariance continues to guarantee a cancellation of the terms that are first-order in q. [Inasmuch as $\omega^{\mu\nu}$ is linear in q, the order of q is equivalent to the order of λ in (5.11).] In the null zone, the radiation vertices in (5.19) vanish up to $O(q^2)$, in the coefficient of $Q/p\cdot q$.

The $O(q^2)$ terms arise when a spinning particle encounters its own derivative coupling,[24] specifically from the product of the spin current and the momentum shift:

$$\text{spin term} = \text{spin current} \times (p\pm q)^\beta \quad (5.34)$$

from (5.31). Thus, second-order terms develop for interactions in which there are derivatives of Dirac or vector fields, as well as those in which higher derivatives of scalar fields occur, and do not cancel in the null zone unless an additional mechanism is operative.

In fact, there is an exceptional case in which such an additional mechanism is present. The quadratic terms cancel under (4.1) for the trilinear single-derivative vector-boson vertex of Fig. 7, as a consequence of both the cyclic symmetry of the vertex and of the specific form (5.12) of the universal transformation, $\omega^{\mu\nu}$. *(See note added in Sec. XI.)* This cancellation is demonstrated explicitly in the next subsection and appears to be intimately related to the question of renormalizability. (The theorem is likewise true for a class of nonrenormalizable interactions. Our arguments also go through for couplings involving *products* of single derivatives of distinct scalar fields and of the triplet Yang-Mills structure as well as of any num-

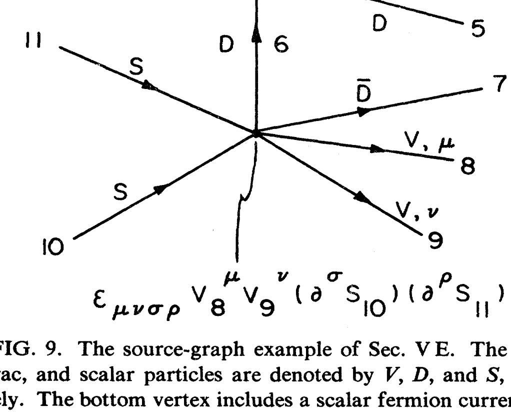

FIG. 9. The source-graph example of Sec. V E. The vector, Dirac, and scalar particles are denoted by V, D, and S, respectively. The bottom vertex includes a scalar fermion current.

ber of scalar, Dirac, or vector fields with constant couplings.) This completes the proof of the radiation theorem.

E. Example with Yang-Mills vertex

The example (Fig. 9) is designed to illustrate the cyclic trilinear Yang-Mills vertex and its seagull, a tensor $\sigma_{\mu\nu}$ Dirac current, multifield vertices, a product of two scalar single-derivative couplings, their seagulls, a Levi-Civita tensor $\epsilon_{\mu\nu\sigma\rho}$ vertex, and vanishing charges (Sec. VII) for internal vector and Dirac particles. In addition, the use of the radiation-vertex expansion is to be demonstrated.

The source-graph amplitude (cf. Fig. 7) is

$$T_G(\text{Fig.}9)=\eta_2^\alpha Y_{\alpha\beta\gamma}(p_2,p_3,-p_1)\eta_1^\gamma V^{\beta\gamma}(p_3)\eta_4^\zeta \bar{u}_5\sigma_{\gamma\zeta}\frac{1}{\not p_6-m_6}v_7\epsilon_{\mu\nu\sigma\rho}\eta_8^\mu\eta_9^\nu p_{10}^\sigma p_{11}^\sigma \ , \tag{5.35}$$

where $\eta_i\equiv\eta(p_i),\bar{u}_5\equiv\bar{u}(p_5),v_7\equiv v(p_7)$, and the vector propagator is $V^{\beta\lambda}(p_3)\equiv(g^{\beta\lambda}-p_3^\beta p_3^\lambda/m_3^2)/(p_3{}^2-m_3{}^2)$. (Overall constants are disregarded.) Before photon emission, the momenta are related by

$$p_1-p_2=p_3=p_4+p_5-p_6\ ,\quad p_6=p_{10}+p_{11}-p_7-p_8-p_9\ . \tag{5.36}$$

Charge conservation leads to the same equations, but with $p_i\rightarrow Q_i$.

The radiation amplitude corresponding to (5.35) has the radiation-vertex expansion

$$M(\text{Fig.}9)=\sum_{v=1}^{3}M(v)R(v)\ , \tag{5.37}$$

where $v=1,2,3$ refers to the vertex at the top, middle, bottom, respectively, of Fig. 9. The vertex radiation amplitude $M(v)$ can be obtained using the appropriate current insertions. The theorem is verified if each $M(v)$ vanishes in the null zone.

The first vertex radiation amplitude is constructed using the external currents, (5.24), and the first (outgoing) internal current in (5.26) with $p'=p_3$, all augmented by contact currents, (5.33), for the momenta in the Yang-Mills vertex. (The contact currents include the quadrilinear γ^{VVV} vertex.) The result is

$$\begin{aligned}M_\beta(1)=Y_{\alpha\tau\gamma}(p_2,p_3,-p_1)&\left[\frac{Q_1}{p_1\cdot q}\eta_2^\alpha(-p_1\cdot\epsilon\eta_1^\gamma+\omega^\gamma{}_\delta\eta_1^\delta)g_\beta^\tau+\frac{Q_2}{p_2\cdot q}(p_2\cdot\epsilon\eta_2^\alpha+\omega^\alpha{}_\delta\eta_2^\delta)\eta_1^\gamma g_\beta^\tau+\frac{Q_3}{p_3\cdot q}\eta_2^\alpha\eta_1^\gamma(p_3\cdot\epsilon g_\beta^\tau+\omega^\tau{}_\beta\right]\\&+\eta_2^\alpha\eta_1^\gamma\left[\frac{Q_1}{p_1\cdot q}(g_{\gamma\alpha}\omega_{\beta\tau}-g_{\beta\gamma}\omega_{\alpha\tau})p_1^\tau+\frac{Q_2}{p_2\cdot q}(g_{\gamma\alpha}\omega_{\beta\tau}-g_{\alpha\beta}\omega_{\gamma\tau})p_2^\tau+\frac{Q_3}{p_3\cdot q}(g_{\alpha\beta}\omega_{\gamma\tau}-g_{\beta\gamma}\omega_{\alpha\tau})p_3^\tau\right]\\&+\frac{Q_1}{p_1\cdot q}\eta_2^\alpha\omega^\gamma{}_\delta\eta_1^\delta(g_{\beta\gamma}q_\alpha-g_{\gamma\alpha}q_\beta)+\frac{Q_2}{p_2\cdot q}\eta_1^\gamma\omega^\alpha{}_\delta\eta_2^\delta(g_{\gamma\alpha}q_\beta-g_{\alpha\beta}q_\gamma)+\frac{Q_3}{p_3\cdot q}\eta_2^\alpha\eta_1^\gamma\omega^\tau{}_\beta(g_{\alpha\tau}q_\gamma-g_{\tau\gamma}q_\alpha)\end{aligned} \tag{5.38}$$

with its common factor in (5.37) given by the remainder of (5.35),

$$R^\beta(1)=V^{\beta\lambda}(p_3)\eta_4^\zeta\cdots\ . \tag{5.39}$$

Here $p_1-p_2-q=p_3$ with the rest of (5.36) unchanged.

$M_\beta(1)$ is easily seen to be gauge invariant. In this regard, note that the decomposition (5.26) produces the same outside factor (5.39) as do the external leg attachments.

In the null zone, we find

$$\begin{aligned}M_\beta(1)=\frac{Q_1}{p_1\cdot q}\{&Y_{\alpha\tau\gamma}(p_2,p_3,-p_1)[\eta_2^\alpha\eta_1^\gamma(-p_1+p_2+p_3)\cdot\epsilon g_\beta^\tau+\eta_2^\alpha\omega^\gamma{}_\delta\eta_1^\delta g_\beta^\tau+\omega^\alpha{}_\delta\eta_2^\delta\eta_1^\gamma g_\beta^\tau+\eta_2^\alpha\eta_1^\gamma\omega^\tau{}_\beta]\\&+\eta_2^\alpha\eta_1^\gamma[g_{\alpha\beta}\omega_{\gamma\tau}(p_3-p_2)^\tau+g_{\beta\gamma}\omega_{\alpha\tau}(-p_1-p_3)^\tau+g_{\gamma\alpha}\omega_{\beta\tau}(p_2+p_1)^\tau]\\&+2\omega_{\delta\beta}\eta_2^\delta\eta_1\cdot q+2\omega_{\gamma\delta}\eta_2^\delta\eta_1^\gamma q_\beta+2\omega_{\beta\delta}\eta_1^\delta\eta_2\cdot q\}\quad(\text{null zone})\ ,\end{aligned} \tag{5.40}$$

grouping the quantities inside the curly brackets according to powers of q.

The fact that $M_\beta(1)=0$ in the null zone can be described order by order in q. First, the zeroth-order convection currents obviously cancel. The next six terms, linear in q, are the first-order universal Lorentz changes in the external vector wave functions, in the internal vector wave function ($\omega^\tau{}_\beta$ term) defined by $M_\beta\equiv M_\tau\eta^\tau(\beta)$ with $\eta^\tau(\beta)=g_\beta^\tau$, and in the four-momenta of the vertex, respectively. Since these are all contracted together, sometimes through the numerically invariant $g_{\mu\nu}$, Lorentz invariance guarantees their cancellation, and an explicit calculation using the antisymmetry of $\omega_{\mu\nu}$ bears this out. Finally, we call special attention to the cancellation of the last three terms, quadratic in q, in (5.40). This goes beyond Lorentz invariance, requiring the cyclic symmetry of the trilinear vertex and the specific structure of $\omega_{\mu\nu}$ in (5.12). *(See note added in Sec. XI.)*

The second gauge-invariant vertex radiation amplitude is similarly constructed (see Appendix B), yielding

$$M_\lambda(2)_\alpha=\bar{u}_5\left\{\frac{Q_4}{p_4\cdot q}(p_4\cdot\epsilon\eta_4^\xi+\omega^\xi{}_\nu\eta_4^\nu)\sigma_{\lambda\xi}+\frac{Q_3}{p_3\cdot q}(-p_3\cdot\epsilon g_\lambda^\nu+\omega^\nu{}_\lambda)\eta_4^\xi\sigma_{\nu\xi}\right.$$
$$\left.+\frac{Q_5}{p_5\cdot q}(p_5\cdot\epsilon+\tfrac{1}{4}[\not\epsilon,\not q])\eta_4^\xi\sigma_{\lambda\xi}+\frac{Q_6}{p_6\cdot q}\eta_4^\xi\sigma_{\lambda\xi}(-p_6\cdot\epsilon-\tfrac{1}{4}[\not\epsilon,\not q])\right\}_\alpha , \tag{5.41}$$

with the contracted remainder

$$R^\lambda(2)_\alpha=\cdots V^{\beta\lambda}(p_3)\left[\frac{1}{\not p_6-m_6}v_7\right]_\alpha , \tag{5.42}$$

and with (5.36) modified by $p_3=p_4+p_5-p_6+q$.

It is easy to see that $M(2)$ vanishes in the null zone. The Dirac spin currents produced the first-order Lorentz transformation of $\sigma_{\lambda\xi}$ [cf. (5.16)],

$$\Delta\sigma_{\lambda\xi}\equiv\tfrac{1}{4}[[\not\epsilon,\not q],\sigma_{\lambda\xi}]=\tfrac{1}{2}\not\epsilon[\not q,\sigma_{\lambda\xi}]+\tfrac{1}{2}[\not\epsilon,\sigma_{\lambda\xi}]\not q$$
$$=\omega_\lambda{}^\beta\sigma_{\beta\xi}+\omega_\xi{}^\beta\sigma_{\lambda\beta} , \tag{5.43}$$

which is cancelled by the (vector wave function) ω terms in (5.41).

