Bag-model calculation of leptonic widths

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We calculate leptonic widths of the light vector mesons in a modified MIT bag model, under the assumption that the total quark-antiquark momentum is identically zero in the center-of-mass system of the meson. All free parameters are determined from spectroscopy. One finds agreement with experiment to within 25%.

I. INTRODUCTION

An experimental result that has puzzled many physicists for some time is that the ratio between the leptonic width of ground-state 1^{--} mesons and the squared mean electric charge of the constituent quarks is approximately flavor independent[1]:

$$\frac{{}_F\Gamma_{e^+e^-}}{{}_F\langle e_q\rangle^2}=12.0\pm1.5 \text{ keV} ,$$

where $F=\rho,\omega,\phi,J/\psi,\Upsilon$. To the best of our knowledge, there is no simple explanation for this fact. One is then tempted to check whether the above result can be obtained from an explicit calculation of $\Gamma_{e^+e^-}$ in a reasonable model of hadrons.

It is widely believed that quantum chromodynamics (QCD) is the underlying theory of strong interactions and hadronic structure. Unfortunately, it has not been possible to derive from the fundamental theory a detailed description of the structure of hadrons, which could enable us to calculate the leptonic widths of 1^{--} mesons in a straightforward way.

Shifman *et al.*[2] have successfully estimated the leptonic widths of both light and heavy mesons through the use of QCD sum rules (or dispersion relations). But to our understanding, these calculations rely upon some assumptions about the contribution of the continuum of states to the sum rules as well as upon assumptions about nonperturbative corrections both of which are not easy to check.

Several authors[3] have calculated the leptonic decay widths of the heavy mesons (J/ψ and Υ) and their excited states, in the context of nonrelativistic potential models for the quark-antiquark system. These calculations are completely analogous to the standard calculation for the decay width of positronium.[4] By adjusting the parameters of the potential to fit the spectroscopy, good results for the leptonic widths can be obtained. An implicit assumption in these calculations is that the relative motion of the quark and antiquark is negligible and their mass is one half of the mass of the meson. Although this might be a reasonable approximation for the heavy-meson systems, it certainly is not for the light mesons (ρ,ω,ϕ). For these mesons, a relativistic model and calculation scheme is clearly needed.

The MIT bag model[5–8] is probably the simplest relativistic model of hadrons that has the basic features that we expect from QCD. Furthermore, it has proven to be a successful model to describe the static properties of light hadrons in their ground state. For these reasons, we choose here to do our calculation of leptonic widths of light 1^{--} mesons in the context of the MIT bag model. Several calculations of leptonic widths have been attempted within this model,[9–11] with limited quantitative success. The main difference in our approach is discussed below.

We consider only the one-photon contribution to the decay process. An essential element of the calculation is the quark-antiquark momentum distribution, which we obtain from the bag-quark wave function and the crucial assumption that the total momentum of the quark-antiquark system is identically zero in the center of mass of the hadron. It is this assumption that distinguishes our calculation from previous ones,[9–11] where the motions of the quark and antiquark inside the bag are assumed to be incoherent. In our view, the picture of a strong correlation between the momenta of the quark and the antiquark in the bag is physically reasonable because the bag is after all an artifice to emulate the confinement dynamics and we should not view it literally as a rigid cavity independent from the quarks. We think of the bag as "following" the quark-antiquark system in such a way that the total quark-antiquark momentum is approximately conserved. If this is a good approximation, then the bag center-of-mass motion should not play an important role in the dynamics of the system. Because we do not know of a better alternative, we still use the static (spherical) cavity approximation to calculate the quark wave functions and the bag parameters. These parameters are all determined from hadron spectroscopy and there is no freedom left in the calculation.

The values obtained for the leptonic widths turn out to be between 10% and 25% high with respect to the average experimental values.

In Sec. II we describe the calculation of the leptonic widths and the results obtained. Our bag-model conventions are defined in the Appendix.

