Analytic solution of the relativistic Coulomb problem for a spinless Salpeter equation

Bernice Durand
Physics Department, University of Wisconsin—Madison, Madison, Wisconsin 53706

Loyal Durand
Theory Group, Fermi National Accelerator Laboratory, Batavia, Illinois 60510
and Physics Department, University of Wisconsin—Madison, Madison, Wisconsin 53706
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We construct an analytic solution to the spinless S-wave Salpeter equation for two quarks interacting via a Coulomb potential, $[2(-\nabla^2+m^2)^{1/2}-M-\alpha/r]\,\psi(r)=0$, by transforming the momentum-space form of the equation into a mapping or boundary-value problem for analytic functions. The principal part of the three-dimensional wave function is identical to the solution of a one-dimensional Salpeter equation found by one of us and discussed here. The remainder of the wave function can be constructed by the iterative solution of an inhomogeneous singular integral equation. We show that the exact bound-state eigenvalues for the Coulomb problem are $M_n=2m/(1+\alpha^2/4n^2)^{1/2}$, $n=1,2,\ldots,$ and that the wave function for the static interaction diverges for $r\to 0$ as $C(mr)^{-\nu}$, where $\nu=(\alpha/\pi)(1+\alpha/\pi+\cdots)$ is known exactly.

I. INTRODUCTION

In this paper, we present an analytic solution for the S-state wave functions of the spinless Salpeter-type equation for a static Coulomb potential,

$$\left[2(-\nabla^2+m^2)^{1/2}-M-\frac{\alpha}{r}\right]\psi(r)=0\ . \tag{1}$$

This equation appears as a natural approximation to the relativistic Bethe-Salpeter-Schwinger equation[1] for two fermions of mass m and total energy M when the interaction kernel is approximated by the instantaneous Coulomb interaction, and spin-dependent effects and the coupling of the "large-large" and "small-small" components of the wave function are neglected. The solution, and the method used to construct it, should therefore be of fairly general interest. We find, for example, that we can determine the exact bound-state eigenvalues of Eq. (1) without actually solving the equation. The result

$$M_n=\frac{2m}{(1+\alpha^2/4n^2)^{1/2}}\ ,\quad n=1,2,\ldots, \tag{2}$$

can probably be generalized to orbital angular momenta $l>0$, and the corresponding result may also be accessible in the spin-dependent problem.

We originally encountered Eq. (1) (with $\alpha=\frac{4}{3}\alpha_s$) in our study of short-range effects in bound quark-antiquark systems.[2] In that work (and later extensions[3]), it was important to know how $\psi(r)$ behaves for $mr\gtrsim 1$. By matching this (free) relativistic Coulomb function to the solution of Eq. (1) with an extra long-range confining interaction and using the known short-range gluonic radiative corrections to $\psi(r)$, we could estimate the radiative corrections to the leptonic decays of $^3S_1$ states in quarkonium[2] and determine an approximate connection between the decay rates calculated relativistically and nonrelativistically.[3] Our results here complete our earlier work by providing rigorous justification for arguments previously made on physical grounds.

The plan of the paper is as follows: In Sec. II A we present an exact solution of a one-dimensional analog of Eq. (1). The one-dimensional wave function appears later as the main component of the three-dimensional wave function for $mr>1$, and determines the normalization of that function. We therefore study the one-dimensional solution in some detail in Secs. II B and II C, and investigate the connection of the one- and three-dimensional problems in Sec. II D.

In Sec. III, we apply the methods and results developed in Sec. II to the solution of the three-dimensional problem defined by Eq. (1). We first show in Sec. III A that the momentum-space form of Eq. (1) can be written as a singular integral equation, and that this equation is equivalent to a mapping or boundary-value problem for analytic functions. We determine the analytic properties of the solutions in Sec. III B, and use this information in Sec. III C to determine the bound-state eigenvalues given in Eq. (2). We complete our reduction of the boundary-value problem in Sec. III D, and show that the momentum-space wave function can be written as the sum of the one-dimensional wave function and a second function which satisfies an inhomogeneous singular integral equation. We show in Sec. III E that the integral equation can be solved iteratively, and obtain exact results for the large-momentum behavior of the solution. We use these results in Sec. III F to show that $\psi(r)$ diverges as $C(mr)^{-\nu}$ for $r\to 0$ where $\nu=(\alpha/\pi)(1+\alpha/\pi+\cdots)$ is known exactly, and conclude in Sec. III G with some comments.

II. THE ONE-DIMENSIONAL COULOMB PROBLEM

A. Solution of the one-dimensional problem

Some time ago one of us (BD) found an exact solution to a one-dimensional relativistic wave equation with a static Coulomb potential, specifically the Salpeter-type equation

$$\left[2\left[-\frac{d^2}{dx^2}+m^2\right]^{1/2}-M-\frac{\alpha}{x}\right]\psi(x)=0 . \quad (3)$$

The solution to this equation appears as an essential part of our solution to the physically interesting three-dimensional problem. The method of solution furthermore suggested methods we will use later. We therefore present the one-dimensional results here.

We observe that Eq. (3) can be reduced to an ordinary first-order differential equation by performing a Fourier transform.[4] If we define $\tilde{\psi}(p)$ by

$$\psi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dp\,\tilde{\psi}(p)e^{ipx} , \quad (4)$$

substitute in Eq. (3), and integrate once by parts, we find that $\tilde{\psi}(p)$ satisfies the equation

$$i\frac{d}{dp}\{[2(p^2+m^2)^{1/2}-M]\tilde{\psi}(p)\}-\alpha\tilde{\psi}(p)=0 . \quad (5)$$

Alternatively, if we define $\chi(p)$ as

$$\chi(p)=(i/\alpha)[2(p^2+m^2)^{1/2}-M]\tilde{\psi}(p) , \quad (6)$$

we find that

$$\frac{d\chi}{dp}(p)+iB(p)\chi(p)=0 , \quad (7)$$

where

$$B(p)=\alpha/[2(p^2+m^2)^{1/2}-M] . \quad (8)$$

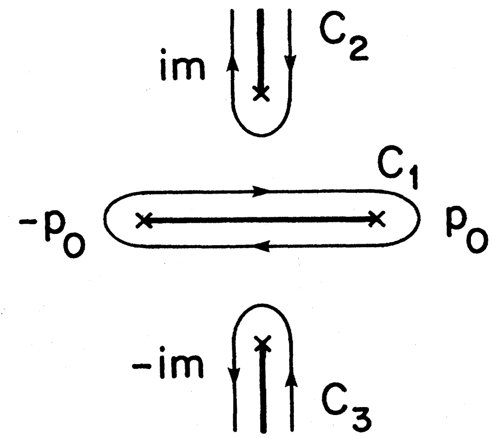

FIG. 1. The branch points and cuts of the function $\chi(p)$, Eq. (11), and the contours used in various integrations.