Finally, the third vertex radiation amplitude is

$$M(3)_\beta=\epsilon_{\mu\nu\sigma\rho}\left\{\eta_8^\mu\eta_9^\nu p_{10}^\sigma p_{11}^\rho\left[\frac{Q_6}{p_6\cdot q}(p_6\cdot\epsilon+\tfrac{1}{4}[\not\epsilon,\not q])+\frac{Q_7}{p_7\cdot q}(p_7\cdot\epsilon-\tfrac{1}{4}[\not\epsilon,\not q])\right]\right.$$
$$+p_{10}^\sigma p_{11}^\rho\left[\frac{Q_8}{p_8\cdot q}(p_8\cdot\epsilon\eta_8^\mu+\omega^\mu{}_\alpha\eta_8^\alpha)\eta_9^\nu+\frac{Q_9}{p_9\cdot q}(p_9\cdot\epsilon\eta_9^\nu+\omega^\nu{}_\alpha\eta_9^\alpha)\eta_8^\mu\right]$$
$$\left.+\eta_8^\mu\eta_9^\nu\left[\frac{Q_{10}}{p_{10}\cdot q}(-p_{10}\cdot\epsilon p_{10}^\sigma+\omega^\sigma{}_\alpha p_{10}^\alpha)p_{11}^\rho+\frac{Q_{11}}{p_{11}\cdot q}(-p_{11}\cdot\epsilon p_{11}^\rho+\omega^\rho{}_\alpha p_{11}^\alpha)p_{10}^\sigma\right]\right\}_\beta v_7 , \tag{5.44}$$

with the factor

$$R(3)_\beta=\left[\cdots\frac{1}{\not p_6-m_6}\right]_\beta . \tag{5.45}$$

Now (5.36) is modified by $p_6=p_{10}+p_{11}-p_7-p_8-p_9-q$.

$M(3)$ is also seen to vanish under (4.1). The direct cancellation of the Dirac spin currents is expected for a scalar fermion coupling. The cancellation of the remaining contact and vector spin currents, expected by the Lorentz invariance of the remaining coupling, follows from the use of the basic identity

$$g^{\mu\nu}\epsilon^{\alpha\beta\gamma\sigma}=g^{\mu\alpha}\epsilon^{\nu\beta\gamma\sigma}+g^{\mu\beta}\epsilon^{\alpha\nu\gamma\sigma}+g^{\mu\gamma}\epsilon^{\alpha\beta\nu\sigma}+g^{\mu\sigma}\epsilon^{\alpha\beta\gamma\nu} . \tag{5.46}$$

VI. RADIATION REPRESENTATION

The conclusions of Sec. V are summarized by the statement that each gauge-theoretic vertex radiation amplitude in (5.19) can be written as

$$M_\gamma(V_G)=\sum_{i=1}^{n_v}\frac{Q_iJ_i}{p_i\cdot q} , \tag{6.1}$$

where

$$\sum_{i=1}^{n_v}\delta_iQ_i=0 , \tag{6.2a}$$

$$\sum_{i=1}^{n_v}J_i=0 , \tag{6.2b}$$

$$\sum_{i=1}^{n_v}\delta_ip_i\cdot q=0 . \tag{6.2c}$$

The source vertex subgraph V_G has n_v internal and external legs, whose propagator factors are not included in (6.1). J_i is the product of the photon-emission current j_i for the ith leg (the j_i rules are summarized in Appendix B) and the remaining factors of the original vertex amplitude. Examples for J_i appear in Sec. V E. The current sum rule (6.2b), a consequence of translational, Lorentz, and Yang-Mills symmetries, is *independent of whether or not the null-zone condition is realized.*

The zeros for identical $Q/p\cdot q$ or identical $J/p\cdot q$ (the latter condition satisfied only under very special circumstances) dictate restrictions on the radiation amplitudes. We wish to use the algebra underlying the theorem and its complement to find a form for the amplitudes that displays explicitly the bilinear expansion in differences of the $Q/p\cdot q$ and $J/p\cdot q$ factors.

The following trivial lemma will help to introduce the algebra.[25]

Lemma 1. If $s=\sum_ia_ib_i$, where $\sum_ib_i=0$, then $s=\sum_i(a_i-a_j)b_i$, for all j. (The sum may omit $i=j$.)

The (easily proven) lemma addressing the specific form of (6.1) is the following.

Lemma 2. If[26]

$$\sum_{i=1}^{l}A_i=\sum_{i=1}^{l}B_i=\sum_{i=1}^{l}C_i=0 , \tag{6.3}$$

then

$$\sum_{i=1}^{l}\frac{A_iB_i}{C_i}=\sum_{i=1}^{l}\left[\frac{A_i}{C_i}-\frac{A_j}{C_j}\right]C_i\left[\frac{B_i}{C_i}-\frac{B_k}{C_k}\right] \tag{6.4}$$

for all j,k. (The sum may omit $i=j,k$.) Writing

$$A_iB_i/C_i=C_i(A_i/C_i)(B_i/C_i)\ ,$$

we see that (6.4) now exhibits the invariance under $A_i/C_i\to A_i/C_i+\text{constant}$ or $B_i/C_i\to B_i/C_i+\text{constant}$.

We may reduce (6.4) to the expected $l-2$ terms by choosing $j\neq k$. In the simplest nontrivial case,

$$\sum_{i=1}^{3}\frac{A_iB_i}{C_i}=\left[\frac{A_1}{C_1}-\frac{A_2}{C_2}\right]C_1\left[\frac{B_1}{C_1}-\frac{B_3}{C_3}\right] \tag{6.5a}$$

$$=\frac{C_1C_2}{C_3}\left[\frac{A_1}{C_1}-\frac{A_2}{C_2}\right]\left[\frac{B_2}{C_2}-\frac{B_1}{C_1}\right]\ , \tag{6.5b}$$

for $j=2$, $k=3$. Any permutation of 123 is permitted in (6.5); (6.3) has been used in passing from (6.5a) to (6.5b), the factorization formula[27] of Ref. 5.

The application of (6.4) to (6.1) yields the *radiation representation* of $M_\gamma(V_G)$,

$$M_\gamma(V_G)=\sum_{i=1}^{n_v}\delta_ip_i\cdot q\Delta_{ij}(Q)\Delta_{ik}(\delta J)\ , \tag{6.6}$$

where we define the differences,

$$\Delta_{ij}(X)\equiv\frac{X_i}{p_i\cdot q}-\frac{X_j}{p_j\cdot q}\ . \tag{6.7}$$

As noted, we may reduce (6.6) to the n_v-2 independent differences[28] among the $\Delta_{ij}(Q)$ and among the $\Delta_{ij}(\delta J)$.

From (5.19) and (6.6) we have a radiation representation for the general radiation amplitude. The bidifference form embodies the consequences of the symmetry properties of the radiation amplitudes. From this perspective, both versions of the radiation theorem are by-products of the radiation representation.

A radiation representation in which only differences in external $Q/p\cdot q$ factors appear can be written for the complete radiation amplitude $M_\gamma(T_G)$. Equation (4.3) and the linearity in the $Q/p\cdot q$ factors imply that

$$M_\gamma(T_G)=\sum_{i=1}^{n}\frac{Q_i}{p_i\cdot q}I_i(T_G)\ , \tag{6.8}$$

where I_i is independent of the charges. It follows from the theorem that

$$\sum_i^n I_i=0\ . \tag{6.9}$$

Hence Lemma 2 applies:

$$M_\gamma(T_G)=\sum_{i=1}^{n}\delta_ip_i\cdot q\Delta_{ij}(Q)\Delta_{ik}(\delta I)\ . \tag{6.10}$$

The I_i appear less convenient for calculation or for physical interpretation, where, for example, there is no gauge-invariant grouping of terms. The representation (6.6), in combination with (5.19), involves the same number of terms, since each M_γ in (6.6) can be reduced to n_v-2 terms and, for any tree graph with V total vertices and n external particles, $\sum_{v=1}^{V}(n_v-2)=n-2$. The organization of the radiation amplitude into only $n-2$ terms can be appreciated when we realize that there are as many as $2n-3$ radiation graphs arising from photon couplings to lines and as many as $3(n-2)$ more seagull terms.

Let us illustrate the radiation representation using the example in (5.4). We find

$$M_\gamma(\text{Fig.}\,6)=\sum_{1}^{2}M(v)R(v)\ , \tag{6.11}$$

where, by choice,

$$M(1)=-p_1\cdot q\left[\frac{Q_1}{p_1\cdot q}-\frac{Q_4}{p_4\cdot q}\right]\left[\frac{p_1\cdot\epsilon}{p_1\cdot q}-\frac{(p_1-p_4)\cdot\epsilon}{(p_1-p_4)\cdot q}\right] \tag{6.12a}$$

$$=-\frac{p_1\cdot qp_4\cdot q}{(p_1-p_4)\cdot q}\left[\frac{Q_1}{p_1\cdot q}-\frac{Q_4}{p_4\cdot q}\right]\left[\frac{p_4\cdot\epsilon}{p_4\cdot q}-\frac{p_1\cdot\epsilon}{p_1\cdot q}\right]\ , \tag{6.12b}$$

$$R(1)=\frac{i\lambda_3{}^2}{(p_3-p_2)^2-m_5{}^2}\ , \tag{6.13}$$

and $M(2),R(2)$ are obtained by relabeling the charges and momenta in (6.12) and (6.13) according to $1\to2$, $4\to3$.

VII. NEUTRAL PARTICLES

We now investigate the role of neutral[29] external particles in the radiation theorem.

A. A view from the radiation representation

The representation of Sec. VI makes it clear that zeros are present in gauge-theoretic radiation amplitudes in tree approximation, even for opposite-sign charges. For example, radiation zeros occur for the reaction $e^+e^-\to e^+e^-\gamma$, albeit in the unphysical region. Charge and momentum conservation, the mass-shell constraints, and Lorentz invariance, which are ingredients of the radiation theorem, can be maintained even for the unphysical energies that the null-zone condition (4.1) may require.

A cursory conclusion, however, from the radiation representation might be that there would be no radiation zero in the presence of an external particle r with *zero charge*, $Q_r=0$. For a set $\{r\}$ of zero external charges in a vertex source graph, (6.6) reduces to

$$M_\gamma(V_G)=\sum_{i\neq r}\delta_ip_i\cdot q\Delta_{ij}(Q)\Delta_{ik}(\delta J)-\frac{Q_j}{p_j\cdot q}\sum_r\delta_rp_r\cdot q\Delta_{rk}(\delta J)\ , \tag{7.1}$$

for some $j,k\neq r$. The null-zone condition $\Delta_{ij}(Q)=0$, for the nonzero charges does not imply that $M_\gamma(V_G)=0$, since only the first term in (7.1) is eliminated. The discrepancy ultimately derives from the fact that terms $Q_rJ_r/p_r\cdot q$ are now missing from the amplitude in (6.1).

This conclusion is wrong. We see that (7.1) vanishes if, in addition to $\Delta_{ij}(Q)=0$ (for $i,j\neq r$), we have

$$\sum_r p_r\cdot q=0\quad\text{(null zone)} \tag{7.2}$$

and

$$\sum_r J_r = 0 \quad \text{(null zone)} . \tag{7.3}$$

In fact, both requirements can be met if each neutral particle is massless and travels parallel to the photon, as expected from the zero-charge limit[30] of the null-zone equations. This is discussed in more detail in Sec. VII B.

Therefore, the radiation theorem is unaltered by the presence of neutral external particles. (We will see in Sec. VII D that neutral internal lines present no problems.) The null zone is simply the corresponding limit of (4.1). Radiation zeros are no longer manifested by Δ_{ij} factors alone, but are also associated with the vanishing or, in the case of neutral internal particles, the cancellation of currents in the radiation representation.