II. CALCULATION OF THE LEPTONIC DECAY WIDTHS

A. General procedure

We assume that the main contribution to the decay process comes from the one-(virtual)-photon annihilation of a valence quark-antiquark pair inside the meson, as shown in Fig. 1. In order to evaluate the amplitude for the process illustrated in Fig. 1, we use a similar approach to the one used by Jauch and Rohrlich[4] for positronium decay, but without taking the nonrelativistic limit.

The spin-dependent amplitude for the process of Fig. 1 in the rest frame of the decaying meson is then given by (Feynman gauge)

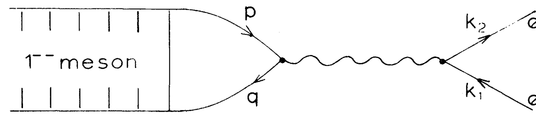

FIG. 1. One-photon contribution to the decay process.

$$-iM_{\lambda_1\lambda_2\delta_1\delta_2}(k_1,k_2)=\sqrt{3}\int\frac{d^3\vec{p}\,d^3\vec{q}\,\delta^{(3)}(\vec{p}+\vec{q})}{(2\pi)^3(16E_pE_qE_{k_1}E_{k_2})^{1/2}}\varphi_{\lambda_2\lambda_1}(\vec{p},\vec{q})\left[\bar{v}_{\lambda_1}(q)\gamma^{\mu}u_{\lambda_2}(p)\frac{-ie_Q(ie)^2}{(k_1+k_2)^2}\bar{v}_{\delta_1}(k_1)\gamma_{\mu}u_{\delta_2}(k_2)\right], \tag{1}$$

where (i) the factor $\sqrt{3}$ expresses the fact that the quark-antiquark pair in the meson is in the color-singlet state, (ii) the spin indices λ_i,δ_i are not to be summed over, (iii) the Dirac spinors are normalized according to

$$v^{\dagger}_{\lambda_1}(p)v_{\lambda_2}(p)=u^{\dagger}_{\lambda_1}(p)u_{\lambda_2}(p)=2E_p\delta_{\lambda_1\lambda_2},$$

(iv) $\phi_{\lambda_2\lambda_1}(\vec{p},\vec{q})$ is the amplitude to find the quark in a state of pure momentum $\vec{p}$, having spin projection λ_2 on the z axis and simultaneously to find the antiquark in a state of pure momentum $\vec{q}$ and spin projection λ_1 inside the bag, and (v) e_Q is the electric charge of the valence quarks in units of the charge of a positron (the values for the relevant mesons are given in Table I). The leptonic decay width of the meson is then simply given by

$$\Gamma_{e^+e^-}=\frac{1}{(2\pi)^2}\int d\vec{k}_1 d\vec{k}_2\,|M_{\rm if}|^2\delta^{(4)}(k_1+k_2-\hat{0}M), \tag{2}$$

where

$$|M_{\rm if}|^2\equiv\sum_{\lambda_1,\lambda_2,\delta_1,\delta_2=1}^{2}|M_{\lambda_1\lambda_2\delta_1\delta_2}(k_1,k_2)|^2, \tag{3}$$

M is the meson mass and $\hat{0}=(1,0,0,0)$. A potentially worrisome aspect of this approach is that energy is not conserved at the quark-photon vertex of Fig. 1. This is because in a bound state the total energy is not equal to the sum of the kinetic energies of the single constituents. We have to assume that when the quark-antiquark annihilation occurs (and the bag disappears), the extra energy $\Delta E=M-2(\vec{p}^{\,2}+m^2)^{1/2}$ is somehow absorbed by the photon.

B. Calculation of the amplitude $\phi_{\lambda_2\lambda_1}(\vec{p},-\vec{p})$

Let $\tilde{g}_{\lambda;\lambda'}(\vec{p})$ be the amplitude that a bag quark of wave function $q^{\alpha}_{\lambda}(\vec{r})$ (spin projection λ) be in a state of definite momentum $\vec{p}$ and spin projection λ'. Then

$$q^{\alpha}_{\lambda}(\vec{r})=\int\frac{d\vec{p}}{(2\pi)^3}\tilde{g}_{\lambda;\lambda'}(\vec{p})\frac{u^{\alpha}_{\lambda'}(\vec{p})}{\sqrt{2E_p}}e^{i\vec{p}\cdot\vec{r}}, \tag{4}$$

where $u(\vec{p})$ is the usual free Dirac spinor with normalization

$$\sum_{\alpha=1}^{4}u^{\alpha\dagger}_{\rho}(\vec{p})u^{\alpha}_{\sigma}(\vec{p})=2E_p\delta_{\rho\sigma}\equiv 2(\vec{p}^{\,2}+m^2)^{1/2}\delta_{\rho\sigma}.$$