The solution of Eq. (7) is straightforward:

$$\chi(p)=A\exp[-i\int^{p}B(p')dp'] \quad (9)$$

and

$$\tilde{\psi}(p)=-iB(p)\chi(p)=\frac{d\chi}{dp}(p) . \quad (10)$$

Explicit evaluation of the integral in Eq. (9) gives the rather complicated expression

$$\chi(p)=A\left[\frac{p}{(p^2+m^2)^{1/2}+m}-\left[\frac{M-2m}{M+2m}\right]^{1/2}\right]^{-i\eta}\left[\frac{p}{(p^2+m^2)^{1/2}+m}+\left[\frac{M-2m}{M+2m}\right]^{1/2}\right]^{i\eta}\left[\frac{(p^2+m^2)^{1/2}+m-p}{(p^2+m^2)^{1/2}+m+p}\right]^{i\alpha/2} , \quad (11)$$

where $\eta=\alpha/2v$ and v is the velocity of a free quark with total energy $M/2$ and momentum p_0,

$$v=\left[1-\frac{4m^2}{M^2}\right]^{1/2} , \quad p_0=Mv/2 . \quad (12)$$

We obtain equivalent results using the customary momentum-space form of the relativistic wave equation

$$[2(p^2+m^2)^{1/2}-M]\tilde{\psi}(p)-\frac{i\alpha}{2}\int_{-\infty}^{\infty}dk\,\epsilon(p-k)\tilde{\psi}(k)=0 . \quad (13)$$

If we replace $\tilde{\psi}$ by $d\chi/dp$ as in Eq. (10), a partial integration reduces this integral equation to the differential equation for χ given in Eq. (7). This approach will be useful in the three-dimensional problem.

It is easily shown that the function $\chi(p)$ has branch points at $p=\pm p_0$ and at $\pm im$ but no other singularities in the (finite) p plane. We choose the branch cuts as shown in Fig. 1. To obtain a space wave function $\psi(x)$ which vanishes for $x=0$, we choose the integration contour in Eq. (4) as the difference between contours which run from $-\infty$ to ∞ just above and below the real axis, that is, use the contour C_1 in Fig. 1. After a partial integration, we can collapse the contour and express $\psi(x)$ as a simple Fourier transform of $\chi(p)$,

$$\psi(x)=xA\int_{-p_0}^{p_0}dp\,e^{ipx}\left[\left[\frac{M-2m}{M+2m}\right]^{1/2}-\frac{p}{(p^2+m^2)^{1/2}+m}\right]^{-i\eta}$$
$$\times\left[\left[\frac{M-2m}{M+2m}\right]^{1/2}+\frac{p}{(p^2+m^2)^{1/2}+m}\right]^{i\eta}\left[\frac{(p^2+m^2)^{1/2}+m-p}{(p^2+m^2)^{1/2}+m+p}\right]^{i\alpha/2} , \quad (14)$$

where we have absorbed various constants into the overall normalization constant A. This expression reduces in the nonrelativistic limit ($m \to \infty$, p_0 fixed) to a standard representation of the S-state Coulomb wave function,

$$\psi(x) = p_0 x A \int_{-1}^{1} dt\, e^{ip_0 x t}(1-t)^{-i\eta}(1+t)^{i\eta}, \quad t = p/p_0 . \tag{15}$$

B. Normalization and asymptotic form of the wave function

The one-dimensional function $\psi(x)$ in Eq. (15) for $x = r > 0$ gives the main term in our three-dimensional solution of the relativistic radial wave equation. We will need this function in three dimensions with the standard plane-wave or unit-amplitude normalization for $r \to \infty$. To determine the appropriate value of the normalization constant A, we therefore let $x \to +\infty$ in Eq. (14) and extract the leading term in x^{-1} in the asymptotic expansion of $\psi(x)$.

The integrand in Eq. (14) has stationary-phase points at

$$p = \pm \left[p_0 + \frac{\eta}{x} \right], \quad p_0 x \gg 1 \tag{16}$$

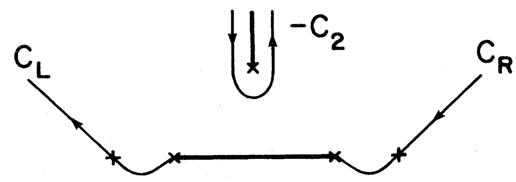

FIG. 2. The integration contours used in the evaluation of the asymptotic form of $\psi(x)$ for $x \to \infty$.

(these points move to $\pm p_0$ for $x \to \infty$) and otherwise oscillates rapidly on the real axis. To take advantage of this observation, we distort the contour of integration in Eq. (14) as shown in Fig. 2 so that it runs through the stationary-phase points in the direction of steepest descent. The contribution from the upper contour (the negative of C_2 in Fig. 1) decreases as e^{-mx} for $x \to +\infty$, and can be neglected. On the right- and left-hand contours, we write p as $\pm p_0(1+t)$ and expand the integrands in powers of t. The result for p_0 real ($M > 2m$) is

$$u(x) \underset{x\to\infty}{\sim} A e^{-\pi\eta} \left[\frac{M}{m} \right]^{i\eta} \left[\frac{(M+2m)^{1/2} - (M-2m)^{1/2}}{(M+2m)^{1/2} + (M-2m)^{1/2}} \right]^{i\alpha/2} p_0 x e^{ip_0 x} \int_{C_R} dt\, e^{ip_0 x t} t^{-i\eta}[1 + O(t)] + \text{complex conjugate} , \tag{17}$$

where C_R is the right-hand contour in Fig. 2. The remaining integral is proportional to $\Gamma(1-i\eta)$, and we find after some algebra that

$$u(x) \underset{x\to\infty}{\sim} 2A e^{-\pi\eta/2} |\Gamma(1-i\eta)| \sin \Big\{ p_0 x + \eta \ln 2p_0 x + \arg\Gamma(1-i\eta) + \eta \ln(M/2m) + \frac{\alpha}{2}\ln[(M+2m)^{1/2} - (M-2m)^{1/2}] - \frac{\alpha}{2}\ln[(M+2m)^{1/2} + (M-2m)^{1/2}] \Big\} \times \left[1 + O\left[\frac{1}{p_0 x} \right] \right] . \tag{18}$$

To obtain unit normalization for $u(x)$ for $x \to \infty$, we therefore take

$$A = \frac{e^{\pi\eta/2}}{2|\Gamma(1-i\eta)|} , \tag{19}$$

and find that the normalized wave function is given for all x by

$$\psi(x) = \frac{x}{2} \frac{e^{\pi\eta/2}}{|\Gamma(1-i\eta)|} \int_{-p_0}^{p_0} dp\, e^{ipx} \left[\left[\frac{M-2m}{M+2m} \right]^{1/2} - \frac{p}{(p^2+m^2)^{1/2}+m} \right]^{-i\eta} \times \left[\left[\frac{M-2m}{M+2m} \right]^{1/2} + \frac{p}{(p^2+m^2)^{1/2}+m} \right]^{i\eta} \left[\frac{(p^2+m^2)^{1/2}+m-p}{(p^2+m^2)^{1/2}+m+p} \right]^{i\alpha/2} ,$$
$$p_0 = (\tfrac{1}{4}M^2 - m^2)^{1/2}, \quad \eta = \alpha/2v = \alpha M/4p_0 . \tag{20}$$

C. Evaluation of $\psi'(0)$

As we will show in Sec. III, the magnitude of the S-state Coulomb wave function near the origin in three dimensions is determined by the value of $\psi'(0)$, or equivalently, by the limiting value of $\psi(x)/x$ for $x \to 0$. [If we were dealing with an ordinary nonrelativistic Schrödinger equation, $\psi(x)$ could be identified with $u(r) = rR(r)$, and $\psi'(0) = R(0)$]. The value of $\psi'(0)$ is determined by Eq. (20). After dividing by x, we can set x equal to zero on the right-hand side of this equation. A change of the variable of integration from p to

$$t = \frac{p}{(p^2+m^2)^{1/2}+m} \tag{21}$$

then gives the expression

$$\psi'(0)=ma\frac{e^{\pi\eta/2}}{|\Gamma(1-i\eta)|}\int_{-1}^{1}dt\left[\frac{1+t}{1-t}\right]^{i\eta}\left[\frac{1-at}{1+at}\right]^{i\alpha/2}\frac{1+a^2t^2}{(1-a^2t^2)^2}\ , \tag{22}$$

where

$$a=\left[\frac{M-2m}{M+2m}\right]^{1/2}=\frac{2p_0}{M+2m}\ . \tag{23}$$

For particles which are not too relativistic, the parameter a is small, $a\sim p_0/2m\sim v/2$, and we can evaluate the integral in Eq. (22) approximately by expanding the last two factors in a power series in at

$$\begin{aligned}\left[\frac{1-at}{1+at}\right]^{i\alpha/2}\frac{1+a^2t^2}{(1-a^2t^2)^2}&=1-i\alpha t+3a^2t^2+\cdots\\&=1-\tfrac{1}{2}i\alpha a[(1+t)-(1-t)]+\tfrac{3}{2}a^2[(1+t)^2+(1-t)^2-2]+\cdots\ .\end{aligned} \tag{24}$$

