Quark-antiquark potential and Q^2 duality

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A detailed study is made of Q^2 duality for the $b\bar{b}$ system and the hypothetical heavy $t\bar{t}$ system using several potentials which include the perturbative QCD results for short distances. It turns out that the QCD corrections are essential in order that the sum rule holds. The Coulomb-type correction should not be neglected, and the gluonic correction to the leptonic width is also important.

I. INTRODUCTION

Sometime ago, Greco *et al.*[1,2] and Sakurai[3] introduced a kind of duality for e^+e^- annihilation into hadrons. This concept, called Q^2 duality, relates the parameters of vector-meson resonances ($\rho,\omega, \ldots$) with the production cross section of "parton" pairs.

With the advent of quantum chromodynamics (QCD), which we believe to be the underlying dynamics for strong interactions, it became possible to calculate the corrections to the parton-model results. A large number of works[4–13] have appeared, a portion of which deal with heavy-flavor production. Using the Q^2 duality, one can determine[5,6,11,12] the mass of the heavy quark and the strong coupling constant from experimental data. The most successful prediction[13] of Q^2 duality would be that of the charmonium spectrum, which was obtained from the moment sum rules for various currents.

With the accumulation of experimental data for heavy quarkonia, we have become aware of the forces between quarks and antiquarks, i.e., the quark-antiquark potential. A number of phenomenological potentials[14–20] which reproduce the data have been proposed, some of which were inspired by QCD.

Ishikawa and Sakurai[9] and Bell and Bertlmann[10] have made a "test" of the Q^2 duality for $c\bar{c}$ system using several potentials and the parton-model cross section. Although they have obtained reasonable agreement, the relativistic correction to the nonrelativistic calculation is not small for the $c\bar{c}$ system. The nonrelativistic description of the quarkonium system becomes better for heavier quarks.

In this paper we make a detailed test of the Q^2 duality for a very heavy t quark of mass around 25 GeV (and also b quark) using several potentials which include the perturbative QCD results for short distances. In Sec. II, we discuss general properties of the potential and give three potentials which show the perturbative QCD behavior at short distances and reproduce the $c\bar{c}$ and $b\bar{b}$ spectra and leptonic widths fairly well. In Sec. III, we introduce a sum rule (Q^2 duality) and discuss both sides of that relation. It is suggested that the Coulomb-type correction to the perturbative calculation of the vacuum polarization is important for very heavy quarks. In Sec. IV, we explicitly check the sum rule using the potentials introduced in Sec. II. Section V contains several concluding remarks. In an appendix, we show the form of the potentials we use in detail.

II. QUARK-ANTIQUARK POTENTIAL

The nonrelativistic description of a quark-antiquark ($Q\bar{Q}$) system becomes better for heavier quarks because of asymptotic freedom. We can obtain the spectra and wave functions (therefore, leptonic widths of vector mesons, etc.) by solving the Schrödinger equation if we know the $Q\bar{Q}$ potential $V(r)$ and the quark mass.

A. General properties

In order to discuss the properties of the $Q\bar{Q}$ potential, it is convenient to consider it in three regions, i.e., at long, short, and intermediate distances.

At long distances, the $Q\bar{Q}$ potential is expected to grow linearly, leading to confinement. This behavior is consistent with the linear Regge trajectories of light mesons. Monte Carlo calculations in the lattice-regularized QCD have confirmed[21] this expectation.

At short distances, perturbative calculations of QCD give reliable predictions for the $Q\bar{Q}$ potential. Since the effective coupling of QCD at short distances is rather small, the one-gluon-exchange term dominates the potential and leads to Coulombic behavior at small r. This $1/r$ potential is modified logarithmically by the running coupling constant.

The two-loop calculation of the $Q\bar{Q}$ potential in massless QCD gives[22]

$$V_{\rm QCD}(r) \sim -\frac{4\pi c_F}{b_0 r \ln(1/\Lambda_{\overline{\rm MS}}^2 r^2)} \left[1 + \left[2\gamma_E + \frac{c}{b_0} \right] \frac{1}{\ln(1/\Lambda_{\overline{\rm MS}}^2 r^2)} - \frac{b_1}{b_0^2} \frac{\ln\ln(1/\Lambda_{\overline{\rm MS}}^2 r^2)}{\ln(1/\Lambda_{\overline{\rm MS}}^2 r^2)} \right], \qquad (2.1)$$

where $c_F = \frac{4}{3}$ is a color factor, $b_0 = 11 - \frac{2}{3}N_f$, $b_1 = 102 - \frac{38}{3}N_f$, $c = \frac{31}{3} - \frac{10}{9}N_f$, $\gamma_E = 0.5772 \cdots$ is Euler's constant, N_f is the effective number of flavors, and $\overline{\rm MS}$ refers to the modified minimal-subtraction scheme.[29] Calculations with massive quarks have not been performed. In the following we take $N_f = 4$ for simplicity.