B. One external neutral particle

Suppose that only one external particle r has zero charge, the rest of the particles with nonzero charges of the same sign. If the charged external particles have equal $Q/p\cdot q$, then

$$p_r \cdot q = 0 \quad \text{(null zone)} , \tag{7.4}$$

from (6.2a) and (6.2c), so that

$$p_r = K_r q \tag{7.5}$$

for constant $K_r > 0$. Therefore, a single external neutral particle of any spin must be massless and must enter or exit the scattering region parallel to the photon, for a physical null zone to exist.[31]

In order to have a zero in the radiation amplitude for the vertex to which r is attached, the partial sum must vanish,

$$\sum_{i\neq r} J_i = 0 \quad \text{(null zone)} , \tag{7.6}$$

since J_r is absent from the sum in (6.1). Equations (6.2b) and (7.6) imply that

$$J_r = 0 \quad \text{(null zone)} . \tag{7.7}$$

Hence, even though $Q_r = 0$, its associated J_r is still relevant as a test of whether (7.6) is satisfied. It suffices to consider the factors j_r which can be calculated independently of Q_r (Appendix B).

The evaluation of j_r for the different spins leads to the following.

Lemma I. A neutral external particle may be included in the radiation theorem if it is massless and, in case the particle is a vector boson, it is coupled to a conserved current in a nonforward direction (defined below).

Proof. Evidently, (7.5) implies that the corresponding (external) convection, Dirac, and contract currents are zero and that the vector spin current can be written as

$$\omega_{\alpha\beta}\eta_r^\beta = q_\alpha \epsilon\cdot\eta_r , \tag{7.8}$$

we may rewrite q_α in (7.8),

$$q_\alpha = (p_r \pm q)_\alpha / (K_r \pm 1) , \tag{7.9}$$

in terms of the momentum transferred to the vertex, which is $p_r \pm q$ for photon emission from a particle in the final/initial state. Therefore, (7.8) does not contribute in the event that the vector particle is attached to a conserved current,[32] with $K_r \neq 1$. In the exceptional case, $K_r = 1$, (7.8) does not vanish and (7.7) does not hold if an initial-state neutral vector particle has momentum identical to the final photon. Such "forward scattering" transfers no momentum to the vertex.

The lemma therefore sanctions additional external photons in the radiation theorem. For an example of $K_r \neq 1$, the reaction $e^-e^- \to e^-e^-\gamma\gamma$ has a null zone where the photons are parallel and which is a simple generalization of the null zone for reaction (4.4), $e^-e^- \to e^-e^-\gamma$. The fact that the "first" photon must be coupled to a conserved current requires a gauge-invariant set of source graphs. For an example[33] of $K_r = 1$, consider Compton scattering, $\gamma + e \to \gamma + e$, where the forward amplitude is nonzero, being proportional to $\epsilon\cdot\epsilon'$. The null zone is the forward direction, where the convection currents cancel, but with zero momentum transfer the spin terms do not.

Further illustrations of the lemma can be found by examining the examples of Figs. 6 and 9 in zero-charge limits. Recall also the forward zero[3,4] for $\bar{\nu}e \to W\gamma$.

Since (6.2b) is based on Poincaré invariance (see Sec. IX), we may look for a simple picture behind (7.7) using momentum and angular momentum conservation. The vanishing of the convection current can be attributed to the fact that a scalar particle cannot emit a unit of helicity collinearly. A massless spinor particle cannot flip its helicity with a vector coupling, and neither can a massless vector particle whose longitudinal component has been eliminated. (This component is not eliminated, however, for $K_r = 1$ which is the exceptional case of forward scattering.)

The calculations showing $J_r = 0$ exhibit the same mechanism whereby collinear mass singularities are found to be suppressed.[34] Related to this is the fact that the $g \neq 2$ photon-emission factors are divergent in the massless limit[35] (see Sec. V). Convergence for $g = 2$ is crucial for the inclusion of neutral particles in the radiation theorem.

The question of gauge dependence arises for the evaluation of J_r in the case of a massless neutral vector particle. Since we are after the defect in (6.2b), where it is only the interactions of the nonzero charges that concern us, the question is irrelevant; the unitary-gauge emission factors (5.24) are sufficient for the purpose of evaluating the partial sum (7.6).

Nevertheless, we can show that the emission factors (5.24) apply in a more general (covariant) gauge,[36] where we replace the propagator factor (5.23) by

$$P_{\mu\nu}(p) = -g_{\mu\nu} + \frac{(1-\xi)p_\mu p_\nu}{p^2 - \xi m^2} . \tag{7.10}$$

The emission factor (5.24a), for example, is replaced by

$$\frac{Q}{p\cdot q}\left[p\cdot\epsilon\eta_\mu + \omega_{\mu\nu}\eta^\nu - \frac{p\cdot q\eta\cdot\epsilon}{2p\cdot q + (1-\xi)m^2}(p+q)_\mu \right] . \tag{7.11}$$

The presence of a conserved current eliminates the $(p+q)_\mu$ term in (7.11).

C. Additional external neutral particles

Lemma II. Lemma I applies independently of the number of neutral external particles.

Proof. If each neutral particle r satisfies the criteria of Lemma I, we have the following null zone specialized to a set of neutral particles $\{r\}$:

$$\Delta_{ij}(Q)=0\ ,\quad i,j\neq r\ , \tag{7.12a}$$

$$p_r\cdot q=0\quad (p=K_r q)\ . \tag{7.12b}$$

(The set of such neutral particles and the photon can be regarded as a massless composite and can easily be included in the discussion of a physical null zone; see Appendix A.) By (7.12b) and the arguments in Sec. VII B, each of the missing currents is zero, so that (7.6) is true for each vertex.

Are the sufficient conditions (7.2) also necessary? Could the null zone be larger? To address this suppose that there are $n_0\leq n-2$ external neutral particles[29] at a given vertex. If the remaining $n-n_0$ particles have the same $Q/p\cdot q$ factor, the generalization of (7.4) is

$$P\cdot q=0\quad(\text{null zone})\ , \tag{7.13}$$

where P is the total neutral momentum

$$P\equiv\sum_r^{n_0}\delta_r p_r\ . \tag{7.14}$$

[Compare (7.2).] Therefore, P must be lightlike, $P\propto q$, if the neutral particles are all in the initial state, or all in the final state. In such cases, such p_r satisfies (7.12b).

Consider the alternative possibility corresponding to neutral particles in both initial and final states, where (7.13) does not lead to (7.4) for the individual particles. Since (7.6) is required for each vertex, the sum over the currents J_r for the neutral particles at each vertex must vanish. Without neutral internal particles, the vanishing for arbitrary photon polarization of the total convection current in this sum, $p\cdot\epsilon$, necessitates $P\propto q$. (It is to be emphasized that a radiation zero, as we have defined it, refers to cancellations that are not peculiar to the various polarization states.) The spin and contact currents could cancel by Lorentz invariance. The conclusion is that we can augment (7.12), but only by configurations where the momentum transfer is lightlike and where the neutral sector in each vertex factorizes in a Lorentz invariant manner such that its spin and contact currents are not needed to cancel the currents in the charge sector.

D. Internal neutral particles

We now verify that the radiation theorem holds without qualification for neutral internal tree lines I, as it might be expected in view of the fact that the null-zone condition involves only the external particles. The limit $Q_I\to 0$ *after* the imposition of the null-zone condition (4.1) obviously shows the standard cancellation within each vertex, in terms of the radiation vertex expansion.

The case of interest, however is $Q_I=0$, *ab initio,* which involves cancellations between vertices:

Lemma III. The defects in the respective terms of the radiation vertex expansion (5.19), due to a given neutral internal particle, cancel each other in the null zone.

Proof. The sum of the two *defects* is proportional in the null zone to

$$D(p')j_{\text{out}}(p')+j_{\text{in}}(p)D(p)\quad(\text{null zone})\ , \tag{7.15}$$

since the remaining factors in the $M_\gamma R$ products are the same. In (7.15) the subscript I is suppressed $(p'=p-q)$ and the currents j refer to the vertices that the internal line has left and entered (and can be found along with the propagator D in Appendix B).

In fact, (7.15) can be seen to vanish from the radiation decomposition identity (Appendix B). From a consideration of the original photon coupling to the internal line (the left-hand side), the decomposition $(D'j'+jD)/p\cdot q$ (the right-hand side) must be regular at $p\cdot q=0$ (Q_I factors out.). As in (7.4),

$$p\cdot q=p'\cdot q=0\quad(\text{null zone}) \tag{7.16}$$

for any neutral internal line p, so (7.15) is zero.

The vanishing of (7.15) establishes the lemma and allows us to regard a neutral line as a short circuit between two vertices, leaving a composite gauge-invariant vertex that could be used in a reorganized radiation vertex expansion. Lemma III may be illustrated by explicit calculation of (7.15) for the various cases. We leave the details to the reader, but note that (7.16) does not imply (7.5), in contrast to external particles. In general, $p\cdot\epsilon=p'\cdot\epsilon\neq 0$ in the null zone.

VIII. EXTENSIONS TO NONGAUGE INTERACTIONS AND CLOSED LOOPS: A LOW-ENERGY THEOREM

We now consider more general interactions including first- or higher-order derivatives of Dirac and vector fields (other than the Yang-Mills form) and/or second- or higher-order derivatives of scalar fields. If we also allow closed loops, the source graphs are entirely arbitrary.

A. A low-energy theorem

Null-zone low-energy theorem. For any source graph S_G, with $g=2$ external legs, the radiation amplitude can be written as[37]

$$\mathscr{M}_\gamma(S_G)=M_\gamma(S_G)+O(q)\ , \tag{8.1}$$

where $M_\gamma=0$ in the null zone and has a radiation representation.

This theorem is the union of the standard low-energy theorem for bremsstrahlung[38,39] and the radiation theorem. In the low-energy expansion the leading (infrared) term vanishes in the null zone; the next-order (spin and contact) term also vanishes in the null zone provided that $g=2$ for the external particles.

We define an effective tree-graph substructure of S_G by contracting all closed loops to points, which implies an effective vertex radiation amplitude

$$\mathscr{M}_\gamma(V_G)=\sum\frac{Q_i\mathscr{J}_i}{p_i\cdot q} \tag{8.2}$$

in direct correspondence with (6.1). The infrared terms in $\mathscr{M}_\gamma$ come from the $O(q^0)$ convection terms in the effective currents $\mathscr{J}_i$ which cancel in $\sum\mathscr{J}_i$ by momentum conservation. The zeroth-order terms in $\mathscr{M}_\gamma$ correspond to the first-order spin and contact terms in $\mathscr{J}_i$ which cancel in the same sum by Lorentz invariance, provided that the photon couplings to the fixed lines in the effective tree graph correspond to $g=2$. In the absence of a general mechanism for the cancellation of higher powers of q, (6.2b) is replaced by

$$\sum \mathscr{J}_i = O(q^2) \ . \tag{8.3}$$

Any terms on the right-hand side of (8.3) must be due to nongauge derivative couplings and closed loops.

The contact currents associated with the nongauge couplings and the closed-loop graphs[40] are straightforward to determine. The term that is linear in q in the expansion of the radiation graph where the photon is attached to an exterior leg of the closed loop or to a leg connected with a derivative coupling yields the momentum-shift part of the contact current. The seagull can be derived by requiring gauge invariance for both cases. (Alternatively, for the closed loop, the linear term from the graph where the photon is attached to the loop itself yields the seagull.) See Sec. V D.

Internal spinning particles are not required to have $g=2$ for the null zone low-energy theorem to hold, since anomalous moments for internal particles contribute only at the $O(q)$ level. [See (5.20) and (5.29).] For example, in the Dirac decomposition (5.18) internal $g \neq 2$ corrections correspond to quadratic terms in the numerators.

The zeroth-order and first-order terms in the $\mathscr{J}_i$ serve to define M_γ in (8.1). It follows from Sec. VI that M_γ has a radiation representation. The quadratic terms associated with the Yang-Mills source vertex, which are the only higher-order terms in gauge-theoretic interactions and which cancel cyclically, could be included either in M_γ or in the $O(q)$ remainder of (8.1). This ambiguity shows that

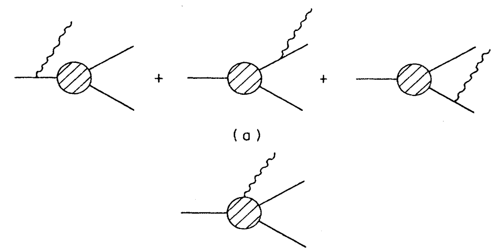

FIG. 10. The amplitude for radiative decay, $1 \to 2+3+\gamma$, separated into (a) radiation from the external legs and (b) internal radiation including seagulls.

the null zone low-energy theorem is not equivalent to the radiation theorem, but is rather its corollary. On the other hand, the content of the radiation theorem is the remark that the $O(q)$ terms in (8.1) are zero for gauge-theoretic couplings and tree graphs.