This relation can be easily inverted to find

$$\tilde{g}_{\lambda;\lambda'}(\vec{p})=\frac{1}{\sqrt{2E_p}}\sum_{\alpha=1}^{4}u^{\alpha\dagger}_{\lambda'}(\vec{p})\int_{\rm bag}d\vec{r}\,q^{\alpha}_{\lambda}(\vec{r})e^{-i\vec{p}\cdot\vec{r}}. \tag{5}$$

Using the bag wave function (A4) we get

$$\int_{\rm bag}d\vec{r}\,q_{\lambda}(\vec{r})e^{-i\vec{k}\cdot\vec{r}}=\frac{2\pi iN(x)R^2}{\sqrt{4\pi}kx}\begin{pmatrix}\left[\dfrac{\omega+m}{\omega}\right]^{1/2}[j_0(kR-x)-j_0(kR+x)]U_{\lambda}\\ \left[\dfrac{\omega-m}{\omega}\right]^{1/2}[j_0(kR+x)+j_0(kR-x)-2j_0(kR)j_0(x)]\vec{\sigma}\cdot\hat{k}U_{\lambda}\end{pmatrix}, \tag{6}$$

where $U_{\uparrow}=\binom{1}{0}$ and $U_{\downarrow}=\binom{0}{1}$. Substituting this into (5) we find

$$\begin{aligned}\tilde{g}_{\lambda;\lambda'}(\vec{p})&=2\pi[q^{\dagger}_{\lambda}(0)q_{\lambda}(0)]^{1/2}\frac{R^2}{px}\left[\frac{E_p+m}{2E_p}\right]^{1/2}\\&\quad\times\left[[j_0(pR-x)-j_0(pR+x)]+\left[\frac{\omega-m}{\omega+m}\right]^{1/2}\frac{p}{E_p+m}[j_0(pR+x)+j_0(pR-x)-2j_0(pR)j_0(x)]\right]\delta_{\lambda\lambda'}\\&\equiv\tilde{g}(\vec{p})\delta_{\lambda\lambda'}.\end{aligned} \tag{7}$$

Here $q_{\lambda}(0)$ is the quark wave function (A4) at the origin. Because we assume that the total momentum $\vec{P}$ of the quark-antiquark system vanishes identically, the amplitude to find a quark of momentum $\vec{p}$ must be equal to the amplitude to find a quark of momentum $\vec{p}$ and an antiquark of momentum $-\vec{p}$ [i.e., $\tilde{g}(\vec{p})=\phi(\vec{p},-\vec{p})$]. Furthermore, the quark-antiquark system is in the $S=1$ (triplet) spin state so that the amplitudes for the different spin configurations are as follows:

$$\uparrow\uparrow: \frac{1}{\sqrt{3}};\quad \uparrow\downarrow: \frac{1}{\sqrt{6}};\quad \downarrow\uparrow: \frac{1}{\sqrt{6}};\quad \downarrow\downarrow: \frac{1}{\sqrt{3}}\ .$$

It then follows that the amplitudes $\phi_{\lambda_2\lambda_1}(\vec{p},-\vec{p})$ are given by

$$\varphi_{\uparrow\uparrow}(\vec{p},-\vec{p})=\varphi_{\downarrow\downarrow}(\vec{p},-\vec{p})=\frac{1}{\sqrt{3}}\tilde{g}(\vec{p})\ , \tag{8}$$

$$\varphi_{\uparrow\downarrow}(\vec{p},-\vec{p})=\varphi_{\downarrow\uparrow}(\vec{p},-\vec{p})=\frac{1}{\sqrt{6}}\tilde{g}(\vec{p})\ , \tag{9}$$

where $\tilde{g}(\vec{p})$ is defined in (7).

