Pion decay constant and effective quark mass from relativistic $q\bar{q}$ wave function

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Expressions for the pion decay constant and $\pi^0\to 2\gamma$ decay amplitude are given in terms of the effective quark mass m_q assuming a $q\bar{q}$ bound-state structure for massless pion. The pion decay constant is found to be given by $f_\pi=(3/2\pi)m_q$, which gives, for $m_q=330$ MeV, $f_\pi=160$ MeV, in good agreement with the measured value of 135 MeV.

The success of the Goldberger-Treiman relation for G_A/G_V in neutron β decay and the smallness of the pion mass compared to the masses of other pseudoscalar mesons and vector mesons shows that $SU(2)\times SU(2)$ is a good symmetry for strong interactions and is spontaneously broken with the appearance of the pion as a Goldstone boson[1]. On the other hand, deep-inelastic lepton-hadron and high-mass μ-pair production data[2] indicate that the pion and other low-lying hadrons can be considered as bound states of quark and antiquark ($q\bar{q}$) moving in a QCD potential with a momentum distribution peaked at a mean transverse momentum $\langle P_T\rangle$ of the order of 300 MeV (the so-called intrinsic quark transverse momentum). Thus the pion seems to behave like a Goldstone boson and at the same time can be described as a $q\bar{q}$ bound state (with some amount of gluons and sea quarks in the wave function). While the motions of quarks in baryons and vector mesons are nonrelativistic to a good approximation as suggested by the success of the SU(6) nonrelativistic quark model in describing low-energy properties of hadrons,[3] this is no longer true for quarks in a massless pion since they are far off the mass shell (constituent quark mass) so that the $q\bar{q}$ component of the pion wave function must be treated as a relativistic Bethe-Salpeter (BS) amplitude. Given this BS wave function for the $q\bar{q}$ component, the pion decay constant f_π, as well as other low-energy properties of the pion, such as quark momentum distribution and $\pi^0\to 2\gamma$ decay rate, can be calculated in terms of a few parameters of the $q\bar{q}$ wave function. In general f_π is a function of m_q (the constituent quark mass) and $\langle P_T\rangle$, and may be written in the limit of massless pion ($p^2\to 0$) as

$$f_\pi=f_\pi(p^2)_{p^2\to 0}=m_q f\left[\frac{\langle P_T\rangle}{m_q}\right].$$

Since m_q and $\langle P_T\rangle$ are independent of the current quark mass, $f_\pi(p^2)$ tends to a finite nonvanishing limit in the limit of the massless pion (vanishing current quark mass); we see immediately that the spontaneous breakdown of chiral symmetry can be realized if a dynamical generation of quark mass and intrinsic quark transverse momentum can be found from nonperturbative effects arising from the motion of quarks in a QCD potential. In this paper we shall give a calculation of f_π assuming that quarks bound in such a potential acquire an effective mass about $\frac{1}{3}$ the nucleon mass, and using a simple spectral representation for the pion $q\bar{q}$ BS amplitude which possesses the scaling properties in Bjorken's limit and the smooth transition to the low-momentum region.[4] By using the PCAC (partial conservation of axial-vector current) expression for the $\pi^0\to 2\gamma$ decay amplitude[5] given by the Adler-Bell-Jackiw anomaly, it is shown that f_π is given by the following relation:

$$f_\pi=\frac{3m_q}{2\pi} \tag{1}$$

which gives $f_\pi=160$ MeV for $m_q=330$ MeV—which is quite close to the measured value of 135 MeV. We now proceed to the derivation of Eq. (1).