B. Example

We first derive the standard low-energy theorem for the decay $1 \to 2+3+\gamma$ where the particles $1-3$ are scalars. The amplitude (Fig. 10) separates into external and internal radiative parts,

$$\mathscr{M}_\gamma\,(\text{Fig. 10}) = \mathscr{M}^{\text{ext}}(q) + \mathscr{M}^{\text{int}}(q) \ . \tag{8.4}$$

If $D(m_1{}^2, m_2{}^2, m_3{}^2)$ is the amplitude for the source decay, $1 \to 2+3$, then

$$\mathscr{M}^{\text{ext}}(q) = -\frac{Q_1}{p_1\cdot q} p_1\cdot\epsilon D((p_1-q)^2, m_2{}^2, m_3{}^2) + \frac{Q_2}{p_2\cdot q} p_2\cdot\epsilon D(m_1{}^2, (p_2+q)^2, m_3{}^2) + \frac{Q_3}{p_3\cdot q} p_3\cdot\epsilon D(m_1{}^2, m_2{}^2, (p_3+q)^2) \ . \tag{8.5}$$

The expansion of (8.6) in q leads to

$$\mathscr{M}^{\text{ext}}(q) = M_\gamma + \Delta\mathscr{M} + O(q) \ , \tag{8.6}$$

where

$$\mathscr{M}_\gamma = \left[-\frac{Q_1}{p_1\cdot q} p_1\cdot\epsilon + \frac{Q_2}{p_2\cdot q} p_2\cdot\epsilon + \frac{Q_3}{p_3\cdot q} p_3\cdot\epsilon \right] D(m_1{}^2, m_2{}^2, m_3{}^2) \ , \tag{8.7}$$

$$\Delta\mathscr{M} = 2\left[Q_1 p_1\cdot\epsilon \frac{\partial}{\partial m_1{}^2} + Q_2 p_2\cdot\epsilon \frac{\partial}{\partial m_2{}^2} + Q_3 p_3\cdot\epsilon \frac{\partial}{\partial m_3{}^2} \right] D(m_1{}^2, m_2{}^2, m_3{}^2) \ . \tag{8.8}$$

$\mathscr{M}^{\text{int}}$ is infrared convergent and can be expanded as

$$\mathscr{M}^{\text{int}}(q) = \mathscr{M}^{\text{int}}(0) + O(q) \ . \tag{8.9}$$

In order to proceed further, we may follow either the approach of Ref. 39 or of Sec. V D. The former approach centers on the observation that if $q^\mu f_\mu = O(q^2)$ for arbitrary q, and if f_μ is independent of q, then $f_\mu = 0$. In our particular case, such an f_μ can be defined by

$$\epsilon^\mu f_\mu \equiv \Delta\mathscr{M} + \mathscr{M}^{\text{int}}(0) \ , \tag{8.10}$$

because M_γ is separately gauge invariant. Therefore,

$$\mathscr{M}^{\text{int}}(0) = -\Delta\mathscr{M} \ , \tag{8.11}$$

so that

$$\mathscr{M}_\gamma\,(\text{Fig. 10}) = M_\gamma + O(q) \ . \tag{8.12}$$

In the approach of Sec. V D, $\Delta\mathscr{M}$ in (8.8) and $\mathscr{M}^{\text{int}}(0)$ in (8.9) correspond to the momentum-shift and seagull terms, respectively, in the contact current (5.33). The fact that the contact current actually vanishes (from $p_i^\mu \omega_{\mu\nu} p_i^\nu = 0$) corresponds to (8.11). [For the external leg i, $p=r=p_i$ in (5.32).]

The leading term in (8.12) vanishes in the null zone, verifying the null-zone low-energy theorem. In more complicated cases, spin currents lead to zeroth-order terms in (8.12) which can also be incorporated into M_γ provided that $g=2$ holds for the external particles with spin.

We note that the gauge-invariant radiation vertex expansion is useful in the general construction of low-energy theorems. In particular, it is well suited for dealing with the complications arising from cancellations between the two ends of a fixed internal line, from the effective-tree organization of graphs with closed loops, and from the definition[37] of $O(q)$.

The null-zone low-energy theorem enlarges the scope of experimental tests, since we are not restricted to perturbative tree graphs. Some of these possibilities are proposed in the conclusion, Sec. XI.

C. Closed loops

The existence of amplitude zeros, central to the radiation theorem, may appear to violate the uncertainty principle. We do not expect, quantum mechanically, to find an exact cancellation in the interference among the various radiators at a specific point in momentum space, unless there is complete uncertainty in the particle positions. Indeed, the theorem refers only to the tree approximation where the radiation is controlled by the classical currents of plane-wave states; $\hbar$ corrections from closed loops which provide coordinate *correlations* are expected to fill in the radiation amplitude zeros. In this respect, radiation zeros are in marked contrast to the exact amplitude zeros due to conservation laws such as angular momentum.

The absence of a radiation zero for particles with $g\neq 2$ (see Sec. V) is an example which can be attributed to quantum effects inasmuch as closed-loop radiative corrections give rise to anomalous magnetic moments. (In fact, the basic content of the Drell-Hearn-Gerasimov sum rule[41] is that deviations from $g=2$ must be due to internal excitations.)

We recall from (8.1) that violations of the radiation theorem appear as $O(q)$ contributions with no radiation zero. In this context, the decay $1\rightarrow 2+\gamma$ provides a simple but instructive example (cf. Sec. IV B). A physical $n=2$ decay automatically satisfies the null-zone condition so that M_γ vanishes identically. However, closed loops and nongauge couplings must lead to nonvanishing $O(q)$ contributions, unless another mechanism intervenes. Indeed, closed-loop amplitudes[42] for $\mu\rightarrow e\gamma$ do not vanish and are $O(q)$ (in theories where lepton number is not conserved). Although the $n=2$ decay amplitude is identically zero to all orders for scalar particles 1 and 2, this is due to angular-momentum conservation.

The existence and position of a radiation zero does not depend on the spin of the external (or internal) particles and, moreover, does not depend on masses, charges, and momenta except in the $Q/p\cdot q$ combinations allowed by the null-zone condition (4.1). By changing these parameters, one may *test* for a radiation zero. In the case of the $n=2$ decay, adding spin eliminates the "angular-momentum" zero. As another example, the general amplitude, including closed loops, for the electron bremsstrahlung reaction (4.4) would vanish by an angular momentum argument in the null zone (4.1), if the electrons were identical scalar bosons. Adding spin removes the angular-momentum zero in (every order of) the amplitude. On the other hand, adding closed loops removes the radiation zero, in general.

The previous remarks suggest two categories of closed-loop amplitudes for which there are amplitude zeros in the null radiation zone:

Category 0. This is the trivial class where the amplitude and its higher-order corrections vanish in the null zone because an additional mechanism is also operative for certain charge, mass, and spin assignments. Such mechanisms may be deactivated by changing the assignments or moving to another part of the null zone. These amplitude zeros are *not* radiation zeros.

Category 1. This is the class of source closed loops that produce no correlations or corrections to $g=2$. We have in mind scalar self-energies, which can be included to all orders (see Sec. V A), and "neutral" closed loops. If a closed loop is completely neutral (meaning there are no photon couplings to its internal lines with no charge transferred to it by external particles at any of its "external" vertices) and if the loop can be factorized so as to leave a Lorentz-invariant tree structure in the remainder, then the null-zone cancellation can proceed according to that tree structure. It is noted that, if Δp_i is the momentum transfer to a neutral loop through its ith neutral leg, $\Delta p_i\cdot\Delta p_j$ is invariant under photon emission from external lines, since $\Delta p_i\cdot q=0$ in the null zone.

Box graphs are closed loops that produce correlations. Self-energy source loops for spinning particles lead to $g\neq 2$. These examples do not belong to category 1.

IX. PHOTON COUPLING: POINCARÉ TRANSFORMATIONS AND BMT

In this section we discuss photon couplings in terms of the Poincaré group of transformations and we make a connection between the BMT equations and the null-zone cancellations.

A. Poincaré transformations

Let us recall the first-order universal Lorentz transformation (5.11),

$$\Lambda_{\mu\nu}=g_{\mu\nu}+\lambda\omega_{\mu\nu}\ , \tag{9.1}$$

where λ represents the freedom in normalization. We express (5.12),

$$\lambda\omega_{\mu\nu}=q_\mu d_\nu-d_\mu q_\nu\ , \tag{9.2}$$

in terms of the spacelike four-vector

$$d_\mu\equiv\lambda\epsilon_\mu\ . \tag{9.3}$$

The generalization of (9.1) to finite λ is $\exp(\lambda\omega)$ or

$$\Lambda_{\mu\nu}=g_{\mu\nu}+\lambda\omega_{\mu\nu}+\frac{\lambda^2}{2}q_\mu q_\nu\ . \tag{9.4}$$

Since

$$\begin{aligned}&\Lambda^\mu{}_\alpha g_{\mu\nu}\Lambda^\nu{}_\beta=g_{\alpha\beta}\ ,\\&\Lambda^\mu{}_\nu q^\nu=q^\mu\ ,\end{aligned} \tag{9.5}$$

the $\Lambda_{\mu\nu}$ form an Abelian subgroup of the little group $E_2(q)$.[43] Also, Λ generates gauge transformations on the polarization vector ϵ,

$$\Lambda^\mu{}_\nu\epsilon^\nu=\epsilon^\mu-\lambda q^\mu\ . \tag{9.6}$$

An important result of Sec. V is that the spin and contact currents can be written in terms of the universal first-order term in the Lorentz transformation (9.1). In addition, the convection current $p\cdot\epsilon$ can be understood as the universal first-order term in the translation ($e^{ip\cdot a}\to 1+ip\cdot a$) in the direction ϵ. Since the relative normalization among the currents is fixed, we must have $a=d$. The length d_μ then appears universally in the generator (9.2) for the spin and contact currents and as the displacement for the convection currents. Thus we consider the full Poincaré transformation $\mathscr{P}=\{d,\Lambda\}$: $x'=\Lambda x+d$. *Each of the current contributions in Appendix B can be expressed universally in terms of the first-order Poincaré transformation $\mathscr{P}$ acting on the particle wave functions.* (The internal currents act through the decomposition identities as transformations on bilinear wave functions.) *(See note added in Sec. XI.)*

The vanishing in the null zone of the radiation amplitude for tree diagrams in gauge theory can be described in terms of Poincaré symmetry: The convection current cancellation by translational invariance and the spin and contact current cancellation by Lorentz invariance.[44] (The Yang-Mills cancellation involves additional symmetry.)

The electromagnetic current J^μ in lowest order has a Gordon decomposition[45] into the separately conserved convection and spin currents,

$$J^\mu_{\rm conv}=\sum_j iQ_j\tilde{\psi}_j\overleftrightarrow{\partial}^\mu\psi_j \ , \tag{9.7}$$

$$J^\mu_{\rm spin}=\sum_j 2iQ_j\partial_\nu(\tilde{\psi}_jS^{\mu\nu}\psi_j) \ , \tag{9.8}$$

where the spin indices of the fields ψ have been suppressed and where $\tilde{\psi}\equiv\psi^+$ or $\bar{\psi}/2m$ as the case may be. The spin tensor in (9.9) is

$$S_{\mu\nu}=\begin{cases}0, \text{ scalar,}\\ -\frac{i}{2}\sigma_{\mu\nu}, \text{ Dirac,}\\ i(g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma}), \text{ vector .}\end{cases} \tag{9.9}$$

[The indices σ,ρ in the vector case are those of the fields in (9.9).] The spin-current Lagrangian $-J^\mu_{\rm spin}A_\mu$, corresponds to the interaction Hamiltonian

$$\mathscr{H}_{\rm int}=\sum_j iQ_j\tilde{\psi}_jS^{\mu\nu}\psi_jF_{\mu\nu} \ , \tag{9.10}$$

which, for the $S_{\mu\nu}$ given by (9.11), implies the gyromagnetic value $g=2$, for each particle with spin.