At present, the $Q\bar{Q}$ potential at intermediate distances cannot be calculated from QCD. Here experimental information from $c\bar{c}$ and $b\bar{b}$ spectroscopy is useful. A number of potentials[14–20] are able to reproduce the spectra of the $c\bar{c}$ and $b\bar{b}$ systems. A remarkable fact is that they all agree[19] approximately at intermediate distances (0.1 fm$\lesssim r \lesssim$1 fm) up to additive constants. (The effect of adding a constant to the potential can be compensated to some extent by changing the quark mass.) The inverse scattering construction of the potential[23] supports this agreement.

B. Choosing a potential

Among the proposed potentials, that of Buchmüller, Grunberg, and Tye (BGT)[18] includes the short-distance behavior of Eq. (2.1). This potential, however, implies a fairly large $\Lambda_{\overline{\rm MS}}$ of about 500 MeV, which is related to the Regge slope parameter ($\alpha' \sim 1$ GeV^{-2}). It is not possible to incorporate a smaller $\Lambda_{\overline{\rm MS}}$ in this scheme.[24] Thus, we have to connect the phenomenological potentials at intermediate (and long) distances and the calculated short-distance potential, Eq. (2.1).

In order to see the effects of changing the short-distance part of the potential (i.e., $\Lambda_{\overline{\rm MS}}$) and changing the intermediate-distance part, we use the following three potentials.

(1) "QCD + Richardson" potential with $\Lambda_{\overline{\rm MS}}=200$ MeV. This potential is constructed from the Richardson potential[16] for $r>0.1$ fm, Eq. (2.1) with $\Lambda_{\overline{\rm MS}}=200$ MeV for $r \lesssim 0.03$ fm, and logarithmic interpolation between 0.1 fm and $\sim$0.03 fm. This method of connection is the same in spirit as that of Ref. 19 ("Buchmüller-Tye potential" with $\Lambda_{\overline{\rm MS}}=200$ MeV).

(2) "QCD + Martin" potential with $\Lambda_{\overline{\rm MS}}=200$ MeV. This potential is obtained by connecting smoothly the Martin potential[20] for $r \gtrsim 0.1$ fm and Eq. (2.1) with $\Lambda_{\overline{\rm MS}}=200$ MeV.

(3) The original BGT potential[18] with $\Lambda_{\overline{\rm MS}}=509$ MeV. These three potentials are shown in Fig. 1.

The potentials (1) and (2) coincide at short distances, while (1) and (3) almost coincide at intermediate distances.

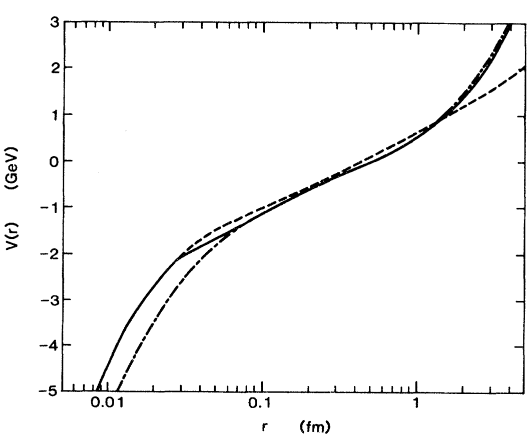

FIG. 1. The three $Q\bar{Q}$ potentials. Solid curve: QCD + Richardson potential (1); dashed curve: QCD + Martin potential (2); and dash-dotted curve: BGT potential (3). Potentials (1) and (2) coincide at short distances, and potentials (1) and (3) almost coincide at intermediate distances.

TABLE I. The calculated nS-$1S$ mass difference in MeV for the $b\bar{b}$ system ($n=2,3,4$).

	Potential (1)	Potential (2)	Potential (3)	Experiment
2S	544	577	554	559±3
3S	871	914	889	891±4
4S	1130	1154	1154	1113±4

Thus, comparison of the results from (1) and (2) is useful to study the uncertainty coming from intermediate (and long) distances which includes the uncertainty arising from the method of connection. The $\Lambda_{\overline{\rm MS}}$ dependence can be obtained by comparing the results from (1) and (3).

C. Predictions for $b\bar{b}$ and $t\bar{t}$ systems

Since we have introduced two new $Q\bar{Q}$ potentials besides an old one, we should check whether they reproduce the spectra and leptonic widths of the existing $c\bar{c}$ and $b\bar{b}$ systems. If they do not, they must be discarded. We have calculated these quantities and found that they are in satisfactory agreement with the data.

For the $c\bar{c}$ system, the short-distance alterations from the original Richardson or Martin potential have little effect for mass spectra, because even the J/ψ has a rather large radius compared with the connecting point.