The integrals which appear can then be reduced to beta functions

$$\int_{-1}^{1}dt\,(1+t)^x(1-t)^y=2^{x+y+1}\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}\ , \tag{25}$$

and we find that

$$\begin{aligned}\psi'(0)&=p_0e^{\pi\eta/2}|\Gamma(1-i\eta)|\frac{4m}{M+2m}[1+2\alpha\eta a+2a^2(1-2\eta^2)+\cdots]\\&=p_0e^{\pi\eta/2}|\Gamma(1-i\eta)|[1+\tfrac{1}{4}\alpha^2+\tfrac{1}{4}v^2+O(\alpha^4,\alpha^2v^2,v^4)]\ .\end{aligned} \tag{26}$$

The leading factor is just the usual Coulomb factor with η calculated using the relativistic velocity of the quark[5]

$$p_0e^{\pi\eta/2}|\Gamma(1-i\eta)|=p_0\left[\frac{2\pi\eta}{1-\exp(-2\pi\eta)}\right]^{1/2}\ ,\quad \eta=\alpha/2v\ . \tag{27}$$

We have not found a simple expression for $\psi'(0)$ in the extreme relativistic case.

D. Comments on the one-dimensional and three-dimensional problems

The expression in Eq. (20) gives the exact solution to the one-dimensional equation

$$\left[2\left[-\frac{d^2}{dx^2}+m^2\right]^{1/2}-M-\frac{\alpha}{x}\right]\psi(x)=0 \tag{28}$$

which vanishes at $x=0$. This solution reduces in the nonrelativistic limit to the $l=0$ solution of the radial Schrödinger equation for an attractive Coulomb potential, $\psi(x)\to u(r)$, $r=x>0$, $E=M-2m$,

$$\left[-\frac{1}{m}\frac{d^2}{dr^2}-E-\frac{\alpha}{r}\right]u(r)=0\ . \tag{29}$$

One might therefore expect $\psi(x)$ to give a solution of the three-dimensional relativistic Coulomb problem as well. This is *not* the case, though the one-dimensional solution *does* give the main part of the three-dimensional solution for $mr\gtrsim 1$. To see why the usual expectation fails, we will consider the connection between the one-dimensional and three-dimensional problems in detail.

We begin with the three-dimensional Salpeter equation

$$\left[2(-\nabla^2+m^2)^{1/2}-M-\frac{\alpha}{r}\right]\psi(\vec{r})=0\ , \tag{30}$$

or equivalently,

$$[2(p^2+m^2)^{1/2}-M]\tilde{\psi}(\vec{k})-\frac{\alpha}{2\pi^2}\int d^3k\frac{\tilde{\psi}(\vec{k})}{(\vec{p}-\vec{k})^2+\epsilon^2}=0\ , \tag{31}$$

where

$$\psi(\vec{r})=\frac{1}{(2\pi)^3}\int d^3p\,e^{i\vec{p}\cdot\vec{r}}\tilde{\psi}(\vec{p})\ . \tag{32}$$

For the S-wave problem, we define radial wave functions by

$$u(r)=\sqrt{4\pi}r\psi(r),\quad \phi(p)=\sqrt{4\pi}p\tilde{\psi}(p)\ . \tag{33}$$

With these conventions

$$u(r)=\frac{1}{2\pi^2}\int_0^\infty dp\,\sin pr\,\phi(p)\ , \tag{34}$$

where $\phi(p)$ is an odd solution of the integral equation

$$[2(p^2+m^2)^{1/2}-M]\phi(p)+\frac{\alpha}{2\pi}\int_{-\infty}^{\infty}dk\,\ln[(p-k)^2+\epsilon^2]\phi(k)=0\ , \tag{35}$$

$$\phi(-p)=-\phi(p),\quad \epsilon\to 0+\ . \tag{36}$$

Equation (35) is *not* equivalent to the one-dimensional integral equation in Eq. (13), or its position-space form, Eq. (28). By calculating the *one-dimensional* Fourier inversion of Eq. (35) we find that it corresponds instead to the position-space equation

$$\left[2\left[-\frac{d^2}{dx^2}+m^2\right]^{1/2}-M-\frac{\alpha}{|x|}\right]\psi(x)=0\ ,$$

$$-\infty<x<\infty\ , \quad (37)$$

that is, to an equation of the form of Eq. (28), but with the potential $V(x)=-\alpha/x$ replaced by $V(|x|)=-\alpha/|x|$. The odd solutions $\psi(x)$ can be identified with the radial wave function $u(r)$ for $x=r>0$. We thus obtain the usual correspondence between the radial wave functions $u(r)$ for a potential $V(r)$ and the odd solutions of the one-dimensional problem with the symmetrical potential $V(|x|)$.

The Coulomb problem solved above involved the asymmetrical potential $-\alpha/x$. In the nonrelativistic case, the replacement of $V(|x|)$ by $V(x)$ is irrelevant since the Schrödinger equation is local (that is, involves only finite-order derivatives) and can be solved locally, e.g., for $x>0$. The solution for $x>0$ does not depend on the form of V for $x<0$ provided we impose the boundary condition $u(0)=\psi(0)=0$. The situation is quite different for the Salpeter equation which involves derivatives of arbitrary order and is nonlocal in position space over regions of size $\sim m^{-1}$. Because of the different behavior of the one-dimensional and three-dimensional potentials [Eqs. (28) and (37)] for $r<0$, the solutions of the two problems will differ slightly for $r\lesssim m^{-1}$ even though the solutions have the same nonrelativistic limits and are essentially identical for $r\gg m^{-1}$.

III. THE THREE-DIMENSIONAL COULOMB PROBLEM

A. The boundary-value form of the problem

The three-dimensional Coulomb problem is defined by the integral equation given above,

$$[2(p^2+m^2)^{1/2}-M]\phi(p)+\frac{\alpha}{2\pi}\int_{-\infty}^{\infty}dk\,\ln[(p-k)^2+\epsilon^2]\phi(k)=0\ , \quad (38)$$

$$\phi(p)=-\phi(-p),\quad \epsilon\to 0+\ .$$

To transform this equation into a more useful form, we make a redefinition suggested by our results in one dimension,

$$\phi(p)=\frac{d\chi}{dp}(p),\quad \chi(-p)=\chi(p)\ , \quad (39)$$

and integrate once by parts. The result is the singular integral equation[6]

$$[2(p^2+m^2)^{1/2}-M]\frac{d\chi}{dp}+\frac{\alpha}{\pi}P\int_{-\infty}^{\infty}dk\frac{\chi(k)}{k-p}=0\ , \quad (40)$$

where P designates the principal-value integral.