From our diagrammatic analysis in Sec. V, we may interpret the photon currents as effective generators of Poincaré transformations in momentum space, even though (9.7) and (9.8) are not space-time Poincaré generators. In particular, (9.9), which is also the set of matrix representations of the generators of Lorentz transformations on spins 0, $\frac{1}{2}$, and 1, respectively, exhibits a direct connection between the spin current and the Lorentz transformation of the fields,[46] but *only for $g=2$.*

B. The BMT analysis and the null zone

Since the radiation amplitude is linear in the photon field, the correspondence principle implies that there should be a classical counterpart for the relationship of $g=2$ to the universal Lorentz transformation. Consider a classical particle with spin moving in a slowly varying external electromagnetic field $F^{\mu\nu}$. Our neglect henceforth of forces dependent upon the gradients of the fields is consistent with the fact that the null-zone cancellation involves only the first two orders in q.

The Lorentz force law for a particle with charge Q, mass m moving in $F^{\mu\nu}$ is[47]

$$\frac{du^\mu}{d\tau}=\frac{Q}{m}F^{\mu\nu}u_\nu \ , \tag{9.11}$$

where u is the four-velocity and τ is the proper time. The BMT equation for the four-polarization s of the particle is[47,48]

$$\frac{ds^\mu}{d\tau}=\frac{Q}{m}\frac{g}{2}F^{\mu\nu}s_\nu+\frac{Q}{m}\left[\frac{g}{2}-1\right]u^\mu s_\lambda F^{\lambda\nu}u_\nu \ . \tag{9.12}$$

A significant and well-known feature of (9.11) and (9.12) is that, for $g=2$, the changes in u and s in time $d\tau$ can be described in terms of the same infinitesimal Lorentz transformation,

$$\Lambda_{\mu\nu}=g_{\mu\nu}+\frac{Q}{m}F_{\mu\nu}d\tau \ . \tag{9.13}$$

Consequently, in proper time $d\tau$, the orbital and precessional frequencies of the particle are identical.

What is of interest is the situation involving a system of particles moving in $F^{\mu\nu}$. In order to compare the Lorentz transformation (9.13) for each particle we refer to a single common observer at a (retarded) time t, which is related to the particle times t' by[49]

$$dt=dt'(1-\hat{n}\cdot\vec{v})=\frac{p\cdot n}{E}dt' \ . \tag{9.14}$$

Here $\vec{v}$ (E) is the velocity (energy) of a given particle and $\hat{n}$ is the unit vector from this particle to the (distant) observer such that $n\equiv(1,\hat{n})$ is a lightlike four-vector proportional to the radiation-wave four-vector.

From (9.13), (9.14), and $dt'=Ed\tau/m$,

$$d\Lambda_{\mu\nu}=\frac{Q}{p\cdot n}F_{\mu\nu}dt \ . \tag{9.15}$$

At a given time, all particles with identical $Q/p\cdot n$ and with $g=2$ are observed to have the identical response to the presence of a constant external field. The condition of identical $Q/p\cdot n$ is equivalent to the null-zone condition since the photon energy can be scaled out of the equations in (4.1). (An initial particle simply corresponds to an earlier t' than does a final particle.) The first-order Lorentz transformation (9.15) can be compared to (9.1), noting that $\omega_{\mu\nu}$ is the Fourier transform of the radiation counterpart to $F_{\mu\nu}$.

Thus all Lorentz invariants constructed of u_i,s_i and their derivatives, such as those that arise in the Lagrangian, are fixed (for $g=2$) in the time interval during which all $Q/p\cdot n$ are equal. [Equivalently, we may think of making an instantaneous Lorentz transformation which cancels (9.15).] In this sense, a system of particles in its null zone experiences no linear response to a slowly varying external field. If we identify $F_{\mu\nu}$ with the radiation field, in semiclassical approximation, then this result corresponds to the radiation interference theorem.

X. EXTENSIONS TO RADIATION OF OTHER GAUGE BOSONS

In this section we extend the radiation theorem to the emission/absorption of other massless gauge bosons. We also briefly discuss the emission of particles with different mass and spin.

A. Other gauge bosons

The radiation theorem, representation, and associated corollaries can be proven for an arbitrary gauge group G where the role of the photon is assumed by the massless gauge boson(s) g assigned to the adjoint representation of G. If the generalized "charges" (calculated from the representation of G to which the particles belong) are conserved, then it is easy to adapt the previous proof. The current for g emission has a dual connection to both internal transformations and space-time transformations and the invariance under each group can be exploited.

Our task is facilitated by the results and notation of Ref. 5 where the four-body amplitude zero is related to factorization for general G, and our first step is to generalize their work to an arbitrary n-vertex source graph.

We assume that g has local gauge couplings to all other particles (possibly including more gauge bosons g), which belong to the various representations of G and whose couplings are invariant under G. If we use factorized Feynman rules, the n-vertex source graph can be written as a product,

$$S_V = \Gamma_{a_1 a_2 \cdots a_n} V(p_1, p_2, \ldots, p_n) , \tag{10.1}$$

with factors invariant under G and Lorentz transformations, respectively. The space-time factor V is the same as in the photon case. The internal-group factor Γ is the Clebsch-Gordan coefficient for the n-particle coupling, labeled by the internal indices a_i which refer to the particle representations.

The corresponding radiation amplitude has the structure of (6.1) with the same space-time current J_i,

$$M_g = \sum_{1}^{n} \frac{Q_i^g J_i}{p_i \cdot q} . \tag{10.2}$$

The gauge-boson couplings,

$$Q_i^g = \Gamma_{a_1 a_2 \cdots a_{i-1} b a_{i+1} \cdots a_n} \Gamma_{\bar{b} a a_i} , \tag{10.3}$$

where a sum over b is understood, generalize the U(1) charge. $\Gamma_{\bar{b} a a_i}$ is the Clebsch-Gordan coefficient for the three-vertex which couples an incoming particle i, the gauge boson g (index a), and an outgoing particle (index b).

Common factors,

$$\frac{Q_i^g}{p_i \cdot q} = \frac{Q_j^g}{p_j \cdot q} , \quad \text{all } i , \quad \text{any } j , \tag{10.4}$$

lead to the vanishing of the amplitude in (10.2) in familiar fashion. The generalized charges also sum to zero [cf. (6.2a)],

$$\sum_{i=1} \delta_i Q_i^g = 0 , \tag{10.5}$$

by G invariance. Since g is in the adjoint representation of G, $\Gamma_{\bar{b} a a_i}$ refers to a matrix representation of the corresponding generator. Therefore, an $n-2$ double-difference radiation representation can also be obtained for (10.2), with the qualifications concerning any derivative couplings in J_i the same as in the photon case.

The above results can be extended to general tree graphs where the emission of g from any given internal line involves the G-space factors,

$$\cdots \Gamma_b^L \Gamma_{\bar{b} a c} \Gamma_{\bar{c}}^R \cdots , \tag{10.6}$$

in which the "left" vertex, with coefficient Γ_b^L, is connected to the "right" vertex, $\Gamma_{\bar{c}}^R$ by the original internal line, $\delta_{\bar{b}c}$, in the source graph. The other source-graph indices and Clebsch-Gordan factors are suppressed in (10.6). The remaining task is to generalize the radiation decomposition identity to include (10.6).

Referring to Fig. 5, we associate $\Gamma_{\bar{b}ac}$ first with $\Gamma_{\bar{c}}^R$ and then with Γ_b^L, respectively, in the corresponding emission terms of the decomposition identity so that there is a complete set of conserved charges, analogous to (10.3), associated with each source vertex. This thus gives a generalized gauge-invariant radiation vertex expansion. The radiation theorem, corollaries, representation, and the other photon results all generalize with the replacement of Q_i by Q_i^g.

Examples of the generalized charges have already been worked out by Zhu[6] for four-body zeros. Suppose that the three-vertex source graph is the spinor-spinor-vector coupling $\bar{\psi}\gamma_\mu T_a \psi V_a^\mu$, where the Dirac particles 1 and 2, and the vector particle 3, belong to the fundamental and adjoint SU(N) representations, respectively. Then the constraint $Q_1^g / p_1 \cdot q = Q_2^g / p_2 \cdot q$ becomes

$$\frac{(T_a T_b)_{ij}}{p_1 \cdot q} = -\frac{(T_b T_a)_{ji}}{p_2 \cdot q} . \tag{10.7}$$

There is a practical limitation to the observation of certain non-Abelian radiation zeros. In the case of QCD, the gluon is coupled to (presumably) unobservable color charges. Therefore, the color-singlet physical states are connected to quark and gluon particles only through color averaging and summing. Since their positions depend on the charges, such amplitude zeros are smeared out in the physical cross sections (as noted previously in the three-vertex case).[6] We emphasize, however, that the radiation representation for the gluon amplitudes can still be utilized.

B. Other spins and masses

The vector character of the gauge boson is essential to the association of the currents with Poincaré invariance. Nevertheless, other spins and relationships should be investigated. We have in mind graviton emission and Riemann invariance, as well as superfield emission and supersymmetry. These questions are not addressed in this paper, but the search for currents that satisfy analogous dualities may be fruitful.

Finally consider vector gauge bosons with $q^2 \neq 0$, addressing the two cases where the radiated boson is virtual (e.g., lepton scattering and e^+e^- annihilation) and where it is real with nonzero mass (e.g., Z^0 production in electroweak theory). Although $q\cdot\epsilon$ still vanishes, it may be kept to exhibit (any) gauge invariance. In the virtual case we assume that ϵ^μ represents a conserved current source.

Upon recalculation, the convection and Dirac spin factors in Appendix B for both external and decomposition-identity emission factors are changed only the replacement

$$\frac{Q}{p\cdot q}\to\frac{Q}{p\cdot q\pm\frac{1}{2}q^2} \qquad (10.8)$$

for outgoing (+) or incoming (−) particles. (Strictly, the gauge-invariant convection current is $\pm p\cdot\epsilon+\frac{1}{2}q\cdot\epsilon$.) The vector-particle spin factor requires two changes, (10.8) and

$$\omega_{\mu\nu}\to\omega_{\mu\nu}+\frac{1}{2m^2}(p\pm q)_\mu(q^2\epsilon_\nu-q\cdot\epsilon q_\nu)\ , \qquad (10.9)$$

where $p\pm q$ is the momentum of the vector particle between the source vertex and the emission. The change in (10.9) does not contribute in the event that the vector particle is itself coupled to a conserved current. However, if gauge invariance requires seagull contributions, the contact current is significantly altered,

$$\omega^{\beta\mu}p_\mu\to\omega^{\beta\mu}p_\mu\mp\tfrac{1}{2}[q^2\epsilon^\beta-(q\cdot\epsilon)q^\beta]\ . \qquad (10.10)$$

Evidently, Lorentz invariance does not also imply the cancellation of the new term, appearing in (10.10), in the $\sum J_i$ sum.

Another difference is that the new factors (10.8) cannot be equal in the physical region, in general.[50] The absence of physical null zones corresponds to the absence of asymptotic radiation fields (r^{-1} behavior). Furthermore, there is no analog to (6.2c) for the denominators, unless the number of particles is unchanged during the collision, so that we cannot generally reduce the number of differences from $n-1$ to $n-2$. Despite these remarks, we can again write a radiation representation, in terms of $n-1$ (or $n-2$) differences or products of differences, depending on whether (6.2a) and (6.2b) are valid. In the case of broken gauge symmetries such as the SU(2)×U(1) electroweak theory, the radiation interference theorem holds in the approximation at high energies where masses are neglected. (See the angular distributions for $q\bar{q}\to W^{\pm}Z^0$ in Ref. 3)

XI. SUMMARY AND FUTURE DIRECTIONS

A. Summary

We have introduced a useful radiation vertex expansion $\sum M_\gamma(V_G)R(V_G)$. The complete set of Feynman diagrams for the photon (or other massless gauge boson) attachments to the source tree graph T_G is expressed in terms of radiation vertex amplitudes $M_\gamma(V_G)$, each of which is a sum $\sum QJ/p\cdot q$ over photon attachments to V_G calculated as if all vertex legs were external. Consequently, each $M_\gamma(V_G)$ is separately gauge invariant. The radiation decomposition identity is instrumental in effecting this reorganization.