The calculated energy levels and leptonic widths relative to the 1S state for the $b\bar{b}$ system is shown in Tables I and II, together with the experimental data. The b-quark mass used is 4.878 (4.837) GeV for the potentials 1 and 3 (2). All three potentials give reasonable agreement with the data.[25]

For the absolute values of the leptonic width, we compare the value obtained from the Van Royen—Weisskopf formula[26] with that from the same formula with $O(\alpha_s)$ corrections.[27] [For details, see Sec. III B.] The $O(\alpha_s)$ correction tends to improve the agreement between the calculated value and the data. The value of the corrected leptonic width of the $b\bar{b}$ 1S state is 1.00 (0.92, 1.07) keV for the potential 1 (2, 3) while the experimental value[25] is 1.15±0.13 keV.

We also show in Tables III and IV our predictions for the energy levels and the leptonic widths for the $t\bar{t}$ system. The mass of t quark is determined in such a way that the 1S mass is equal to 50 GeV.

III. THE Q^2 DUALITY

A. Derivation of the sum rule

The total hadronic cross section in e^+e^- annihilation is related to the absorptive part of the hadronic vacuum po-

TABLE II. The leptonic width for the $b\bar{b}$ system normalized to the 1S width.

	Potential (1)	Potential (2)	Potential (3)	Experiment
2S	0.44	0.49	0.44	0.46±0.05
3S	0.32	0.33	0.32	0.33±0.05
4S	0.26	0.25	0.26	0.22±0.05

larization $\Pi(s)$. [In this paper we normalize $\Pi(s)$ as $R(s)=\text{Im}\Pi(s)$, where $R(s)$ is the ratio of the total hadronic cross section to the $\mu^+\mu^-$ cross section calculated in QED.] For large values of $|s|$ compared with Λ^2, perturbative calculation gives a reliable prediction for $\Pi(s)$ in general. This fact leads to a sum rule for $R(s)$.

In the following, we restrict ourselves to a single flavor of quark, i.e., we consider $c\bar{c}$ production, $b\bar{b}$ production, etc., separately. [Rigorously, this is legitimate up to $O(\alpha_s)$ or in the approximation we will use later.]

Consider an integral of $\Pi(s)$ over the contour C of Fig. 2. Since there are no singularities in the contour, Cauchy's theorem tells us that

$$\oint_C \Pi(s)ds = 0 . \tag{3.1}$$

We separate this integral into two parts, an integral over the large circle C' of radius $\bar{s}$, and an integral over the positive real axis. Thus, the relation

$$\int_{M_1^2}^{\bar{s}} R(s)ds = \frac{i}{2}\int_{C'} \Pi(s)ds \tag{3.2}$$

holds, where M_1 is the mass of the lowest-lying vector meson. The left-hand side of this equation implicitly includes the discrete sum over the resonant states.

If $\bar{s}$ is large enough ($\bar{s} \gg \Lambda^2$) and far from threshold, we can obtain a good approximation to the right-hand side of Eq. (3.2) from perturbation theory. We refer to the perturbative approximation of $\Pi(s)$ as $\Pi^{(\text{pert})}(s)$. A similar argument as above leads to

$$\int_{s_0}^{\bar{s}} R^{(\text{pert})}(s)ds = \frac{i}{2}\int_{C'} \Pi^{(\text{pert})}(s)ds . \tag{3.3}$$

Note that the contour should not enclose any singularity for this relation to be valid. The value of s_0 must be chosen such that this requirement is satisfied, and is different from $M_1{}^2$ in general.

Provided that the right-hand side of Eq. (3.3) is a good approximation to the right-hand side of Eq. (3.2), we can obtain a sum rule by equating the two,

$$\int_{M_1^2}^{\bar{s}} R(s)ds = \int_{s_0}^{\bar{s}} R^{(\text{pert})}(s)ds . \tag{3.4a}$$

We rewrite this equation for notational simplicity as

$$\Omega(\bar{s}) = \Omega^{(\text{pert})}(\bar{s}) , \tag{3.4b}$$

where

$$\Omega(\bar{s}) = \int_{M_1^2}^{\bar{s}} R(s)ds , \tag{3.5a}$$

and

$$\Omega^{(\text{pert})}(\bar{s}) = \int_{s_0}^{\bar{s}} R^{(\text{pert})}(s)ds . \tag{3.5b}$$

The relation Eq. (3.4a) or Eq. (3.4b) represents a kind of duality between a hadronic description (vector-meson resonances plus hadronic continuum; left-hand side) and a quark-gluonic description (perturbative QCD, right-hand side) of e^+e^- annihilation.

TABLE III. The predicted nS-$1S$ mass difference in MeV for the $t\bar{t}$ system. The mass of the $1S$ is set to 50 GeV. The flavor threshold lies between $7S$ and $9S$ depending on the potential.

