PHYSICAL REVIEW D VOLUME 28, NUMBER 2 15 JULY 1983

Comments

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Comment on "On the canonical approach to quantum gravity"

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(Received 4 March 1983)

In the model discussed by Ashtekar and Horowitz, it is shown that the constraint operator applied to the state functional allows support outside the classically allowed region only if no Hermiticity requirements are placed on the fundamental operators. If one requires that the constraint condition be written as the square of the Hermitian operator p_θ on the state function, then the classically forbidden region is also forbidden quantum mechanically, and, furthermore, within the classically allowed region, only a discrete set of points are allowed quantum mechanically. The latter restriction arises because the original manifold is compact, thereby forcing a quantization of the conjugate momenta which, in the model under consideration, forces a quantization of the allowed points. These features are also exhibited in the path-integral quantization of a similar system quantized on a torus.

In classical general relativity, the Hamiltonian constraint is

$$H=-{}^3R+(p^{kl}p_{kl}-p^2/2)=0\ , \tag{1}$$

which is quadratic in the conjugate momenta p^{kl}. This constraint, expressed as a functional differential constraint on the state functional $\Psi(g_{kl})$, implies that

$$\left\{{}^3R+\left[\frac{\delta}{\delta g^{kl}}\frac{\delta}{\delta g_{kl}}-\frac{1}{2}\left[\frac{\delta}{g^{kl}\delta g_{kl}}\right]^2\right]\right\}\Psi(g_{kl})=0, \tag{2}$$

which is formally similar to the Schrödinger equation

$$\left[-\frac{d^2}{dx^2}+V(x)-E\right]\Psi(x)=0\ . \tag{3}$$

It is well known that, in general, the solutions to Schrödinger's equation have support in the regions which are classically disallowed, that is, the wave function oscillates in the classically allowed region and is exponentially damped but nonzero in the classically forbidden region. The complexity of the general-relativistic constraint equation does not allow one to solve explicitly the functional differential equation (2) or, to my knowledge, prove that the support either is or is not restricted to classically allowed geometries; however, Ashtekar and Horowitz[1] have produced a quantum-mechanical model which has a constraint of the form of Eq. (1). In this model, they reported that the support of the wave function does indeed extend into the classically forbidden region, thereby strengthening the argument that the support of the state functional in quantum gravity should include classically disallowed geometries, e.g., geometries with negative mass. In this note, it is shown that the conclusion is critically dependent upon their exclusion of two points from the manifold. If these two points are included or, alternatively, if the boundary conditions at the excluded points are chosen so that the operator p_θ is self-adjoint on the manifold, then the classically forbidden region is also quantum mechanically forbidden. The essential difference between the approach used here and that of Ref. 1 is precisely in the choice of self-adjoint extension of the constraint operator. There, no particular choice was made and, as a consequence, the quantum operator $p_\theta{}^2$ is not positive definite; here, the requirement that p_θ be self-adjoint is imposed, hence its square is positive definite and the system is quantum mechanically restricted to the classically allowed manifold. Furthermore, because the original manifold is compact, the quantum-mechanical constraint restricts the allowed manifold to a finite discrete set of points within the classically allowed region.

The issue of what three-geometries are allowed in quantum gravity is of particular interest when the functional-integral formulation is used.[2] There, the matrix element between states with initial and final three-geometries is given by the integral

$$\langle I\rangle=\int[dg]\delta(H)\exp(iW[g])\ , \tag{4}$$

where the integral is over all three-geometries g_{kl} and conjugate momenta p^{kl}. There are coordinate conditions and a Faddeev-Popov determinant which have been suppressed as well as the Hamiltonian constraint δ functional which has been written explicitly. The constraint δ functional restricts the integral and therefore the support of the state functional to the classically allowed region. Thus, if the arguments of Ref. 1 were correct, the amplitude could not be written as a functional integral of the form of Eq. (4). However, for the model of Ref. 1 slightly modified so that the functional integrals can be evaluated exactly, it will be shown below that the functional integral yields exactly the same constraints as the differential constraints, including the restriction of the support of the wave function to the same discrete set of points.