The general form $\sum QJ/p\cdot q$ for the radiation vertex amplitude clearly shows the basic algebra leading to the radiation theorem and its complement. If $Q/p\cdot q$ ($J/p\cdot q$) is the same for all legs of the vertex, and if $\sum J=0$ ($\sum Q=0$), then $M_\gamma(V_G)=0$.[8] Because $\sum J=\sum Q=\sum p\cdot q=0$, $M_\gamma(V_G)$ can be rewritten as

$$\sum p\cdot q(Q/p\cdot q-A)(J/p\cdot q-B)$$

for any A,B. The radiation representation is obtained by choosing A (B) to be a particular factor $Q/p\cdot q$ ($J/p\cdot q$), exhibiting the radiation theorem(s).

The fundamental relation is $\sum J=0$, which might be called the Poincaré-Yang-Mills sum rule. With $\sum Q=0$, we see a dual role for the electromagnetic (or other gauge-group) current: Generating transformations in the internal space and, also, in effect, transformations in space-time. (After factoring out $Q/p\cdot q$, the convective current effectively generates a universal displacement, the spin current effectively generates a universal Lorentz transformation of its associated wave function, and the contact current effectively generates the same universal Lorentz transformation of its associated derivative coupling. The BMT discussion of Sec. IX shows the classical spinning-particle limit of the universal currents cataloged in Appendix B.) In this way we can view the massless gauge boson as characteristic of the adjoint representation of both the internal group and the relevant little group, whose attachment generates the product of the first-order gauge and Poincaré (displacement and Lorentz) transformations, provided we have the prescribed derivative couplings. Poincaré and Yang-Mills symmetries[51] are thus responsible for the null-zone cancellations.

A physical null-zone theorem has been proven which states that if particles have the same Q/m ratios (more generally, the common value of Q/m for the initial state may be different from that for the final state) then we can always find, at any c.m. energy, physical regions where the radiation zeros occur (i.e., where all $Q/p\cdot q$ are equal). The Q/m restriction can be relaxed for any particle that is massless; we note that the physical null zone is generally smaller for particles with mass. We have also studied physical null-zone limits for more general Q,m values in the $n=3$ case and for equal Q/m in $n=4$.[52]

For a radiation zero, any external neutral particle r must be massless and travel in the same direction as the photon. This leads to $J_r=0$. (The analogous remark for the complementary theorem is that $J_r=0$ would require $Q_r=0$.) Neutral internal particles, however, do not have such restrictions.

The radiation theorem is the statement that gauge interactions preserve the classical zeros in tree approximation. The null-zone condition can be defined equivalently as the condition under which there is complete destructive interference of the classical radiation patterns of the incoming and outgoing charged lines (the infrared limit). In the nonrelativistic limit, this corresponds to the well-known absence of electric dipole radiation for collisions involving particles with the same charge-mass ratio.

The spin independence of the null zone should be emphasized. Reactions which include $u\bar{d}\to W\gamma$ have been examined recently,[53] with the result that the presence of nonradiation zeros depends on the polarization. Only the radiation zero is present in every helicity channel.

Radiation zeros are generally destroyed by closed loops. The existence of these short-range quantum corrections can be anticipated from the uncertainty principle. One cannot expect *exact* amplitude zeros for subregions of angles and energies except in the violation of a conservation law. The special class of closed loops, where there are no correlations and no $g=2$ corrections, is an exception. Thus, we can include certain neutral closed loops defined in Sec. VIII. We can also include scalar self-energies in the source graph since there the radiation decomposition identity is correct to all orders.

Indeed, in a recent study of scalar particles in the null zone[54] it is shown that first-order bubbles preserve the radiation zero while a triangle source graph does not. In the context of our discussion, the former example introduces neither a correlation nor an anomalous moment, while the latter generates a correlation.

A null-zone low-energy theorem is based on the fact that the radiation theorem can be applied to the leading terms in photon momentum q. The infrared term is guaranteed to vanish in the null zone for arbitrary amplitudes. The $O(q^0)$ term also vanishes there provided that the external particles have $g=2$. Therefore, low-energy theorems automatically separate out terms that have radiation zeros. We have also presented a useful formalism for the study of low-energy theorems and the null zone by means of a generalized radiation vertex expansion for an arbitrary source graph.

B. Remarks

It is well-known that gauge-theory couplings can be derived by imposing a unitarity constraint on the high-energy limit of tree amplitudes.[55] Since minimal couplings can also be inferred by the requirement that the radiation theorem hold, we seem to be building a bridge from the classical infrared limit to high-energy behavior. Note also that the Drell-Hearn-Gerasimov sum rule for anomalous moments implies $g=2$ for all spins at the tree level (classical limit), given a high-energy condition on the spin-flip Compton amplitude. The same conclusion follow for the existence of null radiation zones.

Furthermore, it has been suggested to us that the radiation theorem could possibly be stated directly in terms of renormalizability[56]: "The necessary and sufficient condition for a tree amplitude with one or more external massless gauge particles to have a zero independent of spin is that the model be renormalizable, where the renormalizability may be disguised by a Higgs mechanism or by heavy particles whose exchange looks like a point interaction (tree segments of zero length)." In this sense, gauge-theoretic interactions may be called *quasirenormalizable.*

The most striking experimental implication of the radiation zeros involves the original reaction, $q\bar{q}\rightarrow W\gamma$, which may be measurable[57] in future proton-collider experiments. Although the actual external legs are hadrons with anomalous moments, the high-transverse-momentum photon, recoiling against the W, couples in leading twist only to the hard-scattering subprocess; diagrams involving radiation from spectators, etc., are suppressed by powers of $m^2/M_W{}^2$ where m is the hadronic mass scale. In addition there is transverse-momentum smearing and gluon radiative corrections of order $\alpha_s(M_W{}^2)/\pi$. To this accuracy, gauge-theory couplings can be probed. The investigation of null zones in bremsstrahlung reactions such as hard-quark scattering, $eq\rightarrow eq\gamma$, $qq\rightarrow qq\gamma$, or in radiative decays may give a measure of heavy-quark and heavy-lepton magnetic moments.

In principle, a measure of neutrino masses can be found in the decay, $A\rightarrow B+\nu+\gamma$, since its null zone requires $m_\nu=0$.[58] (But examples such as $\pi\rightarrow e\nu\gamma$ do not have physical null zones.) It has also been suggested that corrections to PCAC (...) (partial conservation of axial-vector current) may be similarly studied.[59] In general, the deviations from zero in the null zone provide estimates of higher-order corrections [which must also be $O(q)$ by the null-zone low-energy theorem] in any process, from the standard reactions such as $e^-e^-\rightarrow e^-e^-\gamma$ to exotic processes involving new particles.

The null-zone condition can be applied very simply to composite particles with arbitrary spin and with collinear constituents i (momenta $p_i=x_ip$ in terms of the composite momentum p), such as hadrons involved in hard-scattering QCD processes. In the region where $x_i\propto Q_i$, the tree-graph approximation with gauge couplings for the constituents implies that the composite has the same $Q/p\cdot q$ factor as its constituents, and its resultant effective current follows the description in Appendix B, corresponding to an effective gauge coupling for the composite. The null zone is preserved. More generally, we may use a composite picture to understand the null zone in any radiative reaction. Both the initial and final states can be considered to be composites, and, in the null zone, the reaction is equivalent to $1\rightarrow 2+\gamma$ whose tree amplitude vanishes for $Q_1/p_1\cdot q$, irrespective of the spin of the composites.

Another interest is whether the radiation representation could be used to simplify computations. Recent calculations in QED and QCD have shown that lowest-order radiative amplitudes reduce to simple forms. In particular, massless five-body results factorize.[60] We have verified that radiation zeros are present in these forms. For example, both reactions $e^+e^-\rightarrow e^+e^-\gamma$ and $e^+e^-\rightarrow\mu^+\mu^-\gamma$, which have the same (unphysical) null zone when lepton masses are neglected, yield a common amplitude factor in which the zeros reside. The symmetries inherent in the concept of radiation zeros can be instrumental in understanding the simplicity of the forms obtained.

Finally, it is important to determine the extent to which currents in theories of higher spins such as supersymmetry play an analogous role. Do they also generate transformations in both internal and external spaces in the manner of the massless-vector-gauge-boson currents? Will they also lead to equations which relate variables in both spaces like $Q_i/p_i\cdot q=Q_j/p_j\cdot q$?

Note added in proof. Double derivatives of scalar fields can be included in the gauge-theoretic couplings by replacing a vector field by a derivative of a scalar field, in the manner of Higgs, in the Yang-Mills vertex. See the discussion of "radiation symmetry" by R. W. Brown, in Electroweak Effects at High Energies, proceedings of the Europhysics Study, Erice, 1983 (unpublished). The finite Poincaré transformation is related to classical plane-wave interactions. The Yang-Mills cancellation can be described in terms of the Bianchi identity (cf. Ref. 51). Also, $g=2$ is necessary but not sufficient for gauge-theoretic couplings. See R. W. Brown and K. L. Kowalski (unpublished). The radiation representation has recently been used to simplify certain polarization calculations. See C. L. Bilchak, R. W. Brown, and J. D. Stroughair (unpublished).

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APPENDIX A

We address the location of the null radiation zone and, in particular, some details behind the equations of Sec. IV.

1. The $n=3$ decay

We begin with the boundary limits for the decay $1\to 2+3+\gamma$. The lower (upper) limit on the range in (4.9) is derived from $p_3\cdot q \geq 0$ $(E_2\geq m_2)$. The range in (4.10) is obtained from $q^2=0$ and $(\vec{p}_2\cdot\vec{p}_3)^2 \leq \vec{p}_2{}^2\vec{p}_3{}^2$, or

$$y^2(x+\mu_3{}^2)+yx(x+\mu_2{}^2+\mu_3{}^2-1)+\mu_2{}^2x^2\leq 0 \ . \tag{A1}$$

In terms of the relative charge

$$Q\equiv\frac{Q_2}{Q_3} \ , \tag{A2}$$

(A1) and (4.8) yield

$$Q^2\mu_3{}^2+Q(\mu_2{}^2+\mu_3{}^2-1)+\mu_2{}^2\leq 0 \ . \tag{A3}$$

Therefore,

$$Q_-\leq Q\leq Q_+ \ ,$$
$$Q_\pm=\{1-\mu_2{}^2-\mu_3{}^2 \pm[(1-\mu_2{}^2-\mu_3{}^2)^2-4\mu_2{}^2\mu_3{}^2]^{1/2}\}(2\mu_3{}^2)^{-1} \ . \tag{A4}$$

Given some masses, m_2 and m_3, only those charges that lie in the range (A4) can lead to a physical null zone. The massless limit is $0\leq Q\leq\infty$, giving a physical null zone for all same-sign charges.

To see the range allowed for $\mu_i\neq 0$, we calculate the mass limits for a given Q, using (A3),

$$0\leq\mu_2{}^2\leq\frac{Q}{Q+1}<1 \ , \tag{A5a}$$

$$0\leq\mu_3{}^2\leq\frac{1}{Q+1}-\frac{\mu_2{}^2}{Q} \ , \tag{A5b}$$

consistent with

$$m_2+m_3\leq m_1 \ . \tag{A6}$$

The nonrelativistic limit is the upper limit of (A6),

$$\mu_2+\mu_3=1 \ . \tag{A7}$$

From (A5) and (A7),

$$\frac{Q_2}{m_2}=\frac{Q_3}{m_3}=\frac{Q_1}{m_1} \ . \tag{A8}$$

Assume

$$\frac{Q_2}{m_2}=\frac{Q_3}{m_3} \ , \tag{A9}$$

but not the nonrelativistic limit. [Equation (A8) holds only in that limit.] Then (A5) leads to

$$\mu_2=Q\mu_3\leq\frac{Q}{Q+1} \ . \tag{A10}$$

However, this is equivalent to (A6) and (A9) alone. Thus, all values of m_2/m_3 consistent with (A6) and (A9) produce a null zone.