	Potential (1)	(2)	(3)
2S	625	565	723
3S	927	885	1045
4S	1135	1111	1263
5S	1301	1288	1436
6S	1443	1434	1583
7S	1569	1558	1713
8S	1684	1667	1831
9S	1791	1763	1942

TABLE IV. The predicted leptonic width (for $1S$) and the ratio to that of $1S$ (for $2S$–$9S$) for the $t\bar{t}$ system. The mass of the $1S$ is set to 50 GeV.

	Potential (1)	(2)	(3)
1S	3.0 keV	3.0 keV	5.5 keV
2S	0.42	0.47	0.31
3S	0.27	0.32	0.19
4S	0.21	0.24	0.14
5S	0.175	0.195	0.115
6S	0.155	0.165	0.100
7S	0.14	0.14	0.091
8S	0.13	0.125	0.084
9S	0.12	0.115	0.079

B. Hadronic side of the sum rule

Since our aim is to check the consistency of the $Q\bar{Q}$ potentials and the finite-energy sum rule (FESR) [Eq. (3.4)], we prefer heavier quarks for which relativistic and higher-order corrections should be smaller. We decide to use mainly a hypothetical t quark of charge $\frac{2}{3}$ and mass around 25 GeV (which may be discovered in the next generation of experiments) and also the existing b quark.

By making use of the $Q\bar{Q}$ potentials discussed in Sec. II, we have solved the Schrödinger equation and obtained energy eigenvalues and wave functions of the vector mesons. (See Sec. II C.) The leptonic width $\Gamma(V\to e^+e^-)$ can be calculated from these quantities. In the narrow-width approximation, $R(s)$ can be written as

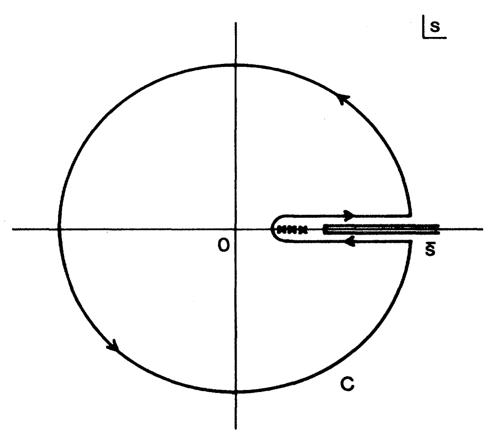

FIG. 2. The integration contour C in the complex s plane.

$$R(s)=\frac{9\pi}{\alpha^2}\sum_n M_n\Gamma_n\delta(s-M_n{}^2)\ , \quad (3.6)$$

where $M_n(\Gamma_n)$ is the mass (leptonic width) of the nth vector meson. The left-hand side of Eq. (3.4) may thus be approximated as

$$\Omega(\bar{s})=\int_{M_1{}^2}^{\bar{s}}R(s)ds=\frac{9\pi}{\alpha^2}\sum_{M_n{}^2<\bar{s}}M_n\Gamma_n\ . \quad (3.7)$$

The leptonic width is given in the lowest order by the Van Royen—Weisskopf formula,[26]

$$\Gamma_n^{(0)}=4\alpha^2 e_Q{}^2\frac{|R_n(0)|^2}{M_n{}^2}\ , \quad (3.8)$$

where e_Q is the charge of the quark in units of e, $R_n(0)$ the radial wave function at the origin (of the nth meson). The $O(\alpha_s)$ correction to this formula has been calculated as[27]

$$\Gamma_n^{(1)}=4\alpha^2 e_Q{}^2\frac{|R_n(0)|^2}{M_n{}^2}\left[1-\frac{16}{3\pi}\alpha_s\right]\ . \quad (3.9)$$

It is easy to see that this correction factor is very large for the $c\bar{c}$ system (~ 0.5) and is not small for the $b\bar{b}$ system (~ 0.3). This is one reason that we study the $t\bar{t}$ system of mass around 50 GeV. Another reason is that the nonrelativistic approximation is excellent for these heavy quarkonia.

C. Quark-gluonic side of the sum rule

The right-hand side of the sum rule Eq. (3.4) can be calculated in perturbation theory. In the zeroth order in α_s, it represents the production of a free quark pair [see Fig. 3(a)]:

$$R^{(0)}(s)=3e_Q{}^2\frac{\beta(3-\beta^2)}{2}\ . \quad (3.10a)$$

Thus,

$$\Omega^{(0)}(\bar{s})=\int_{4m^2}^{\bar{s}}R^{(0)}(s)ds=3e_Q{}^2\bar{s}\bar{\beta}^3\ , \quad (3.10b)$$

where

$$\beta=\left[1-\frac{4m^2}{s}\right]^{1/2},\quad \bar{\beta}=\left[1-\frac{4m^2}{\bar{s}}\right]^{1/2}, \quad (3.11)$$

and m is the quark mass.