The model discussed in Ref. 1 consists of a single quantum-mechanical particle moving in three dimensions. Spherical coordinates are used and the Hamiltonian and the constraint are independent of both the radial coordinate r and its conjugate variable p_r. Since the dynamics only involve the two-dimensional motion, the particle may be viewed as moving on a two-sphere. The coordinates are then θ and ϕ with conjugate momenta p_θ and p_ϕ. The constraint is

$$C = p_\theta^2 - R(\phi) = 0 \ , \tag{5}$$

and the Hamiltonian is

$$H = V(\phi) \ , \tag{6}$$

where

$$R(\phi)V(\phi) \geq 0 \ . \tag{7}$$

The operator p_θ which appears in the constraint, Eq. (5), must be self-adjoint with respect to the measure $\sin\theta d\theta d\phi$, hence

$$p_\theta = \sin^{-1/2}\theta \frac{\partial}{i\partial\theta} \sin^{1/2}\theta \ , \tag{8}$$

and the differential constraint reads

$$\left[-\frac{1}{\sin^{1/2}\theta} \frac{\partial^2}{\partial\theta^2} \sin^{1/2}\theta - R(\phi) \right] \Psi(\theta,\phi) = 0 \ . \tag{9}$$

This equation has the general solution

$$\Psi(\theta,\phi) = (\sin^{1/2}\theta)[A(\phi)\sin\omega(\phi)\theta + B(\phi)\cos\omega(\phi)\theta] \ , \tag{10}$$

where

$$\omega(\phi) = R^{1/2}(\phi)$$

is real and positive in the classically allowed region and positive imaginary in the classically forbidden region. The functions A and B must be determined by the boundary conditions at $\theta = 0, \pi$.

The requirement that the operator p_θ be self-adjoint on the open interval $(0,\pi)$ is

$$\int_0^{2\pi} \int_0^{\pi} [p_\theta f(\theta,\phi)]^\dagger \Psi(\theta,\phi) \sin\theta d\theta d\phi = \int_0^{2\pi} \int_0^{\pi} [f(\theta,\phi)]^\dagger p_\theta \Psi(\theta,\phi) \sin\theta d\theta d\phi \ , \tag{11}$$

where f is an arbitrary periodic C^∞ function of ϕ and an arbitrary C^∞ function of θ which satisfies the boundary conditions

$$\begin{aligned} \sin^{1/2}\theta f(\theta) &= \alpha \frac{d}{d\theta} \sin^{1/2}\theta f(\theta) \ , \quad \theta = 0+ \ , \\ \sin^{1/2}\theta f(\theta) &= \beta \frac{d}{d\theta} \sin^{1/2}\theta f(\theta) \ , \quad \theta = \pi- \ . \end{aligned} \tag{12}$$

The fixed constants α and β define the boundary conditions at the end points. In order for the operator p_θ to be Hermitian, the wave function Ψ must also satisfy the same boundary conditions with $\alpha = 0 = \beta$, whence

$$\tan\omega\pi = 0 \ . \tag{13}$$

This equation is an eigenvalue for $\omega\pi$ with eigenvalues

$$\omega = [R(\phi)]^{1/2} = n \ , \tag{14}$$

or

$$R(\phi) = n^2 \ . \tag{15}$$

In order for the equation to have a solution, the azimuthal angle ϕ must lie in the classically allowed region $R(\phi) > 0$, and ϕ must take one of the finite number of values such that $R(\phi)$ is the square of an integer [note that if $\max(R) < 1$, then no allowed points exist]. In Ref. 1, the requirement that the operator p_θ be Hermitian was not imposed; the only requirement was that the resultant wave function $\Psi(\theta,\phi)$ be normalizable. Here, we find that the requirement that the operator be Hermitian entails the restriction to the classically allowable region and, in addition, forces a further restriction, quantizing the allowable points in that region. Of course, one would not expect such a restriction in general, it is a result of the compact range of the variables plus the somewhat unphysical nature of the constraint.

The essential feature in the preceding model is the fact that the (θ,ϕ) space is compact and that the constraint depends only on p_θ and ϕ. The curvature of the space plays no essential role. A simpler model which exhibits these features and which can be solved somewhat more easily is obtained by changing the manifold to a two-torus so that the locally Euclidean coordinates x and y are defined modulo X and Y, respectively. The constraint and Hermitian then become

$$C = p_x^2 - R(y) \tag{16}$$

and

$$H = V(y) \ ,$$

where $R(y)$ and $V(y)$ are periodic with period Y, and

$$R(y)V(y) \geq 0 \ . \tag{17}$$

The constraint equation for the state wave function $\Psi(x,y)$ is

$$\left[-\frac{\partial^2}{\partial x^2} - R(y) \right] \Psi(x,y) = 0 \ . \tag{18}$$

The solutions to this equation are

$$\Psi(x,y) = A(y)\exp[ix\omega(y)] + B(y)\exp[-ix\omega(y)] \ , \tag{19}$$

where $\omega(y) = R(y)^{1/2}$. The solution must be periodic in x with period X, hence $\omega(y) = 2\pi n$, and, just as in the preceding case, y is restricted to the classically allowed region and, in addition, is restricted to the finite discrete set of points, $\{y_n | R(y_n) = (2\pi n)^2\}$.