This integral equation can be transformed into a boundary-value problem as follows. We represent $\chi(p)$, p real, as the difference of the boundary values of two functions $\Phi^+(p)$ and $\Phi^-(p)$ which are analytic, respectively, in the upper and lower halves of the complex p plane and vanish at infinity,[7]

$$\chi(p)=\Phi^+(p+i\epsilon)-\Phi^-(p-i\epsilon)\ . \quad (41)$$

Use of the Plemelj relations for the principal-value integral[8] (or direct calculation) then shows that

$$\frac{P}{\pi}\int_{-\infty}^{\infty}dk\frac{1}{k-p}[\Phi^+(k+i\epsilon)-\Phi^-(k-i\epsilon)]=i[\Phi^+(p+i\epsilon)+\Phi^-(p-i\epsilon)]\ , \quad (42)$$

and we can rewrite Eq. (40) as

$$\left[\frac{d}{dp}+iB(p)\right]\Phi^-(p-i\epsilon)=\left[\frac{d}{dp}-iB(p)\right]\Phi^+(p+i\epsilon)\ . \quad (43)$$

Here $B(p)$ is the function defined in Eq. (8),

$$B(p)=\alpha/[2(p^2+m^2)^{1/2}-M]\ . \quad (44)$$

This function is symmetric in p, $B(-p)=B(p)$, and has poles at $p=\pm p_0$ [Eq. (12)] and branch points at $\pm im$. We will assume initially that $M>2m$ so that p_0 is real.

The radial wave function $u(r)$ can be expressed in terms of the Φ's using Eqs. (34), (39), and (41). We first use the antisymmetry of $\phi(p)=d\chi/dp$ and Eq. (41) to rewrite Eq. (34) [with the factor $(2\pi)^{-2}$ absorbed in χ] as

$$u(r)=\int_{-\infty}^{\infty}dp\,\sin pr\frac{d\chi}{dp}(p)$$

$$=-\tfrac{1}{2}r\int_{-\infty}^{\infty}dp(e^{ipr}+e^{-ipr})[\Phi^+(p+i\epsilon)-\Phi^-(p-i\epsilon)]\ . \quad (45)$$

We then observe that the integrals which involve $e^{ipr}\Phi^+$ and $e^{-ipr}\Phi^-$ vanish for $r>0$ (the functions are analytic in the upper and lower half planes, respectively, and the integration contours can be pushed to $\pm i\infty$ where the integrands vanish exponentially). Thus

$$u(r)=\tfrac{1}{2}r\int_{-\infty}^{\infty}dp[e^{ipr}\Phi^-(p)-e^{-ipr}\Phi^+(p)],\quad r>0\ , \quad (46)$$

where the integration contour in the first (second) term runs just below (above) the real axis. Changing from p to $-p$ as the integration variable in the second term, we find that

$$u(r)=\tfrac{1}{2}r\int_{-\infty}^{\infty}dp\,e^{ipr}[\Phi^-(p)-\Phi^+(-p)]\ ,\quad r>0\ , \quad (47)$$

where the integration contour now runs just below the real axis in both terms. Similarly, for $r<0$,

$$u(r)=-\tfrac{1}{2}r\int_{-\infty}^{\infty}dp\,e^{ip|r|}[\Phi^-(p)-\Phi^+(-p)]$$

$$=-u(|r|),\quad r<0\ , \quad (48)$$

and $u(r)$ has the proper symmetry.

The function $[\Phi^-(p)-\Phi^+(-p)]$ which appears in Eqs. (47) and (48) is analytic in the lower half of the complex p plane. It must be an even function of p for p real if $u(r)$ is to have a Fourier sine representation as in Eq. (45). These observations suggest that the Φ's satisfy the symmetry relation

$$\Phi^+(p)=-\Phi^-(-p) \quad (49)$$

for p complex. This symmetry is consistent with the boundary-value equation, Eq. (43), and will be seen later to hold for the solutions of Eq. (43) near the poles of $B(p)$, hence to hold in general. The wave function $u(r)$ is therefore expressible in terms of $\Phi^-(p)$ alone,

$$u(r)=r\int_{-\infty}^{\infty}dp\, e^{ipr}\Phi^-(p), \quad r>0$$
$$u(r)=-|r|\int_{-\infty}^{\infty}dp\, e^{ip|r|}\Phi^-(p)\ , \quad r<0\ . \tag{50}$$

The problem we now face is that of finding a solution of Eq. (43) subject to the symmetry requirement in Eq. (49). Our technique will be to use the analytic properties of the Φ's and $B(p)$ to transform the boundary-value problem into an equivalent problem which we can solve.

B. Analytic properties of $\Phi^-(p)$

The analytic properties of $\Phi^-(p)$ are easily deduced from the boundary-value equation. $\Phi^-(p)$ is analytic in the lower half p plane by construction. $\Phi^+(p)$ is similarly analytic in the upper half p plane. The function $B(p)$ in Eq. (43) has poles at $p=\pm p_0$ and branch points at $p=\pm im$. These singularities will be reflected in singularities of the Φ's. We first examine the effect of the poles.

The most singular part of the boundary-value equation for $p\to\pm p_0$ is of the form

$$\left[\frac{d}{dp}-\frac{i\eta}{p_0\mp p}\right]\Phi^-(p)=\left[\frac{d}{dp}+\frac{i\eta}{p_0\mp p}\right]\Phi^+(p)\ , \tag{51}$$

where $\eta=\alpha/2v$. The two sides of this equation involve functions which are analytic in the lower and upper half planes, respectively, and vanish at infinity. The only nontrivial solution to this reduced boundary-value problem is obtained when the two sides of the equation vanish independently. We therefore conclude that

$$\Phi^-(p)\propto(p_0\mp p)^{\mp i\eta},$$
$$\Phi^+(p)\propto(p_0\mp p)^{\pm i\eta}, \quad p\to\pm p_0\ , \tag{52}$$

hence that the functions Φ^- (Φ^+) have the *same* branch points at $\pm p_0$ as the function $\chi(p)$ encountered in the one-dimensional problem for an attractive (repulsive) Coulomb interaction. We will choose the cut in Φ^- (Φ^+) to connect the branch points as shown in Fig. 3. We can then continue Φ^- (Φ^+) around the cut into the upper (lower) half plane. We remark also that the expressions in Eq. (52) have the symmetry $\Phi^+(-p)=-\Phi^-(p)$, Eq. (49), for complex p if the constants of proportionality are chosen to be equal in magnitude and opposite in sign.

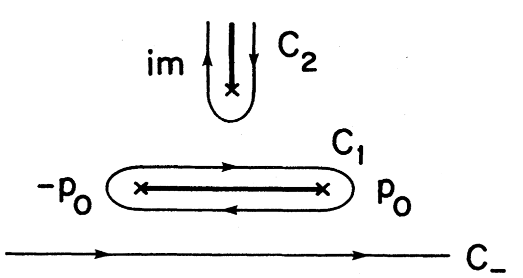

FIG. 3. The branch points and cuts of the function $\Phi^-(p)$, and the contours used in various integrations.

We next consider the continuation of Eq. (43) into the upper half plane starting on the real axis with $p>p_0$, that is, to the right of the cuts in $\Phi^{\pm}(p)$. The two sides of the equation are independently analytic near the real axis for $p>p_0$, so are equal as analytic functions and may be continued together. Since Φ^+ is analytic in the upper half plane by construction, the only singularity of the right-hand side of Eq. (43) for $\mathrm{Im}p>0$ is the branch point of $B(p)$ at $p=im$. The left-hand side therefore has only this singularity, and we conclude that the only singularity of Φ^- in the upper half plane is a branch point at $p=im$. [Φ^- *must* have this branch point for a nontrivial solution to exist since $\Phi^-(p)\neq\Phi^+(p)$, see, e.g., Eq. (52).] We choose the branch cut in Φ^- to run from im to $i\infty$ as shown in Fig. 3. The symmetry relation in Eq. (49) then implies that Φ^+ has a branch point at $p=-im$ with a cut which runs from $-im$ to $-i\infty$.