Next we consider the effect of the strong interactions in the lowest order $[O(\alpha_s)]$. Relevant diagrams are shown in Fig. 3(b). In this order we may use the interpolation formula of Schwinger:[28]

$$R^{(1)}(s)=3e_Q{}^2\frac{\beta(3-\beta^2)}{2}[1+c_F\alpha_s f(\beta)]\ , \quad (3.12)$$

where

$$f(\beta)=\frac{\pi}{2\beta}-\frac{3+\beta}{4}\left[\frac{\pi}{2}-\frac{3}{4\pi}\right], \quad (3.13)$$

and $c_F=\frac{4}{3}$ is a color factor.

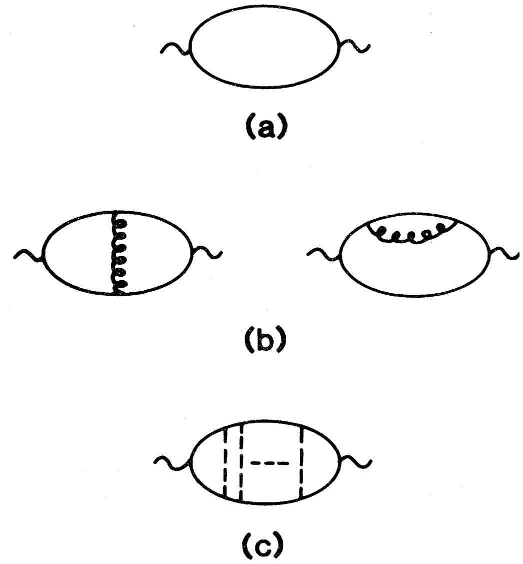

FIG. 3. Contributions to $\Pi(s)$. The solid line denotes the quark, curly line the gluon, dashed line the Coulomb part of the gluon, wavy line the photon. (a) Lowest-order diagram. (b) $O(\alpha_s)$ diagrams. (c) Diagrams which give $O((\alpha_s/\beta)^n)$ contribution.

In this equation, the quark mass m (which is implicit in β) is to be understood as the position of the pole in the quark propagator. We call it the "pole mass" hereafter. There enters another quark mass in the discussion, i.e., the quark mass M used in solving the Schrödinger equation. We call it the "constituent mass" from now on.

The order $\alpha_s{}^2$ correction to the cross section has been calculated[28a] only for massless quarks (i.e., for $\beta=1$). In the $\overline{\text{MS}}$ scheme,

$$R^{(2)}(s)\,|_{m=0}=3e_Q{}^2\left[1+\frac{\alpha_s}{\pi}+(1.99-0.12N_f)\left[\frac{\alpha_s}{\pi}\right]^2\right]. \quad (3.14)$$

Typically this $O(\alpha_s{}^2)$ term contributes only less than 1% to R and we might be tempted to conclude that the $O(\alpha_s{}^2)$ correction can be neglected in the general case. In fact, the form of $R^{(0)}(s)$ or $R^{(1)}(s)$ has been used in most of the previous studies of the $c\bar{c}$ and $b\bar{b}$ systems. Nevertheless, it turns out that the $O(\alpha_s{}^2)$ term cannot be neglected for small β.

Inspection of Eqs. (3.11) and (3.12) indicates that at small β with $\beta \lesssim \alpha_s$, the correction term of order α_s dominates the zeroth-order term. This is due to the $1/\beta$ singularity in $f(\beta)$. Generally, in the nth order in α_s, the strongest singularity comes from the ladder diagram of the Coulomb type, Fig. 3(c), and is proportional to $(\alpha_s/\beta)^n$. These terms in fact have a dominant contribution to $R^{(\text{pert})}$ at small β. Summing up these leading $(\alpha_s/\beta)^n$ terms, we are led to[7,8]

$$R^{(c)}(s)=3e_Q{}^2\left\{\frac{\beta(3-\beta^2)}{2}\left[\frac{\pi c_F\alpha_s/\beta}{1-\exp(-\pi c_F\alpha_s/\beta)}-\frac{3+\beta}{4}\left[\frac{\pi}{2}-\frac{3}{4\pi}\right]c_F\alpha_s\right]+3\pi\zeta(3)(c_F\alpha_s)^3 m^2\delta(s-4m^2)\right\}. \quad (3.15)$$

In this expression, all $O(\alpha_s)$ corrections and all $O((\alpha_s/\beta)^n)$ corrections are both included. The last term of this equation is the contribution of the artificial Coulomb poles which have no direct correspondence to the actual resonance. These poles should be considered as technical objects. For a heavy quarkonium such as the $t\bar{t}$ system we consider, it is easy to see that β is very small in the resonance region and the Coulomb-type correction turns out to be very important.