2. The $n=3$ scattering

In terms of the initial c.m. speeds v_1 and v_2, (4.11) may be written

$$\cos\theta=(Q_2/v_1-Q_1/v_2)/Q_3 \ . \tag{A11}$$

From $|\cos\theta|\leq 1$, we obtain for given v_i,

$$\frac{v_2{}^{-1}-1}{v_1{}^{-1}+1}\leq Q\leq\frac{v_2{}^{-1}+1}{v_1{}^{-1}-1} \ , \tag{A12}$$

where

$$Q\equiv\frac{Q_2}{Q_1} \ . \tag{A13}$$

On the other hand, given Q,

$$1\leq v_1{}^{-1}\leq\infty \ , \tag{A14a}$$

$$\max[1,Q(v_1{}^{-1}-1)-1]\leq v_2{}^{-1}\leq Q(v_1{}^{-1}+1)+1 \ . \tag{A14b}$$

In the equal-mass case, (A13) reduces to

$$\frac{|Q-1|}{Q+1}\leq v\leq 1 \ , \tag{A15}$$

with $v_1=v_2\equiv v$.

The limits of the above equations are by now familiar. For example, the nonrelativistic limit of (A12) is

$$Q=\frac{v_1}{v_2} \tag{A16}$$

or

$$\frac{Q_1}{m_1}=\frac{Q_2}{m_2} \ . \tag{A17}$$

Note that the third particle is not required to be nonrelativistic and thus Q_3/m_3 is not necessarily equal to the ratios in (A17). (We only require equal $Q/p\cdot q$.) In lowest order, (4.11) places no restriction on $\cos\theta$, implying totally destructive interference in the nonrelativistic limit, whereas (A11) and (A16) give the first-order correction, which is satisfied by $\cos\theta=0$.

A physical null zone is guaranteed for all energies by (A17), since this combines with (A11) to yield

$$\cos\theta=\frac{1}{m_1+m_2}\left[\frac{m_2}{v_1}-\frac{m_1}{v_2}\right] \ . \tag{A18}$$

Equation (A18) always satisfies $|\cos\theta|\leq 1$.

3. An $n=4$ example

If all particles have the same charges and the same masses, (4.11) leads to a photon c.m. direction perpendicular to the beams (Fig. 2), $\theta=\pi/2$. The other null-zone equation (4.13) reduces to $E_3=E_4\equiv E'$, momentum conservation obviously demands that $\theta_3=\theta_4\equiv\theta$, and the photon energy is given by

$$\omega=E-2E'=-2E'v'\cos\theta' , \tag{A19}$$

where $v_1=v_2\equiv v'$. As a check, the third null-zone equation also leads to (A19).

4. Null-zone theorems

We first prove the physical null-zone theorem of Sec. IV D for the decay $1\to n-1+\gamma$ in the parent rest frame. If the $n-2$ null-zone equations are chosen to be

$$\frac{Q_i}{p_i\cdot q}=\frac{Q_2}{p_2\cdot q} , \quad i=3,\ldots,n-1 , \tag{A20}$$

we find

$$\frac{Q_i}{m_i}\frac{1}{\gamma_i(1-v_i\cos\theta_i)}=\frac{Q_2}{m_2}\frac{1}{\gamma_2(1-v_2\cos\theta_2)} , \tag{A21}$$

in terms of particle speeds v_i and angles θ_i (relative to the photon). We are given that all Q_i/m_i are equal for $i\geq 2$. Therefore, if the particles travel together, opposite to the photon ($\theta_i=\pi, v_i=v_2$), (A21) is satisfied. This corresponds to the maximum energy for the photon and resembles the two-body decay $m_1\to M+\gamma$, where $M=\sum_2^{n-1}m_i\leq m_1$. More generally, (A21) is satisfied by some finite neighborhood, but we have already proven that the physical null zone is not empty, without resorting to zero photon energy.[52]

We next consider the reaction $1+2\to n-2+\gamma$ in the c.m. frame. One null-zone equation is taken as $Q_1/p_1\cdot q=Q_2/p_2\cdot q$, which can always be satisfied, for $Q_1/m_1=Q_2/m_2$, at some physical photon angle [cf. (A18)]. The remaining $n-3$ equations can be satisfied when the $n-2$ final particles travel together opposite to the (fixed) photon direction.

Finally, k particles in the initial state can be arbitrarily separated into two bunches with equal and opposite three-momenta (c.m. frame), choosing the initial phase-space region where each particle in a given bunch has the same velocity (same rest frame). These two composites have the same Q/m ratio by virtue of the identity (4.3). Thus $k-2$ equations are satisfied within the bunches, arguing as in the decay case, and another equation is satisfied for some photon angle, as in (A18). The final particles may be again clumped together opposite to the photon, satisfying another $n-k-1$ null-zone equations, for a total of $n-2$. The case where the photon is in the initial state is a simple reversal of this discussion.

The physical null-zone theorem for massless charges has a similar proof (and we can consider it to be a corollary to the previous theorem). In the general decay, $1\to n-1+\gamma$, a physical null zone exists where all the final-state particles are massless, and travel together opposite to the photon. So (A20) reduces to

$$\frac{Q_i}{E_i}=\frac{Q_2}{E_2} , \tag{A22}$$

and it is only necessary that the energy $m_1/2$ may be divided up according to the fraction of the total charge Q_1 that each particle carries. For more general initial states, Eq. (4.12) applies to two initial particles and, by construction, to the bunched initial states for $k>2$.

In a null zone, neutral particles must be massless and travel along with the photon (cf. Sec. VII). As such, they are easily incorporated into the physical null-zone theorem and its corollary. [It is intriguing that all known neutral structureless (elementary) particles have mass measurements consistent with zero.]

5. General equations and remarks

To prove the physical null-zone theorem, we only needed to show that the null-zone condition is satisfied somewhere in the physical region. We outline below an analytical approach that may be useful in the full determination of physical null zones for more particles (larger n) and general mass and charge values.

The $n-2$ constraints are to be superimposed on phase space. For general decay, $1\to n-1+\gamma$, the $3n-7$ final state variables imply a null zone with $2n-5$ dimensions. For two-body collisions, $1+2\to n-2+\gamma$, the $3n-8$ variables imply a null zone with $2n-6$ dimensions. ($n=3$ corresponds to a single point.) A given k-particle initial state, *with no symmetry axis,* corresponds to $3(n-k)-1$ final variables and $2n-3k+1$ null-zone dimensions.

We discuss an inductive analysis where we build larger-n null zones from smaller-n results by systematically replacing a particle by a composite of particles. For definiteness, consider the replacement of particle 3, in the $n=3$ decay, by a composite of $n-2$ particles. Denoting composite variables by the subscript c, we may replace one of the $n-2$ null-zone equations by

$$\frac{Q_c}{p_c\cdot q}=\frac{Q_2}{p_2\cdot q} . \tag{A23}$$

Equations (4.6)–(4.10) and (A1)–(A4) can be adapted by the change $3\to c$. The lower (upper) limit of $p_c{}^2$ corresponds to the constituents traveling together (particle 2 at rest with zero photon energy), but for a fixed x and y these limits are changed. The limits on x,y, and $Q\equiv Q_2/Q_c$ are found by the substitution $3\to c$ in Eqs. (4.9), (4.10), and (A4) using the minimum value of $p_c{}^2$. The original discussion can be repeated here, but it must be kept in mind that

TABLE I. The (only) modifications of source graphs necessary for the construction of the amplitudes $M_\gamma(V_G)$ in (5.19).

Radiator	Factor	Position
Vertex leg with charge Q along momentum p (or $p+q$) before emitting photon with momentum q and polarization ϵ, seagull included (if any)	$\frac{Q}{p\cdot q}j$	Factor goes between wave function and vertex in source graph (internal wave functions are Kronecker δ functions in spin space)

TABLE II. Feynman rules for (spin ≤ 1) propagators and single-photon vertices.

	Scalar	Dirac	Dirac antiparticle	Vector
Propagator $D(p)$	$i(p^2-m^2)^{-1}$	$i(\not p-m)^{-1}$	$i(-\not p-m)^{-1}$	$iP_{\mu\nu}(p)(p^2-m^2)^{-1}$ Eq. (5.27)
Photon vertex $\Gamma(p-q,q,p)$	$-iQ(2p-q)\cdot\epsilon$	$-iQ\not\epsilon$	$+iQ\not\epsilon$	$iQY_{\alpha\beta\gamma}(p-q,q,-p)\epsilon^{\beta}$ Fig. 7

the other null-zone equations are not yet satisfied.

We may regard c as a two-body system made up of particle 3 and another composite d with momentum p_d and charge Q_d. To (A23) we add

$$\frac{Q_d}{p_d\cdot q}=\frac{Q_3}{p_3\cdot q} . \tag{A24}$$

This procedure can be continued, peeling away constituents from the composite and adding the null-zone restrictions. At the second stage of telescoping we are led to define variables analogous to (4.6). The next stage is to regard d as made up of particle 4 and another composite e, and so on. There remains the task of determining the nested sequence of limits on the independent variables.[61]

An alternative procedure for smaller n or for the selection of points in the null zone, if not the whole null zone, is to rewrite (4.1) in c.m. coordinates:

$$E_i(1-v_i\cos\theta_i)=\frac{Q_i}{Q}E . \tag{A25}$$

For $e_i\equiv 2E_i/E, q_i\equiv Q_i/Q$, the relativistic version is

$$e_i\sin^2(\theta_i/2)=q_i . \tag{A26}$$

We observe that smaller charges must have less energy and/or smaller angles with the photon. It is essentially these equations and their implications that were used in the proof of the null-zone theorem.

APPENDIX B

In this appendix we present the rules for the construction of the radiation vertex expansion (5.19) for radiation amplitudes generated by any source tree graph with gauge-theoretic couplings. The factors in Table I modify the external or internal leg of each source vertex and are derived in Sec. V. All propagators are included in the factors R in (5.19), where the momentum assignment follows photon emission from vertex v. There is no momentum shift from derivative couplings in the coefficient of the convection current since this product is included in the contact current (see Ref. 23). In the Yang-Mills vertex, however, the coefficient of the spin currents includes the momentum shift, yielding the quadratic terms discussed in Sec. V E. Internal-leg factors are derived from the radiation decomposition identity, generalized to include possible contact currents.

The current j in Table I is

$$j=j_{\rm conv}+j_{\rm spin}+j_{\rm cont} , \tag{B1}$$

where

$$j_{\rm conv}=\text{(first-order coefficient in) universal displacement of wave function}=\pm p\cdot\epsilon \text{ for outgoing } (+) \text{ or incoming } (-) , \tag{B2a}$$

$$j_{\rm spin}=\text{(first-order coefficient in) universal Lorentz transformation of wave function } =(0;+\frac{i}{4}\sigma^{\alpha\beta}\omega_{\alpha\beta};-\frac{i}{4}\sigma^{\alpha\beta}\omega_{\alpha\beta};g_{\alpha\beta}\rightarrow\omega_{\alpha\beta}) \text{ for (scalar; spinor } \bar u,\bar v\text{; spinor } u,v\text{; vector } \eta_\alpha=g_{\alpha\beta}\eta^{\beta} , \ \eta^{\dagger}_{\alpha}=g_{\alpha\beta}\eta^{\dagger\beta}) , \tag{B2b}$$

$$j_{\rm cont}=\text{(first-order coefficient in) universal Lorentz transformation of derivative coupling, } g_{\alpha\beta}\rightarrow\omega_{\alpha\beta} \text{ for } p_\alpha=g_{\alpha\beta}p^{\beta} , \tag{B2c}$$

with

$$\omega_{\alpha\beta}=q_\alpha\epsilon_\beta-\epsilon_\alpha q_\beta . \tag{B3}$$

The radiation decomposition identity is

$$D(p-q)\Gamma D(p)+\text{seagulls (if any)}=D(p-q)j\frac{Q}{p\cdot q}+\frac{Q}{p\cdot q}jD(p) , \tag{B4}$$

where the various propagators and photon vertices are exhibited in Table II.