It will be convenient at this point to write Φ^- as a sum of two functions,

$$\Phi^-(p)=\Phi_1(p)+\Phi_2(p)\ , \tag{53}$$

where $\Phi_1(p)$ has only the "short" cut on the real axis from $-p_0$ to p_0, and $\Phi_2(p)$ has only the "long" cut from im to $i\infty$. Since $\Phi^-(p)$ vanishes at infinity, we can express the separate functions in terms of their discontinuities across the cuts using the Cauchy integral formula,

$$\Phi_1(p)=\frac{1}{2\pi i}\int_{C_1}\frac{dp'}{p'-p}\Phi_1(p')$$
$$=\frac{1}{2\pi i}\int_{-p_0}^{p_0}\frac{dp'}{p'-p}\mathrm{disc}\Phi_1(p')\ , \tag{54}$$

$$\Phi_2(p)=\frac{1}{2\pi i}\int_{C_2}\frac{dp'}{p'-p}\Phi_2(p')$$
$$=\frac{1}{2\pi i}\int_{im}^{i\infty}\frac{dp'}{p'-p}\mathrm{disc}\Phi_2(p')\ . \tag{55}$$

In these expressions the point p lies outside the integration contours C_1 and C_2 shown in Fig. 3, and the discontinuities are defined by

$$\mathrm{disc}\Phi_1(p)=\Phi_1(p+i\epsilon)-\Phi_1(p-i\epsilon)\ ,$$
$$-p_0<p<p_0, \quad \epsilon\to 0+\ ,$$
$$\mathrm{disc}\Phi_2(p)=\Phi_2(p-\epsilon)-\Phi_2(p+\epsilon)\ ,$$
$$m<-ip<\infty, \quad \epsilon\to 0+\ . \tag{56}$$

The function $\mathrm{disc}\Phi_1(p)$ is determined as follows. We note that the result obtained by continuing Eq. (43) from the region $p>p_0$ (where both sides are analytic) to the upper edge of the short cut must be consistent with the original equation for $-p_0<p<p_0$. Since Φ^- is given on the upper edge of the cut by

$$\Phi^-(p+i\epsilon)=\Phi^-(p-i\epsilon)+\mathrm{disc}\Phi^-(p)\ , \tag{57}$$

while only $\Phi^-(p-i\epsilon)$ appears in the original equation, the discontinuity function must satisfy the equation

$$\left[\frac{d}{dp}+iB(p)\right]\mathrm{disc}\Phi^-(p)=0, \quad -p_0<p<p_0\ . \tag{58}$$

This is precisely the equation satisfied by the one-dimensional wave function $\mathcal{X}(p)$, Eq. (7). The solution is given by Eq. (11).

The boundary-value problem has been reduced at this point to one of determining the single function $\mathrm{disc}\Phi_2(p)$. We will postpone our discussion of this problem to Sec. III D, and will first use the present results to determine the exact bound-state eigenvalues for the relativistic Coulomb problem.

C. Wave functions and bound-state eigenvalues

The radial wave function $u(r)$ can be expressed in terms of Φ^- through the Fourier transform [Eq. (50)]

$$u(r)=r\int_{C_-} dp\, e^{ipr}\Phi^-(p), \quad r>0\ , \tag{59}$$

where the contour C_- is shown in Fig. 3. By writing Φ^- in terms of Φ_1 and Φ_2, deforming C_- into the sum of two contours $-C_1, -C_2$ surrounding the cuts in Φ_1 and Φ_2, and using the definition of the discontinuity functions in Eq. (56), we find that

$$u(r)=u_1(r)+u_2(r)\ , \tag{60}$$

where

$$u_1(r)=-r\int_{C_1} dp\, e^{ipr}\Phi_1(p) = -r\int_{-p_0}^{p_0} dp\, e^{ipr}\mathrm{disc}\Phi_1(p)\ , \tag{61}$$

$$u_2(r)=-r\int_{C_2} dp\, e^{ipr}\Phi_2(p) = -r\int_{im}^{i\infty} dp\, e^{ipr}\mathrm{disc}\Phi_2(p) = -ir\int_m^\infty d|p|\, e^{-|p|r}\mathrm{disc}\Phi_2(i|p|)\ . \tag{62}$$

While $\mathrm{disc}\Phi_2$ has yet to be determined, we observe that $u_2(r)$ is always an exponentially decreasing function for r large, $u_2(r)\sim O(e^{-mr})$. As a result, if we normalize $u(r)$ to unit amplitude for $r\to\infty, p_0$ real, we must take

$$\mathrm{disc}\Phi_1(p)=-\mathcal{X}(p)\ , \tag{63}$$

where $\mathcal{X}(p)$ is given by Eqs. (11) and (19). In this case, $u_1(r)$ is just the one-dimensional wave function studied in Sec. II,

$$u_1(r)=\frac{r}{2}\frac{e^{\pi\eta/2}}{|\Gamma(1-i\eta)|}\int_{-p_0}^{p_0} dp\, e^{ipr}\left[\left[\frac{M-2m}{M+2m}\right]^{1/2}-\frac{p}{(p^2+m^2)^{1/2}+m}\right]^{-i\eta} \times\left[\left[\frac{M-2m}{M+2m}\right]^{1/2}+\frac{p}{(p^2+m^2)^{1/2}+m}\right]^{i\eta}\left[\frac{(p^2+m^2)^{1/2}+m-p}{(p^2+m^2)^{1/2}+m+p}\right]^{i\alpha/2}. \tag{64}$$

We can also determine the exact $l=0$ bound-state energies for the Coulomb problem without determining $\mathrm{disc}\Phi_2$. For $M<2m$, p_0 is imaginary,

$$p_0=i|p_0|=i(m^2-\tfrac{1}{4}M^2)^{1/2}\ , \tag{65}$$

and the contour C_1 in Eq. (61) surrounds the segment $(-i|p_0|, i|p_0|)$ of the imaginary axis. The upper part of the contour gives an exponentially decreasing contribution to $u_1(r)$; the lower part of the contour gives an exponentially increasing contribution. To obtain an acceptable (normalizable) bound-state wave function, we must be able to eliminate the lower part of the contour. This requires that there be no branch point or pole at $p=-i|p_0|$, hence from Eq. (64), that $i\eta$ be a positive integer, $i\eta=1,2,\ldots$ (the wave function vanishes for $i\eta=0$). Using the relation $i\eta=\alpha M/4|p_0|$, we find that the exact bound-state energies for $l=0$ (S states) are

$$M_n=2m\Big/\left[1+\frac{\alpha^2}{4n^2}\right]^{1/2} =2m-\frac{\alpha^2 m}{4n^2}+\frac{3}{64}\frac{\alpha^4 m}{n^4}+\cdots\ , \quad n=1,2,\ldots\ . \tag{66}$$

The α^4 term agrees with the correction to the Schrödinger energy $E_n=M_n-2m$ obtained by expanding the square root in Eq. (30) to order ∇^4/m^3, and treating that term as a perturbation.