Finally, we should make a comment concerning the strong coupling constant α_s. Which α_s should be used in Eqs. (3.12) and (3.15) is a difficult question. In the following, we decided to use $\alpha_s(4m^2)$ in the $\overline{\text{MS}}$ scheme[29] with $N_f=4$. Rigorously speaking, we should have used a coupling including the effect of the quark mass. The most important reason for our choice is our inability to include the quark-mass effect in the short-distance behavior of the $Q\bar{Q}$ potential. Consistency requires that the same approximation should be used in the perturbative approach as the potential approach. We believe that this approximation causes little error on the results.

IV. TESTING THE SUM RULE

In the previous section we have derived the sum rule, Eq. (3.4). Now we are in a position to check the sum rule explicitly.

A. The $t\bar{t}$ system with the potential (1)

We begin with the $t\bar{t}$ system of mass around 50 GeV using the QCD + Richardson potential (1). The value of the constituent mass M is 25.47 GeV in this case. We take[30] the pole mass m as half of the mass of the 1S vector meson, i.e., 25 GeV.

Several approximations to the left- and right-hand sides of Eq. (3.4) are plotted in Fig. 4. For the left-hand side, $\Omega(\bar{s})$ of Eq. (3.7), both with and without $O(\alpha_s)$ correction to the leptonic width [Eqs. (3.9) and (3.8)], are shown. For the right-hand side, the three functions

$$\Omega^{(\text{pert})}(\bar{s}) \sim \begin{cases} \Omega^{(0)}(\bar{s}) \\ \Omega^{(1)}(\bar{s}) \\ \Omega^{(c)}(\bar{s}) \end{cases}$$

are plotted. These functions are obtained by integrating $R^{(0)}(s)$, $R^{(1)}(s)$, and $R^{(c)}(s)$ in Eqs. (3.10a), (3.12), and (3.15). The left-hand side of Eq. (3.4) with $O(\alpha_s)$ correction to the width formula is in good agreement with the right-hand side with Coulomb-type correction, $\Omega^{(c)}(\bar{s})$. Other curves do not agree with each other. Thus, we may conclude that the QCD correction is essential in order that the sum rule holds.[31] The Coulomb-type correction should not be neglected, and the $O(\alpha_s)$ correction to the leptonic width is also important to obtain the agreement. Another conclusion is that the difference between the pole mass and the constituent mass is about 500 MeV.

B. The $t\bar{t}$ system: potential dependence

From now on we use the leptonic-width formula with the $O(\alpha_s)$ corrections and $\Omega^{(c)}(\bar{s})$ only.

Dependence on the potential at intermediate distances

In Figs. 5, we show the two sides of the sum rule obtained from potentials (1) and (2). Both of these potentials show the QCD behavior with $\Lambda_{\overline{\text{MS}}}=200$ MeV at short distances. Thus, we can check the uncertainty arising from

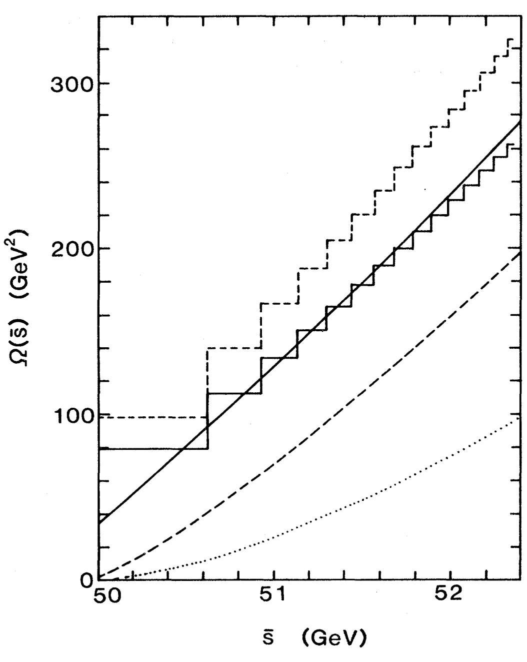

FIG. 4. $\Omega(\bar{s})$ and $\Omega^{(\text{pert})}(\bar{s})$ for the $t\bar{t}$ system using the potential (1). Solid (dashed) step-line is $\Omega(\bar{s})$ with (without) the $O(\alpha_s)$ correction to the width formula. Solid (dashed, dotted) curve is $\Omega^{(c)}(\bar{s})$ ($\Omega^{(1)}(\bar{s})$,$\Omega^{(0)}(\bar{s})$).