The same result may be obtained with the use of functional-integration techniques; the configuration-space amplitude is given by

$$\langle x'',y'',T | x',y',0 \rangle = \int d[x]d[p_x]d[y]d[p_y]\delta[c(p_x,y)]\delta[x-\xi]\mathrm{Det}[p_x]\exp(iW) \ , \tag{20}$$

where

$$W=\int_0^T [p_x dx/dt + p_y dy/dt - V(y(t))]dt \ ,$$

and the integral is over all paths such that

$$x(T)=x'', \ \ y(T)=y'', \ \ x(0)=x', \ \text{ and } y(0)=y' \ ,$$

while the momenta are unconstrained at the end points. The δ functionals, respectively, enforce the constraint and the conjugate coordinate condition, $x(t)-\xi(t)$, mod$X$. Note that, just as in general relativity, the constraint and the coordinate condition both commute with the Hamiltonian (when the constraint is satisfied) and do not commute with each other. The Faddeev-Popov determinant associated with the constraint and the accompanying coordinate choice is Det$[p_x]$.

The only appearance of the momentum p_y is as a linear term in the action, hence that integral may be evaluated explicitly. Let

$$y(t)=[y''t+y'(T-t)]/T+\sum_{n=1}^{\infty} y_n(2/T)^{1/2}\sin(n\pi t/T) \ , \tag{21}$$

and

$$p_y(t)=p_0T^{-1/2}+\sum_{n=1}^{\infty} p_n(2/T)^{1/2}\cos(n\pi t/T) \ .$$

Then, the p_y term in W becomes

$$p_y dy/dt = p_0(y''-y')T^{-1/2}+\sum_{n=1}^{\infty} p_n y_n(n\pi/T) \ ,$$

and the p_y and y integrations can be done, yielding

$$\langle x'',y'',T|x',y',0\rangle = \delta(y''-y')e^{-iTV(y')}G(x'',x',y',T) \ , \tag{22}$$

where

$$G=\int d[x]d[p_x]\delta[p_x^2-R(y')]\delta[x-\xi]\text{Det}[p_x]\exp(iW) \tag{23}$$

and

$$W=\int_0^T dt p_x dx/dt \ .$$

The p_x integration can now be done with use of the constraint; the result is

$$G=\int d[x]\delta[x-\xi][\exp(iW_+)+\exp(iW_-)] \ , \tag{24}$$

where

$$W_\pm = \pm R(y')^{1/2}\int_0^T dx/dt = \pm R(y')^{1/2}(x''-x') \ .$$

The functional integral now depends on x only through the δ functional which sets the gauge (coordinate) condition. The x integration is over paths whose end points are fixed at x'' and x' (modulo X). This may not be consistent with the gauge specification $x=\xi$, because the specification that x have a definite value at $t=T$ and at $t=0$ is itself a gauge specification at the initial and final times; the requirement that the two specifications be consistent yields $\xi(T)=x''$ and $\xi(0)=x'$. Then, the integration may be taken over all paths with the end points being allowed to vary freely. In order to impose periodicity in x, the integrals must be taken over all paths which end at the same point, but x is only defined modulo X, hence the result must be summed over all paths with end points at $x''+nX$. The resultant n dependence is of the form

$$\sum_{n=-\infty}^{\infty}\exp[inXR(y')^{1/2}] = \sum_{m=-\infty}^{\infty}\delta[XR(y')^{1/2}-2\pi m] \ , \tag{25}$$

hence y has a discrete spectrum and the amplitude becomes

$$\langle x'',y_n,T|x',y_m,0\rangle = \delta(y_n,y_m)\exp[-iTV(y_m)] \times\cos[2\pi(x''-x')/X] \ , \tag{26}$$

where

$$R(y_m)=(2\pi m)^2 \ ,$$

exactly the same condition as was found by imposing the differential constraint.

In conclusion, on the torus, both the differential constraint and the functional-integral formulation yield a quantum-mechanical restriction to the classically allowed region. If the range of the quantum variable is finite, then there may be further, quantum-mechanical, restriction to a subset of the classically allowed region. If the model is defined on the sphere then the restriction to the classically allowed region follows from the requirement that the constraint involve the square of the self-adjoint operator p_θ. The functional-integral and operator formulations are equivalent only when the constraint is expressible in terms of the fundamental operators of the theory; if the constraint is not defined as above then the amplitude cannot be written as the functional integral over the canonical variables. In the case of general relativity, the range of the metric is not finite (although the manifold on which it is defined may be) and there is no reason to expect a quantization of the allowed values of the metric.

It is a pleasure to thank Gary Horowitz and Abhay Ashtekar for helpful discussions on this topic. This work was supported in part by the U. S. Department of Energy under Contract No. DE-AC06-81ER40048.

[1]A. Ashtekar and G. Horowitz, Phys. Rev. D 26, 3342 (1982).
[2]L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967); *Gauge Fields: Introduction to Quantum Theory* (Benjamin, New York, 1980).