D. Integral equation for $\mathrm{disc}\Phi_2$

We showed in Secs. III A and III B that the solution of the three-dimensional relativistic Coulomb problem can be expressed in terms of a function $\Phi^-(p)=\Phi_1(p)+\Phi_2(p)$ which is analytic in the entire complex p plane except for cuts from $-p_0$ to $+p_0$ (Φ_1) and from im to $i\infty$ (Φ_2). We found, furthermore, that $\mathrm{disc}\Phi_1$ (the discontinuity of Φ^- or Φ_1 across the "short" cut) satisfied the same differential equation as was encountered in the one-dimensional problem, and was given explicitly by $\mathrm{disc}\Phi_1(p)=-\mathcal{X}(p)$ where $\mathcal{X}(p)$ is defined in Eqs. (11) and (19). In this section, we will derive an integral equation for $\mathrm{disc}\Phi_2(p)$. This equation relates $\mathrm{disc}\Phi_2$ to the known function $\mathrm{disc}\Phi_1$, and can be solved by iteration. The solution will be discussed in the following sections.

We begin by deriving an equation for $\Phi^-(p)$ which displays the analytic structure of that function. To do this, we note that the condition on $\mathrm{disc}\Phi_1$ in Eq. (58) allows us to write the boundary-value equation, Eq. (43), as an analytic expression in the cut p plane,

$$\left[\frac{d}{dp}+iB(p)\right]\Phi^-(p)=\left[\frac{d}{dp}-iB(p)\right]\Phi^+(p)\ . \tag{67}$$

We multiply this equation by the Cauchy denominator and integrate on the contour C_2 shown in Fig. 1 to obtain a identity valid for general complex p,

$$\frac{1}{2\pi i}\int_{C_2}\frac{dp'}{p'-p}\left[\frac{d}{dp'}+iB(p')\right]\Phi^-(p') =\frac{1}{2\pi i}\int_{C_2}\frac{dp'}{p'-p}\left[\frac{d}{dp'}-iB(p')\right]\Phi^+(p') . \tag{68}$$

The integral of the right-hand side is simple to evaluate: Φ^+ is analytic in the upper half plane, while $B(p)$ has a cut from im to $i\infty$. Thus

$$\frac{1}{2\pi i}\int_{C_2}\frac{dp'}{p'-p}\left[\frac{d}{dp'}-iB(p')\right]\Phi^+(p') =-\frac{1}{2\pi}\int_{im}^{i\infty}\frac{dp'}{p'-p}\Phi^+(p')\mathrm{disc}B(p') , \tag{69}$$

where

$$\mathrm{disc}B(p)=B(p-\epsilon)-B(p+\epsilon) , \quad \epsilon\to 0+,\ \ m\le |p| <\infty ,$$
$$=i\alpha\frac{(|p|^2-m^2)^{1/2}}{|p|^2+p_0{}^2} . \tag{70}$$

To evaluate the integral on the left-hand side of Eq. (68), we deform the contour of integration downward, pick up the residue of the pole at $p'=p$, and contributions from the contours $-C_1, -C_3$ in Fig. 1. The contribution from $-C_1$ vanishes by Eq. (58). The contribution from $-C_3$ can be expressed in terms of disc$B$ [Φ^- is analytic in the lower half p plane, and $B(-p)=B(p)$], and we obtain

$$\frac{1}{2\pi i}\int_{C_2}\frac{dp'}{p'-p}\left[\frac{d}{dp'}+iB(p')\right]\Phi^-(p')=-\left[\frac{d}{dp}+iB(p)\right]\Phi^-(p)+\frac{1}{2\pi}\int_{-im}^{-i\infty}\frac{dp'}{p'-p}\Phi^-(p)\mathrm{disc}B(p) . \tag{71}$$

Combining Eqs. (70) and (71) and using the symmetry relation $\Phi^+(p)=-\Phi^-(-p)$, we find that Φ^- satisfies the equation

$$\left[\frac{d}{dp}+iB(p)\right]\Phi^-(p)-\frac{1}{2\pi}\int_m^\infty dp'\left[\frac{1}{p'-ip}+\frac{1}{p'+ip}\right]\Phi^-(-ip')\mathrm{disc}B(ip')=0 . \tag{72}$$

It is easily checked that this expression has no cuts in the lower half plane despite the appearance of $B(p)$.

To obtain an equation for the function discΦ_2, we now write $\Phi^-=\Phi_1+\Phi_2$ in Eq. (72) and evaluate the discontinuity of the equation across the upper cut. Both $\Phi_2(p)$ and $B(p)$ have nonvanishing discontinuities. Using the definitions in Eqs. (56) and (70) and noting in particular that

$$\mathrm{disc}[B(p)\Phi^-(p)]=\Phi^-(p)\mathrm{disc}B(p)+B(p)\mathrm{disc}\Phi^-(p)-\mathrm{disc}B(p)\mathrm{disc}\Phi^-(p) , \tag{73}$$

where all functions are evaluated on the left-hand edge of the cut, we find that

$$\left[\frac{d}{dp}+iB(p)-i\,\mathrm{disc}B(p)\right]\mathrm{disc}\Phi_2(p)+i\,\mathrm{disc}B(p)[\Phi_2(p)-\Phi_2(-p)]+i\,\mathrm{disc}B(p)[\Phi_1(p)-\Phi_1(-p)]=0 ,$$
$$p=i|p|-\epsilon,\ \ m\le|p|<\infty . \tag{74}$$

Finally, using Eq. (55), the identity (Plemelj[8])

$$\Phi_2(p)=\frac{1}{2\pi i}\int_{im}^{i\infty}\frac{dp'}{p'-p}\mathrm{disc}\Phi_2(p')$$
$$=\tfrac{1}{2}\mathrm{disc}\Phi_2(p)+\frac{P}{2\pi i}\int_{im}^{i\infty}\frac{dp'}{p'-p}\mathrm{disc}\Phi_2(p') ,$$
$$p=i|p|-\epsilon,\ \ m\le|p|<\infty,\ \ \epsilon\to 0+ \tag{75}$$

and the explicit expressions for $B(p)$ and disc$B(p)$, Eqs. (8) and (70), we find that

$$\left[\frac{d}{d|p|}+\frac{1}{4}\frac{\alpha M}{|p|^2+p_0{}^2}\right]\mathrm{disc}\Phi_2(i|p|)-\frac{\alpha}{\pi}\frac{|p|(|p|^2-m^2)^{1/2}}{|p|^2+p_0{}^2}P\int_m^\infty\frac{d|p'|}{|p'|^2-|p|^2}\mathrm{disc}\Phi_2(i|p'|)$$
$$=i\alpha\frac{(|p|^2-m^2)^{1/2}}{|p|^2+p_0{}^2}[\Phi_1(i|p|)-\Phi_1(-i|p|)],\ \ m\le|p|<\infty . \tag{76}$$

This is an inhomogeneous singular integral equation for discΦ_2 with a "driving term" which involves the known function Φ_1. In the next section we will study the iterative solution of this equation.