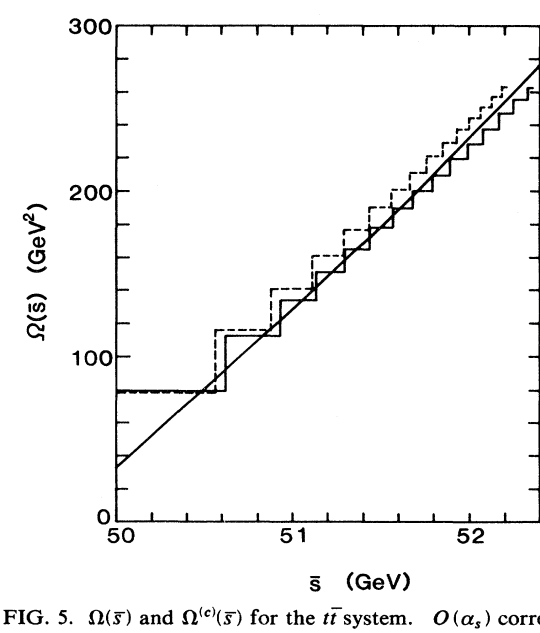

FIG. 5. $\Omega(\bar{s})$ and $\Omega^{(c)}(\bar{s})$ for the $t\bar{t}$ system. $O(\alpha_s)$ correction to the width formula is included. Solid [dashed] step-line is for the potential (1) [(2)]. Solid curve is $\Omega^{(c)}(\bar{s})$ and common to the two potentials.

the form of the potential at intermediate distances. Results from the two potentials agree with each other fairly well. In the case of potential (2), we can obtain better agreement by reducing m by 100 MeV. Thus, we can conclude that changing the intermediate potential does not cause large uncertainty in the results.

$\Lambda_{\overline{MS}}$ *dependence*

Curves for $\Omega(\bar{s})$ derived from the potentials (1) and (3), as well as $\Omega^{(c)}(\bar{s})$ for $\Lambda_{\overline{MS}}=200$ MeV and 500 MeV are shown in Fig. 6. Since these two potentials almost agree at intermediate distances, we can extract the $\Lambda_{\overline{MS}}$ dependence from this comparison. The agreement for $\Lambda_{\overline{MS}}=500$ MeV is not as good as that for $\Lambda_{\overline{MS}}=200$ MeV. We should use $m \sim M_1/2-150$ MeV in this case. This corresponds to $M-m\sim 700$ MeV, which is larger than 500 MeV from $\Lambda_{\overline{MS}}=200$ MeV. Thus, the larger value of $\Lambda_{\overline{MS}}$ leads to a larger difference between the pole mass and the constituent mass.

C. The $b\bar{b}$ system

Curves for $\Omega(\bar{s})$ from the three potentials and corresponding $\Omega^{(c)}(\bar{s})$ are plotted in Figs. 7 and 8. Again the pole mass is taken as $m=M_1/2$. Better agreement is obtained for $\Lambda_{\overline{MS}}=200$ MeV if we decrease m by ~ 100 MeV. Since the difference between M_1 and $2M$ is smaller (binding energy is smaller) for the $b\bar{b}$ system, the difference $M-m$ is in the range of 150–250 MeV. Although this difference is smaller than that for the $t\bar{t}$ system, the ratio $r=(M-m)/M$ is larger:

$$r=\begin{cases}0.03\text{–}0.05 & (b\bar{b}\text{ system}),\\ 0.01\text{–}0.015 & (t\bar{t}\text{ system}).\end{cases}$$

This is consistent with the naive expectation that various corrections are smaller for heavier quarks.

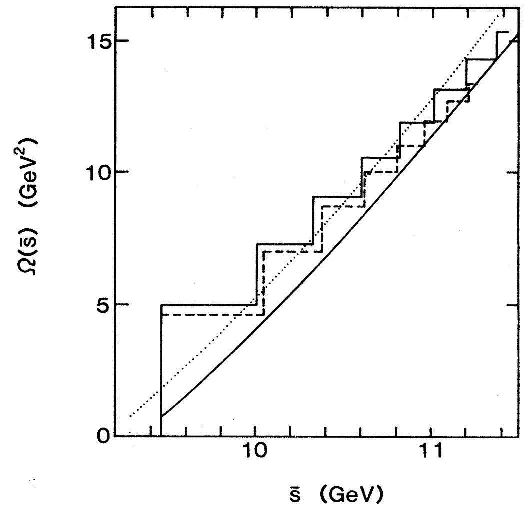

FIG. 7. $\Omega(\bar{s})$ and $\Omega^{(c)}(\bar{s})$ for the $b\bar{b}$ system. Notations are the same as in Fig. 5. Dotted curve shows $\Omega^{(c)}(\bar{s})$ with adjusted pole mass m for potential (1).