C. Results

Now we have all the ingredients needed to calculate the spin-dependent amplitude (1) and therefore also the leptonic decay width (2). This entails a straightforward but lengthy calculation. We content ourselves with giving the final answer for the leptonic decay width:

$$\Gamma_{e^+e^-}=\frac{2\pi q^\dagger(0)q(0)e_Q{}^2\alpha^2}{M^2}\left[\frac{[x^2/3R^2+(m+\omega)^2]\{m+\omega+(x/R)[(\omega-m/\omega+m)]^{1/2}\}}{\omega^{3/2}(m+\omega)^{3/2}}+\Delta_1+\Delta_2\left[\frac{\omega-m}{\omega+m}\right]^{1/2}\right]^2\ , \tag{10}$$

where

$$\Delta_1(mR,x)\equiv\frac{2}{\pi}\int_{mR}^{\infty}\frac{y\,dy\,e^{-y}[\cos x+y(\sin x)/x]}{(x^2+y^2)(y^2-m^2R^2)^{3/4}}[(\tfrac{2}{3}y-mR)(y+mR)^{1/2}-(\tfrac{2}{3}y+mR)(y-mR)^{1/2}]\ , \tag{11}$$

$$\Delta_2(mR,x)\equiv\frac{2\sqrt{2}}{3\pi}\int_{mR}^{\infty}\frac{y^2dy\,e^{-y}[xj_0(x)(1/y^2+1/y)+j_1(x)]}{(x^2+y^2)(y^2-m^2R^2)^{3/4}}[y-(y^2-m^2R^2)^{1/2}]^{3/2}\ . \tag{12}$$

The integrals (11) and (12) were evaluated by computer, obtaining $\Delta_1=\Delta_2=0$ for the ω and ρ mesons, and $\Delta_1=0.023$; $\Delta_2=0.028$ for the ϕ meson. Thus, one finds that the nonalgebraic terms Δ_1 and Δ_2 contribute only about 3% to the leptonic width of the ϕ meson and their contribution to the other mesons is zero. [If one would "continue" the quark wave function (A4) smoothly outside the bag, then the momentum distribution $\phi(\vec{p},-\vec{p})$ is proportional to $\delta(p-x/R)$ and the result for $\Gamma_{e^+e^-}$ is identical to (10), with Δ_1 and Δ_2 set to zero.]

Using the meson bag parameters given in Table I, one can evaluate the leptonic widths from Eqs. (10)—(12). The results of this calculation, together with the experimental values, are given in Table II.

We see that the calculated leptonic widths are between 10% and 25% high with respect to the experimental values. On the other hand, the results obtained in Ref. 9 are between 30% and 40% low with respect to the experimental values. It thus seems that the actual physical situation should lie somewhere between. The quark and antiquark momenta are neither completely correlated as we assumed, nor completely uncorrelated as was assumed in Ref. 9.

TABLE I. Bag-model results for the relevant mesons. The notation is as follows: m is the quark mass, R is the bag radius, x is the quark "momentum parameter" defined in the Appendix, ω is the energy of the quark mode, $q^\alpha(0)$ is the quark wave function at the origin, $M_{\rm bag}$ is the meson mass as calculated in the bag model, $M_{\rm exp}$ is the meson experimental mass, and $e_Q{}^2$ is the square of the mean electric quark charge in units of a positron charge. We assume the quark composition given at the end of the Appendix.

	ρ	ω	ϕ
m (GeV)	0	0	0.24
R (GeV^{-1})	4.71	4.71	4.636
x	2.04	2.04	2.42
ω (GeV)	0.433	0.433	0.574
$4\pi q^\dagger(0)q(0)$ (GeV^3)	0.0491	0.0491	0.0791
$M_{\rm bag}$ (GeV)	0.783	0.783	1.023
$M_{\rm exp}$ (GeV)	0.770	0.783	1.020
$e_Q{}^2$	$\frac{1}{2}$	$\frac{1}{18}$	$\frac{1}{9}$

APPENDIX: BAG-MODEL CONVENTIONS USED IN OUR CALCULATIONS

Here we will stick closely to the assumptions made in Ref. 6. The surface of the bag is assumed to be static and

TABLE II. Calculated leptonic widths and comparison with experiment (experimental values obtained from Ref. 1).