The pion decay constant f_π which is measured in $\pi^+\to\mu^++\nu_\mu$ ($\pi^+\to e^++\nu_e$) decays is usually defined as

$$\langle 0\,|\,A_{1-i2}{}_\mu(0)\,|\,\pi^+\rangle=if_\pi p_\mu\ , \tag{2}$$

where $A_{i\mu}(x)$ ($i=1,2,3$) is the isovector axial-vector current, written in terms of the quark field operators as

$$A_{i\mu}=\bar{q}\gamma_\mu\gamma_5\frac{\lambda_i}{2}q\ ,\quad i=1,2,3\ . \tag{3}$$

The matrix element defined by Eq. (2) appears in the Wilson short-distance expansions of the $q\bar{q}$

meson wave function

$$\langle 0 \,|\, T(\psi(x)\bar{\psi}(0)) \,|\, \pi \rangle$$

and the matrix element

$$\langle 0 \,|\, T(J_\mu(x)J_\nu(0)) \,|\, \pi^0 \rangle$$

with $J_\mu(x)$ the electromagnetic current. We have[6] for $x^2 \to 0$, $x \to 0$:

$$\langle 0 \,|\, T(J_\mu(x)J_\nu(0)) \,|\, \pi^0 \rangle \sim C(x^2) i\epsilon_{\mu\nu\rho\sigma}\partial_\rho D(x) \langle 0 \,|\, A_{3\sigma}(0) \,|\, \pi^0 \rangle \,, \tag{4}$$

$$\langle 0 \,|\, T(\psi(x)\bar{\psi}(0)) \,|\, \pi \rangle \sim \widetilde{C}(x^2)\gamma_\mu\gamma_5 \langle 0 \,|\, :\bar{\psi}(0)\gamma_\mu\gamma_5\psi(0): \,|\, \pi \rangle + \text{other terms} \,. \tag{5}$$

A calculation of $\langle 0 \,|\, A_{i\mu}(0) \,|\, \pi \rangle$ is thus equivalent to making a short-distance expansion of the matrix elements (4) or (5). The electromagnetic currents are defined as

$$J_\mu(x) = \bar{q}(x)\gamma_\mu Q q(x)$$

with Q the charge operator acting on the quark field in SU(3) space. Other terms not relevant to f_π have been omitted in (4) and (5). In the following we shall work with the pion BS wave function given by (5). The Wilson coefficients $C(x^2)$ and $\widetilde{C}(x^2)$ are given by the canonical dimension of $J_\mu(x)$ and $\psi(x)$ apart from logarithmic scaling violation effects due to QCD interactions at short distances. Since the matrix element $\langle 0 \,|\, A_{i\mu}(0) \,|\, \pi \rangle$ depends only on the long-distance part of the BS wave function, one can subtract from the true wave-function scaling violation effects without modifying the matrix element. Thus we can assume a canonical behavior for the wave function as $x^2 \to 0$ and set $\widetilde{C}(x^2) = 1$. In momentum space we have, from (5),

$$\langle 0 \,|\, A_\mu(0) \,|\, \pi \rangle = \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}[\gamma_\mu\gamma_5\psi_p(q, q-p)] \tag{6}$$

which is the usual expression for the pion decay constant obtained from a quark-loop Feynmann diagram provided that the integral over quark momentum converges. $\psi_p(q_1,q_2)$ $(q_1 = q,\ q_2 = q - p)$ is the pion $q\bar{q}$ BS wave function in momentum space defined as

$$\psi_p(q_1,q_2) = \int d^4x \exp(iq\cdot x) \times \langle 0 \,|\, T(\psi(x)\bar{\psi}(0)) \,|\, \pi \rangle \,. \tag{7}$$

For a $q\bar{q}$ bound state of constituent quark with mass $m_q = m$, $\psi_p(q_1,q_2)$ satisfies the BS equation

$$\Gamma_p(q_1,q_2) = (\not{q}_1 - m)\psi_p(q_1,q_2)(\not{q}_2 - m) = \int \frac{d^4q_1'}{(2\pi)^4} K(q_1,q_2;q_1',q_2')\psi_p(q_1',q_2') \,. \tag{8}$$