E. Iterative solution for discΦ_2

It will be convenient to make the change of variables $|p|=mx$, $1\le x<\infty$, in Eq. (76), and to write discΦ_2 as

$$\mathrm{disc}\Phi_2(i\,|p\,|)=e^{\zeta(x)}\Psi(x)\ , \tag{77}$$

$$\zeta(x)=\frac{\alpha}{2v}\tan^{-1}\frac{x_0}{x},\ \ x_0=p_0/m\ . \tag{78}$$

This substitution eliminates the second term in Eq. (76), and leaves us with the equation

$$\frac{d\Psi}{dx}(x)-\frac{\alpha}{\pi}\frac{x\sqrt{x^2-1}}{x^2+x_0{}^2}e^{-\zeta(x)}P\int_1^\infty\frac{dx'}{x'^2-x^2}e^{\zeta(x')}\Psi(x')=i\alpha\frac{\sqrt{x^2-1}}{x^2+x_0{}^2}e^{-\zeta(x)}[\Phi_1(imx)-\Phi_1(-imx)]\ . \tag{79}$$

This equation can be solved by iteration. We let

$$\frac{d\Psi^{(0)}}{dx}=i\alpha\frac{\sqrt{x^2-1}}{x^2+x_0{}^2}e^{-i\zeta(x)}[\Phi_1(imx)-\Phi_1(-imx)]\ , \tag{80}$$

and define functions $\Psi_2^{(n)}$ recursively by

$$\frac{d\Psi^{(n)}}{dx}=\frac{\alpha}{\pi}\frac{x\sqrt{x^2-1}}{x^2+x_0{}^2}e^{-\zeta(x)}P\int_1^\infty\frac{dx'}{x'^2-x^2}e^{\zeta(x')}\Psi^{(n-1)}(x')\ , \tag{81}$$

$$\Psi^{(n)}(x)=-\int_x^\infty\frac{d\Psi^{(n)}}{dx'}dx'\ . \tag{82}$$

Then

$$\Psi(x)=\sum_{n=0}^{\infty}\Psi^{(n)}(x)\ . \tag{83}$$

We note that each successive term in Eq. (83) involves an extra overall factor of α/π. However, Eq. (83) is not a simple power series in α because of the α dependence introduced by the factors $e^{\pm\zeta}$ and by Φ_1. The second relevant parameter in Eq. (79) is $x_0{}^2/x^2$. This is of order v^2 for nonrelativistic quarks.

The function $[\Phi_1(imx)-\Phi_1(-imx)]$ is given from Eq. (54) by

$$\Phi_1(imx)-\Phi_1(-imx)=\frac{x}{\pi}\int_{-x_0}^{x_0}\frac{dx'}{x'^2+x^2}\mathrm{disc}\Phi_1(mx')\ . \tag{84}$$

We will be primarily interested in the behavior of $\Psi(x)$ for $x\to\infty$. In this limit,

$$\Phi_1(imx)-\Phi_1(-imx)\underset{x\gg1}{\to}\frac{1}{\pi x}\int_{-x_0}^{x_0}dx'\mathrm{disc}\Phi_1(mx')\ , \tag{85}$$

where the corrections are of relative order $x_0{}^2/x^2$. The integral in Eq. (85) can be identified through Eqs. (61) and (64) with $-u_1'(0)/m=-\sqrt{4\pi}\psi_1(0)/m$, where

$$\psi(r)=[u_1(r)+u_2(r)]/\sqrt{4\pi}r\ . \tag{86}$$

The same integral was encountered in Sec. III C in the evaluation of $\psi'(0)$, the derivative of the one-dimensional Coulomb wave function at the origin. To avoid confusion, we define $C=\sqrt{4\pi}\psi_1(0)$. Then

$$\Phi_1(imx)-\Phi_1(-imx)\underset{x\gg1}{\to}-C/\pi mx\ , \tag{87}$$

where

$$C=-\int_{-p_0}^{p_0}dp\ \mathrm{disc}\Phi_1(p)=\int_{-p_0}^{p_0}dp\ \chi(p)=p_0e^{\pi\eta/2}\,|\,\Gamma(1-i\eta)\,|\,[1+\tfrac{1}{4}\alpha^2+\tfrac{1}{4}v^2+\cdots]\ . \tag{88}$$

The first approximation for $\Psi(x)$ for $x_0{}^2/x^2\ll1$ and $\alpha^2\ll1$ is given by

$$\frac{d\Psi^{(0)}}{dx}\sim-\frac{i\alpha C}{\pi m}\frac{\sqrt{x^2-1}}{x^3}\left[1-\frac{\alpha M}{2mx}+O(\alpha^2,x_0{}^2/x^2)\right]\ , \tag{89}$$

$$\Psi^{(0)}\sim\frac{i\alpha C}{2\pi m}\left\{\frac{\sqrt{x^2-1}}{x^2}+\tan^{-1}\frac{1}{\sqrt{x^2-1}}-\frac{\alpha M}{6m}\left[\frac{\sqrt{x^2-1}}{x^3}+\frac{1}{x(x+\sqrt{x^2-1})}\right]+\cdots\right\}\underset{x\gg1}{\sim}\frac{\alpha}{\pi}\frac{iC}{mx}\left[1-\frac{1}{3x^2}+\cdots-\frac{\alpha M}{8mx}+\cdots\right]\ . \tag{90}$$

Iterating once, we find that $d\Psi^{(1)}/dx$ is given to order α^2 by

$$\frac{d\Psi^{(1)}}{dx}=\left[\frac{\alpha}{\pi}\right]^2\frac{iC}{2m}\frac{\sqrt{x^2-1}}{x}P\int_1^\infty\frac{dx'}{x'^2-x^2}\left[\frac{\sqrt{x'^2-1}}{x'^2}+\tan^{-1}\frac{1}{\sqrt{x'^2-1}}\right]\underset{x\gg1}{\sim}-\left[\frac{\alpha}{\pi}\right]^2\frac{iC}{m}\frac{1}{x^2}\left[\ln x+\ln 2-\frac{\pi}{4}+O(x^{-2}\ln x)\right]\ , \tag{91}$$

hence that

$$\Psi^{(1)}\underset{x\gg1}{\sim}\left[\frac{\alpha}{\pi}\right]^2\frac{iC}{m}\frac{1}{x}\left[\ln x+1+\ln 2-\frac{\pi}{4}+O(x^{-2}\ln x)\right]\ . \tag{92}$$

It is relatively straightforward to show by further iteration that the leading term in $\Psi^{(n)}$ in powers of $\ln x$ behaves as

$[(\alpha/\pi)\ln x]^n/n!$ These terms sum in Eq. (83) to the simple power $x^{(\alpha/\pi)-1}$, a result which suggests that Ψ varies asymptotically as $x^{\nu-1}$ with $\nu\approx\alpha/\pi$,

$$\Psi \underset{x\gg 1}{\sim} Ax^{\nu-1}\left[1+\frac{c_1}{x}+\cdots\right]. \tag{93}$$

To show that this is the case and determine ν and A, we take $x\gg 1$ in Eq. (79) and write the resulting equation as

$$\frac{d\Psi}{dx}-\frac{\alpha}{\pi}P\int_1^\infty \frac{dx'}{x'^2-x^2}Ax'^{\nu-1}\left[1+\frac{c_1}{x}'+\cdots\right]$$
$$=-\frac{\alpha}{\pi}\frac{iC}{m}\frac{1}{x^2}+\frac{\alpha}{\pi}P\int_1^\infty\frac{dx'}{x'^2-x^2}\left\{e^{\zeta(x')}[\Psi(x')-Ax'^{\nu-1}\cdots]+[e^{\zeta(x')}-1]Ax'^{\nu-1}\cdots\right\}+\cdots . \tag{94}$$

If Ψ has the asymptotic expansion in Eq. (93) with $\nu<1$, the integral on the right-hand side of Eq. (94) converges without the factor $(x'^2-x^2)^{-1}$ in the integrand, and the result is proportional in leading order in x to x^{-2}. The principal-value integral on the left-hand side can be evaluated exactly. Keeping only the leading powers in x and replacing $d\Psi/dx$ by its asymptotic expansion, we obtain the equation

$$A\left[\nu-1+\frac{\alpha}{2}\cot\frac{\pi\nu}{2}\right]x^{\nu-2}-A\frac{\alpha}{\pi\nu}\frac{1}{x^2}$$
$$=-\frac{\alpha}{\pi}\frac{iC}{m}\frac{1}{x^2}+O\left[\frac{\alpha}{\pi}\frac{A}{x^2}\right], \tag{95}$$