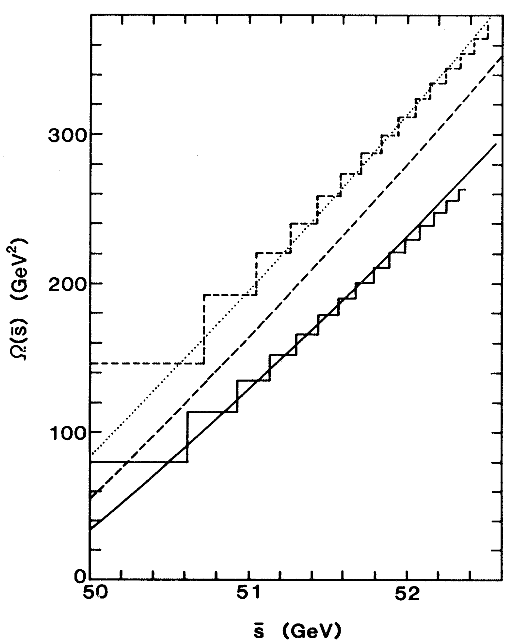

FIG. 6. $\Omega(\bar{s})$ and $\Omega^{(c)}(\bar{s})$ for the $t\bar{t}$ system. $O(\alpha_s)$ correction to the width formula is included. Solid [dashed] lines are for the potential (1) [(3)]. Dotted curve gives $\Omega^{(c)}(\bar{s})$ with adjusted pole mass m for $\Lambda_{\overline{MS}}=500$ MeV.

V. CONCLUDING REMARKS

We can draw the following conclusions about the $b\bar{b}$ and $t\bar{t}$ systems from studies of the Q^2 duality.

(i) If we use the leptonic widths without $O(\alpha_s)$ corrections [Eq. (3.8)] on the left-hand side and $\Omega^{(0)}(\bar{s})$ without $O(\alpha_s)$ corrections [Eq. (3.10b)] on the right-hand side of

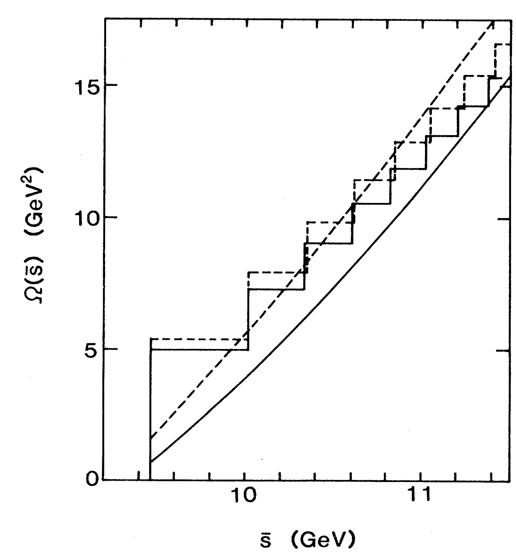

FIG. 8. $\Omega(\bar{s})$ and $\Omega^{(c)}(\bar{s})$ for the $b\bar{b}$ system. Notations are the same as in Fig. 6.

FESR, Q^2 duality in the form of FESR cannot be satisfied.

(ii) If, however, we include corrections to the leptonic widths and Coulomb-type corrections to the perturbative calculation, we can obtain a reasonable agreement by taking the quark pole mass around a half of the 1S vector mesons.

(iii) We would also like to point out that the above results are fairly insensitive to the choice of the potential so far as the potential reproduces the correct mass spectra and leptonic widths.

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APPENDIX

In this appendix, we show the detailed form of the potentials (1) and (2).

Potential (1): QCD+Richardson potential. This potential is constructed by connecting the Richardson potential[16] $V_R(r)$ and the two-loop QCD potential $V_{\rm QCD}(r,\Lambda_{\overline{\rm MS}}=200\ {\rm MeV})$, Eq. (2.1), with logarithmic interpolation:

$$V(r)=\begin{cases} V_R(r), & r>r_2\,, \\ 0.7824\ln r-0.5717, & r_1<r<r_2\,, \\ V_{\rm QCD}(r,\Lambda_{\overline{\rm MS}}=200\ {\rm MeV}), & r<r_1\,, \end{cases} \tag{A1}$$

where r is in GeV^{-1}, $r_1=0.1428$ GeV^{-1}, $r_2=0.5$ GeV^{-1}.

Potential (2): QCD+Martin potential. This potential is obtained by connecting the Martin potential[20] and the two-loop QCD potential. A constant is added to the Martin potential in order to connect the above two smoothly:

$$V(r)=\begin{cases} -7.392+6.870r^{0.1}, & r>r_0\,, \\ V_{\rm QCD}(r,\Lambda_{\overline{\rm MS}}=200\ {\rm MeV}), & r<r_0\,, \end{cases} \tag{A2}$$

where $r_0=0.3665$ GeV^{-1} and r is in GeV^{-1}.

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[30] If we set $m=M$, the sum rule cannot be satisfied. Our choice ($m=M_1/2$) is somewhat arbitrary and may be modified in order to improve the agreement in later cases.

[31] If we multiply the width formula (3.9) by another factor $M_n{}^2/4m^2$ as suggested in Ref. 4, the agreement in slope is improved.