	$\Gamma_{e^+e^-}$ (keV) (experimental)	$\Gamma_{e^+e^-}$ (keV) (calculated)
ρ	6.5±0.8	7.8
ω	0.76±0.17	0.84
ϕ	1.34±0.08	1.69

spherical of radius R. Each meson has a different radius R, which is determined from energy minimization.

1. Individual quark wave functions

We will assume that the quarks (antiquarks) will all occupy the lowest mode for the free Dirac equation inside the bag. This would not be so if we were dealing with excited states.

The equation obeyed by the quark wave function $q(\vec{r})$ is

$$(-i\vec{\gamma}\cdot\vec{\nabla}+\gamma^0\omega+m)q(\vec{r})=0,\quad r<R \tag{A1}$$

with the boundary condition

$$-i\vec{\gamma}\cdot\hat{r}q(\vec{r})=q(\vec{r}),\quad r=R \tag{A2}$$

and

$$q(\vec{r})\equiv 0,\quad r>R\ . \tag{A3}$$

Here ω is the energy of the mode, m is the "current mass" of the quark, and R is the radius of the bag. The lowest-mode solution has the form

$$q(\vec{r})=\frac{N(x)}{\sqrt{4\pi}}\begin{bmatrix}\left[\dfrac{\omega+m}{\omega}\right]^{1/2} ij_0\left[\dfrac{xr}{R}\right]U\\ -\left[\dfrac{\omega-m}{\omega}\right]^{1/2} j_1\left[\dfrac{xr}{R}\right]\vec{\sigma}\cdot\hat{r}U\end{bmatrix}, \tag{A4}$$

where $\omega\equiv(x^2/R^2+m^2)^{1/2}$ defines x. By virtue of the boundary condition (A2) and our assumption that the quark occupies the lowest cavity mode, $x=x(mR)$ is the lowest positive eigenvalue of the equation

$$\tan x=\frac{x}{1-mR-[x^2+(mR)^2]^{1/2}}\ . \tag{A5}$$

In Eq. (A4), j_n are spherical Bessel functions, U are two-component spinors [$U=\binom{1}{0}$ and $U=\binom{0}{1}$ correspond, respectively, to $S_z=+\frac{1}{2}$ and $S_z=-\frac{1}{2}$], and $N(x)$ is a normalization constant such that

$$\int d\vec{r}\,q^\dagger(\vec{r})q(\vec{r})=1\ . \tag{A6}$$

Equation (A6) implies

$$N^{-2}(x)=R^3 j_0^2(x)\frac{2\omega(\omega-1/R)+m/R}{\omega(\omega-m)}\ . \tag{A7}$$

2. The free parameters of the model

We set the light-quark masses m_u, m_d equal to zero, $m_u\equiv m_d\equiv 0$.

The parameters Z_0, B, and α_c, which are related, respectively, to the zero-point energy ($E_0=-Z_0/R$), the bag pressure ($P=B$), and the strong-coupling constant ($\alpha_s=4\alpha_c$), are fixed by exactly fitting the masses of the Δ, P, ω. The values obtained are $Z_0=1.84$, $B^{1/4}=0.145$ GeV, $\alpha_c=0.55$.[6]

The mass of the s quark is determined from fitting to the mass of the ϕ meson. We obtain $m_s=0.24$ GeV.

All other quantities are determined in terms of these parameters. Table I shows some bag-model results for the relevant mesons which are used in the calculations of Sec. II. The following quark composition has been assumed for the mesons:

$$\rho=\tfrac{1}{2}(u\bar{u}-d\bar{d}),\quad \omega=\tfrac{1}{2}(u\bar{u}+d\bar{d}),\quad \phi=s\bar{s}\ .$$

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