$\Gamma_p(q_1,q_2)$ is the BS vertex function (quark propagators removed) and $K(q_1,q_2;q_1',{}_2')$ is the two-particle-irreducible BS kernel. In the scaling limit ($q^2 \,| \to \infty$, $2p\cdot q \to \infty$, $q^2/2p\cdot q$ fixed) if $K(q_1,q_2;q_1',q_2')$ is given by the one-gluon-exchange kernel, then $\Gamma_p(q_1,q_2)$ will have a canonical behavior apart from scaling-violation logarithmic factor $(\ln q^2/\Lambda^2)^{-\gamma}$ (similar to those in deep-inelastic lepton-hadron scattering). Ignoring this factor we have

$$\Gamma_p(q,q-p) \sim \left[\frac{1}{q^2}\right] F\left[\frac{q^2}{2p\cdot q}\right] \quad (\,|q^2| \to \infty,\ 2p\cdot q \to \infty,\ q^2/2p\cdot q \text{ fixed}) \,. \tag{9}$$

The behavior of $\Gamma_p(q,q-p)$ given by (9) can also be obtained by using conformal invariance.[7] To extrapolate the wave function to the low-momentum region, we shall assume a simple spectral representation for $\Gamma_p(q,q-p)$ of the form

$$\Gamma_p(q,q-p) = a(q^2,p\cdot q)\gamma_5 \,, \quad a(q^2,p\cdot q) = Nm^2 \int_0^1 \frac{d\beta\, g(\beta)}{\beta(q^2-m^2) + (1-\beta)[(q-p)^2 - m^2]} \tag{10}$$

with $g(\beta)$ normalized such that

$$\int_0^1 d\beta\, g(\beta) = 1 \,.$$

N is a normalization constant and m is the effective quark mass which gives rise to the quark intrinsic $\langle P_T \rangle$ in hadrons. Since the $\pi^0 \to 2\gamma$ decay amplitude should not depend on other $q\bar{q}$ glue components in the meson as a consequence of the PCAC anomaly[8] (in the limit $p^2 \to 0$), the simplified form for Γ_p in (10) enables us to compute the low-energy properties of the pion, such as f_π and the $\pi^0 \to 2\gamma$ amplitude, in terms of a minimum number of parameters. Note that the absence of $\not{p}$, $\not{q}$, and $\not{p}\not{q}$ terms in Γ_p as given by Eq. (10) may be justified to some extent with the help of the Ward identity as shown[9] by Pagels and Stokar. Using (6) and (10), a straightfor-

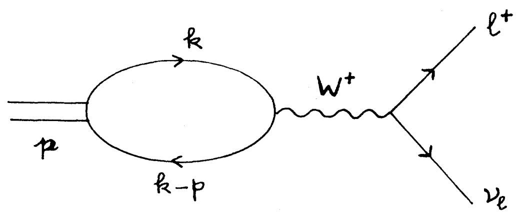

FIG. 1. Contribution to the $\pi^+ \to l^+ + \nu_l$ decay amplitude of the pion $q\bar{q}$ wave function.

ward calculation gives (Fig. 1)

$$f_\pi = \frac{N\sqrt{3m}}{8\pi^2} \tag{11}$$

in the massless-pion limit.

For the $\pi^0 \to 2\gamma$ decay, let us define the decay amplitude in the usual way as

$$\mathcal{M}(\pi^0 \to 2\gamma) = i\left[\frac{e^2}{4\pi^2}\right]\epsilon_{\mu\nu\rho\sigma}\epsilon_\mu k_\nu \epsilon'_\rho k'_\sigma F(p^2,0,0) \tag{12}$$

with (ϵ,k) and (ϵ',k') the polarization vectors and momenta of the two photons. A similar quark-loop calculation (Fig. 2) gives, in the limit of massless pion,

$$F(0,0,0) = \frac{N}{3\sqrt{6}}\left[\frac{1}{m}\right]. \tag{13}$$

On the other hand, from the PCAC anomaly we have

$$F(0,0,0) = -\frac{2S\sqrt{2}}{f_\pi} \tag{14}$$

which reproduces the $\pi^0 \to 2\gamma$ decay rate in a theory with quarks having three colors as extra degrees of freedom and $S = \frac{1}{2}$. Since the ratio $f_\pi / F(0,0,0)$ is rather insensitive to the detailed form of the $q\bar{q}$ wave function, it is convenient to divide both the left-hand side and the right-hand side of (11) by $F(0,0,0)$ given in (14) and (13) to obtain