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[1]S. J. Brodsky and R. W. Brown, Phys. Rev. Lett. 49, 966 (1982).

[2]We have been informed by Mark Samuel that he has independently derived the null-zone condition for the interactions of scalar particles at a point. See also M. A. Samuel, Phys. Rev. D 27, 2724 (1983).

[3]R. W. Brown, D. Sahdev, and K. O. Mikalian, Phys. Rev. D 20, 1164 (1979). For a recent review, see R. W. Brown, in *Proton-Antiproton Collider Physics—1981,* proceedings of the Workshop on Forward Collider Physics, Madison, Wisconsin, edited by V. Barger, D. Cline, and F. Halzen (AIP, New York, 1982), p. 251.

[4]K. O. Mikaelian, M. A. Samuel, and D. Sahdev, Phys. Rev. Lett. 43, 746 (1979). The zeros and factorization of the cross sections for $q\bar{q}\to W\gamma$ and $\bar{\nu}e\to W\gamma$ can also be seen in the curves and formula of Ref. 3. Cross-section factorization was first shown for $\gamma q\to Wq$ by K. O. Mikaelian, Phys. Rev. D 17, 750 (1978).

[5]C. J. Goebel, F. Halzen, and J. P. Leveille, Phys. Rev. D 23, 2682 (1981).

[6]Zhu Dongpei, Phys. Rev. D 22, 2266 (1980). See also K. O. Mikaelian, *ibid.* 25, 66 (1982).

[7]T. R. Grose and K. O. Mikaelian, Phys. Rev. D 23, 123 (1981).

[8]As an alternative to using δ_i, all particles could be defined as outgoing, with the replacement $Q\to -Q$ and $p\to -p$ to be made for incoming particles. Note that $Q/p\cdot q$ is invariant under this replacement.

[9]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), Chap. 9.

[10]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), Chap. 14 and 15. See problems 15.3 and 15.5 of this reference.

[11]The magnetic dipole radiation depends upon the second (first) time derivative of the magnetic moment if one is dealing with orbital (intrinsic) magnetic moments (cf. Ref. 9, p. 189 and Ref. 10, p. 672).

[12]D. R. Yennie, *Lectures on Strong and Electromagnetic Interactions* (Brandeis, Massachusetts, 1963), p. 165.

[13]Where needed, the complex conjugation of polarization vectors is left understood.

[14]Various names for the $Q/p\cdot q$ factor might be given based on the relation of the denominator $p\cdot q$ to retarded time (see Sec. IX), to Doppler shifts, or to light-cone variables.

[15]Restrictions, such as the positivity condition (4.5) and the conditions for neutral particles (Sec. VII), have strongly limited the historical appearance of radiation zeros.

[16]S. Weinberg and G. Feinberg, Phys. Rev. Lett. 3, 111 (1959).

[17]For particles with spin the Ward-Takahashi identity does not suffice to determine $\epsilon\cdot\Gamma$ in terms of the appropriate full propagators $\hat{\Delta}$. In general [see A. Salam, Phys. Rev. 130, 1287 (1963)], $\Gamma^\mu=\Gamma^\mu_A+\Gamma^\mu_B$, where

$$\Gamma^\mu_A=(p'+p)^\mu(p'^2-p^2)^{-1}[\hat{\Delta}(p'^2)^{-1}-\hat{\Delta}(p^2)^{-1}]$$

and $(p'-p)\cdot\Gamma_B=0$. However, $\epsilon\cdot\Gamma_B\neq 0$, although $\epsilon\cdot\Gamma_A$ clearly satisfies a spin-indexed version of (5.3).

[18]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Chap. 2.

[19]Even though the outer product $\Gamma_1\cdots\Gamma_D$ in (5.7) is not tied together in spin space, it must be an overall rank-zero Lorentz tensor

$$S^{-1}\Gamma_1SS^{-1}\Gamma_2S\cdots S^{-1}\Gamma_DS=\Gamma_1\Gamma_2\cdots\Gamma_D$$

or, in first order,

$$\sum_i^D\Delta\Gamma_i\prod_{j\neq i}^D\Gamma_j=0\ .$$

[20]V_G is a multi-spinor-indexed matrix $(V_G)_{\alpha\beta}\dots$, in general, with an index for each internal leg and where each index may be regarded as an internal spinor wave function. For example, $(V_G)_\alpha=(V_G)_\delta w^\delta(\alpha)$ for $w^\delta(\alpha)=g^\delta_\alpha$. In this way we may say that each wave function of the vertex v, external or internal, is transformed with the same $\omega_{\mu\nu}$ by the photon emission associated with the corresponding vertex leg.

[21]Reference 18, Chap. 10.

[22]The Feynman rules are listed, for example, in Refs. 3 and 6. For comparison, note that $Q=-e<0$ for the quanta of the W field, the W^- particles, in Ref. 3.

[23]The reader is warned that, in the separation of terms in (5.31) leading to the contact current in (5.33), the residual convection current has the coefficient $p\cdot r$ and *not* $(p\pm q)\cdot r$. The momentum shift is included in (5.33).

[24]They also arise for couplings with higher derivatives, irrespective of spin.

[25]No bidifference sum in $(a_i-a_j)(b_i-b_k)$ exists for both $\sum_i a_i=\sum_i b_i=0$, since it is now impossible to have either identical a_i or identical b_i. However, bidifference forms can be constructed via a multiplier λ_i, e.g., such that $\sum_i(a_i-a_j)\lambda_i=0$. Then by Lemma 1,

$$s=\sum_i(a_i-a_j)\lambda_i(b_i/\lambda_i-b_k/\lambda_k)\ .$$

The bidifference expansion in Lemma 2 can be obtained in this way noting that identical A_i/C_i, identical B_i/C_i, and Eq. (6.3) can all coexist.

[26]We borrow notation from Ref. 5 in developing what is essentially a generalization of their factorization formula.

[27]A naive generalization of (6.5b) would be

$$\sum_1^l A_iB_i/C_i=C_k{}^{-1}\sum_{i<j}C_iC_j(A_i/C_i-A_j/C_j)\times(B_j/C_j-B_i/C_i)$$

for $i,j\neq k$, but has $(l-1)(l-2)/2$ terms. The minimal form is (6.4).

[28]The linear relationship is

$$\sum_i^{n_v}\Delta_{ij}(Q\text{ or }\delta J)\delta_ip_i\cdot q=0\ .$$

[29]A neutral particle has *no* photon couplings. Particles with zero charge but nonzero higher moments have non-gauge-theoretic interactions.

[30]An amplitude zero obviously occurs if the null-zone condition is satisfied first. We are concerned in this section with the reverse order, $Q_r=0$, *ab initio*, the physically relevant limit.

[31]Of course, (7.5) may lie outside the physical region. (See Appendix A.)

[32]The massless limit of a vector particle is generally singular. Conserved currents eliminate the extraneous helicity state, as required by Lorentz invariance [S. Weinberg, Phys. Lett. 9, 357 (1964); Phys. Rev. 138, B988 (1965)].

[33]As another example, there is no radiation zero in Fig. 9 when particle 1 has no charge. Although it is possible to construct a conserved current by considering particle 3 to be external and to have the same mass as particle 2, the null zone is in the forward direction.

[34]A classic paper on mass singularities is T. Kinoshita, J. Math. Phys. 3, 650 (1962).

[35]See the remarks by K. A. Johnson, MIT Report No. CTP977, 1982 (unpublished).

[36]G. 't Hooft and M. Veltman, DIAGRAMMAR, CERN Report No. 73-9, 1973 (unpublished).

[37]The limit $q \to 0$ implied by $O(q)$ can be taken within the physical region, although it requires a threshold in a two-body reaction (e.g., $q\bar{q} \to W\gamma$). In the case of additional massless particles, we can have nonanalytic behavior such as $O(q\ln q)$ in place of $O(q)$ in (8.1). This infrared problem is left understood in the standard low-energy theorem.

[38]F. E. Low, Phys. Rev. 110, 974 (1958).

[39]S. L. Adler and Y. Dothan, Phys. Rev. 151, 1267 (1966).

[40]A closed-loop graph can be expanded in external momenta, leading to an effective derivative-coupling series.

[41]S. D. Drell and A. C. Hearn, Phys. Rev. Lett. 16, 908 (1966); S. B. Gerasimov, Yad. Fiz. 2, 598 (1965) [Sov. J. Nucl. Phys. 2, 430 (1966)].

[42]See, for example, T.-P. Cheng and L.-F. Li, Phys. Rev. Lett. 38, 381 (1977).

[43]S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966), Chap. 4. We thank W. Bardeen and G. Mack for discussions about the little group.

[44]We therefore have an explanation for the "spatial generalized Jacobi identity" in Eq. (14) of Ref. 5. (A factor of $2p\cdot p_1$ is missing.) Only the single instance of the quadratic Yang-Mills term, implicit in their identity and discussed in Sec. V, goes beyond the linear Poincaré invariance argument. *(See note added in Sec. XI.)* Also, from our general arguments we see that additional $n=3$ vertices beyond the class of interactions considered in Ref. 5 qualify for factorization (the radiation theorem in the $n=3$ case).

[45]See, for example, R. J. Hughes, Phys. Lett. 97B, 246 (1980); Nucl. Phys. B186, 376 (1981).

[46]Notice that the spin currents in Appendix B can be written in terms of the finite transformation (9.4). Furthermore, the Dirac transformation (5.14) is correct for finite λ.

[47]Reference 10, Chap. 11.

[48]V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Lett. 2, 435 (1959); S. J. Brodsky and J. R. Primack, Ann. Phys. (N.Y.) 52, 315 (1969).

[49]Reference 10, Chap. 14.

[50]See Mikaelian, Ref. 6, for virtual-photon effects in the $n=3$ case.

[51]The vanishing of the q^0 and q^1 terms, due to Poincaré invariance, can be considered as a cancellation in flat-space. The q^2 Yang-Mills cancellation is at the basis of gauge theory and can be interpreted to be a curved-space symmetry.

[52]Additional studies of physical null zones have now been made by G. Passarino, Report No. SLAC-PUB-3024, 1982 (unpublished); M. A. Samuel, A. Sen, G. S. Sylvester, and M. L. Laursen, Oklahoma State University Research Note 144, 1983 (unpublished); S. G. Naculich, Case Western Reserve University Report No. CWRUTH-83-4 (unpublished).

[53]M. Hellmund and G. Ranft, Z. Phys. C 12, 333 (1982). See also K. J. F. Gaemers and G. J. Gounaris, Z. Phys. C 1, 259 (1979).

[54]M. L. Laursen, M. A. Samuel, A. Sen, and G. Tupper, Oklahoma State University Research Note 137, 1982 (unpublished). The zeros found in this study that persist to all orders are due to angular-momentum constraints. See also M. L. Laursen, M. A. Samuel, and A. Sen, following paper, Phys. Rev. D 28, 650 (1983).

[55]C. H. Llewellyn-Smith, Phys. Lett. 46B, 233 (1973); J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Phys. Rev. Lett. 30, 1268; (1973); 31, 572(E) (1973).

[56]C. J. Goebel (private communication).

[57]D. Cline and C. Rubbia (private communication).

[58]This has been suggested by R. Decker (private communication).

[59]E. A. Paschos (private communication).

[60]R. Gastmans, lecture delivered at the 18th Winter School of Theoretical Physics, at Karpacz, Poland, 1981 (unpublished); F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, W. Troost, and T. T. Wu, Nucl. Phys. B206, 61 (1982) and references therein.

[61]An expanded version of Appendix A and of other portions of this paper can be found in R. W. Brown, K. L. Kowalski, and S. J. Brodsky, Report No. Fermilab-82/102, 1982 (unpublished).