$$f_\pi{}^2 = (2S)\frac{9m^2}{4\pi^2} \tag{15}$$

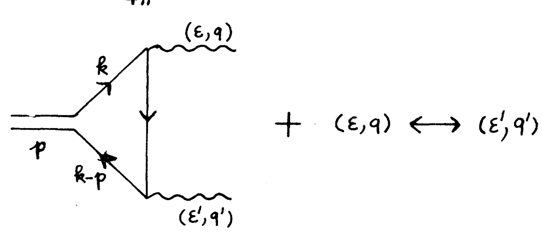

FIG. 2. The $\pi^0 \to 2\gamma$ decay amplitude in terms of the $q\bar{q}$ wave function.

which is the result for f_π given in Eq. (1) with $S = \frac{1}{2}$ and $m = m_q$ (the constituent quark mass).

In obtaining (15) we have used the PCAC-anomaly expression for the $\pi^0 \to 2\gamma$ amplitude to eliminate N so that f_π is given in terms of only the effective quark mass—the unknown parameter. This is a matter of convenience and the PCAC expression for $\pi^0 \to 2\gamma$ is not essential in the derivation of (15). One would get the same result by simply using the measured value for the $\pi^0 \to 2\gamma$ amplitude. The good agreement with experiment for f_π indicates that the effective mass for a quark bound in a pion is the same as the constituent quark mass m_q ($m_q \approx \frac{1}{3} m_p$) in the nonrelativistic quark model. This is consistent with the fact that the mean quark transverse momentum $\langle P_T \rangle$ which is given by the effective quark mass is also of the order 300—500 MeV and is the same for mesons, and nucleons, as measured in μ-pair production experiments.

From (13) and (11) we find for $F(0,0,0)$:

$$F(0,0,0) = -\frac{N^2\sqrt{2}}{48\pi^2 f_\pi} \tag{16}$$

which satisfies the PCAC-anomaly condition for $N^2 = 48\pi^2$. A derivation of the PCAC expression from (16) thus requires a determination of N from some independent sources. Normalization of the $q\bar{q}$ wave function from the pion charge form factor $F_\pi(q^2)$ at $q^2 = 0$ cannot be used to obtain N since $F_\pi(0)$ is sensitive to the form of the $q\bar{q}$ wave function. In fact one can always add a scalar piece $\varphi(q^2, p\cdot q)\gamma_5$ to the wave function $\psi_0(q, q-p)$ given by Eqs. (8) and (10) without changing the expressions for f_π and $F(0,0,0)$ given above since the traces involving $\varphi(q^2, p\cdot q)\gamma_5$ vanish in the expressions for f_π and $F(0,0,0)$. This added piece, however, does contribute to the pion charge normalization which may also receive contributions from possible $q\bar{q}$ glue components in meson. Note that the above expressions for f_π and $\pi^0 \to 2\gamma$ amplitudes are only valid in the chiral-symmetry limit (i.e., $p^2 \to 0$ or $p^2 \ll m^2$). In this limit, one can set $p\cdot q = 0$ in all expressions involving the $q\bar{q}$ wave function and the quark propagator. The expressions thus obtained are rather insensitive to the form of the propagator one uses. In fact, in this limit, for a reasonable behavior of the quark propagator, the integration over virtual quark momenta is limited to the spacelike region ($q^2 < 0$) with $|q^2|$ peaked at some mean value

$$|\langle q^2 \rangle| \sim O(m^2).$$

In the spacelike region no discontinuity is present so that no large error can be expected by using the free-quark propagator,

$$S_F(q)=\frac{\not{q}+m}{q^2-m^2}\ ,$$

with the effective quark mass m.

Finally, it should be stressed that our approach is purely phenomenological and is the usual relativistic calculation of f_π using a simple parametrization for the BS $q\bar{q}$ amplitude. This is in contrast with theoretical calculations by Pagels and Stokar and by Cornwall who obtain a closed expression for f_π using only the Ward identity for the off-shell $q\bar{q}$ amplitude.[9,10] The qualitative agreement between the two calculations for f_π indicates that quarks bound in hadrons possess an effective mass roughly equal to the constituent quark mass.

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