The coefficients of $x^{\nu-2}$ and x^{-2} in Eq. (95) must vanish independently. We therefore conclude that ν is determined *exactly* by the equation

$$\nu=\nu^2+\frac{\alpha}{2}\nu\cot\frac{\pi\nu}{2}. \tag{96}$$

Expanding the cotangent for $\pi\nu/2$ small, we find that

$$\nu=\frac{\alpha}{\pi}+\nu^2-\frac{\alpha}{\pi}\frac{\pi^2\nu^2}{12}-\frac{\alpha}{\pi}\frac{\pi^4\nu^4}{720}+\cdots$$
$$=\frac{\alpha}{\pi}+\frac{\alpha^2}{\pi^2}+\left[2-\frac{\pi^2}{12}\right]\frac{\alpha^3}{\pi^3}+O\left[\frac{\alpha^4}{\pi^4}\right]. \tag{97}$$

We find similarly that

$$A=\frac{\alpha}{\pi}\frac{iC}{m}[1+O(\alpha)], \tag{98}$$

and in a separate calculation in which we retain higher-order terms in x^{-1}, that the coefficient c_1 in Eq. (93) is

$$c_1=-\frac{\alpha M}{8m}[1+O(\alpha)]. \tag{99}$$

These results agree to the relevant order with the iterative results given in Eqs. (90) and (91),

$$\Psi(x)\underset{x\gg 1}{\sim}\frac{\alpha}{\pi}\frac{iC}{mx}\left[1+\frac{\alpha}{\pi}\left[1+\ln 2-\frac{\pi}{4}\right]+\frac{\alpha}{\pi}\ln x\right.$$
$$\left.-\frac{\alpha M}{8mx}+\cdots\right]. \tag{100}$$

We see, in fact, that

$$A=\frac{\alpha}{\pi}\frac{iC}{m}\left[1+\frac{\alpha}{\pi}\left[1+\ln 2-\frac{\pi}{4}\right]+O(\alpha^2,\nu^2)\right]. \tag{101}$$

We conclude that the behavior of $\Psi(x)$ for $x\to\infty$ is given by the asymptotic expansion in Eq. (93) with ν, c_1, and A given by Eqs. (97), (99), and (101). The behavior of $\Psi(x)$ for $x\sim 1$ is given by the iterative series in Eq. (83). For $\alpha^2,\nu^2\ll 1$, the first two terms in the series, Eqs. (90) and (91), should give a satisfactory approximation to the exact result. The function discΦ_2 is given by Eq. (77).

F. Behavior of $\psi(r)$ for $r\to 0,\infty$

We consider finally the behavior of the space wave function $\psi(r)=[u_1(r)+u_2(r)]/\sqrt{4\pi}r$ for $r\to 0$. The behavior of $u_1(r)/r$ is easily determined from Eq. (64),

$$u_1(r)/r=C(1-\tfrac{1}{2}\alpha mr+\cdots), \tag{102}$$

where C is given in Eq. (88), and we have omitted corrections of orders α^2,υ^2 in the second term.

To determine the behavior of $u_2(r)/r$ for $r\to 0$, we use Eq. (62) and the results just obtained for discΦ_2 for $|p|\to\infty$,

$$u_2(r)/r=-i\int_m^\infty d|p|\,e^{-|p|r}\text{disc}\Phi_2(i|p|)$$
$$=-im\int_1^\infty dx\,e^{-mrx}e^{\zeta(x)}\Psi(x)$$
$$\underset{mr\ll 1}{\sim}\frac{\alpha}{\pi}C\int_1^\infty dx\,e^{-mrx}x^{\nu-1}+O(\alpha^2C)$$
$$=\frac{\alpha}{\pi}C\Gamma(\nu)(mr)^{-\nu}-\frac{\alpha}{\pi\nu}C+O(\alpha mr,\alpha^2)C$$
$$\approx C[(mr)^{-\nu}-1]+O(\alpha,mr,\alpha/\pi)C. \tag{103}$$

In the last step, we have used the relation $\nu\approx\alpha/\pi$. Combining Eqs. (102) and (103), we find that

$$\psi(r)\underset{mr\ll 1}{\sim}C(mr)^{-\nu}+O(mr,\alpha/\pi)C. \tag{104}$$

A more complete calculation using the leading approximation to Ψ given in Eq. (90) gives

$$u(r)/r=C\left[1-\frac{\alpha}{\pi}\ln mr-\left[\gamma+\frac{\pi}{4}-\ln 2\right]\frac{\alpha}{\pi}+O(\alpha^2,\alpha mr)\right], \tag{105}$$

where $\gamma=0.5772\ldots$ is Euler's constant. The logarithm in Eq. (105) can be identified with the $O(\alpha)$ term in the expansion of $(mr)^{-\nu}$, and we conclude that

$$\psi(r)\underset{r\to 0}{\longrightarrow}\frac{1}{\sqrt{4\pi}}C\left[(mr)^{-\nu}-\left[\gamma+\frac{\pi}{4}-\ln 2\right]\frac{\alpha}{\pi}+O(\alpha^2,\alpha mr)\right]. \tag{106}$$

We have checked this result by showing that it satisfies a position-space version of the Salpeter equation which we will discuss elsewhere.

We note for completeness that $\psi(r)$ has the standard plane-wave normalization for $r\to\infty$,

$$\psi(r)\underset{r\to\infty}{\sim}\frac{1}{\sqrt{4\pi}r}\sin\left\{p_0r+\eta\ln 2p_0r+\arg\Gamma(1-i\eta)+\eta\ln(M/2m)+\frac{\alpha}{2}\ln[\sqrt{M+2m}-\sqrt{M-2m}\,]-\frac{\alpha}{2}\ln[\sqrt{M+2m}+\sqrt{M-2m}\,]\right\}$$
$$\sim\frac{1}{\sqrt{4\pi}r}\sin[p_0r+\eta\ln 2p_0r+\arg\Gamma(1-i\eta)+O(\alpha v^2)]\;. \tag{107}$$

G. Comments

The mild divergence of $\psi(r)$ for $r\to 0$ is a consequence of the static Coulomb singularity in the Salpeter equation. The divergence is not present in the complete Bethe-Salpeter-Schwinger wave function because of radiative and retardation effects which modify the Coulomb singularity and the wave function at distances $r\lesssim m^{-1}$. While it would be useful to calculate the radiative corrections beginning with the reduction of the Bethe-Salpeter-Schwinger equation to Eq. (1), we have not done so. We can nevertheless use the static solution for $mr\gtrsim 1$ where the radiative corrections and the divergent function $\psi_2(r)=u_2(r)/\sqrt{4\pi}r$ are small. For example, if we take $\alpha=0.25$, a value appropriate for charmonium, the logarithmic term in Eq. (105) is less than 20% of the leading term for $mr>0.08$. The entire contribution from the "extra" function $\psi_2(r)$ is less than 2% of $\psi_1(r)$ for $mr=1$, and decreases exponentially for mr large. The complete wave function is therefore well approximated for $mr\gtrsim 1$ by $\psi_1(r)$, and this function is known exactly. (In fact, for α and v not too large, it is essentially the standard S-state Coulomb wave function[5] evaluated for the relativistic velocity of the quarks.)

The present results are also of interest for quarkonium systems in which the Coulomb potential in Eq. (1) is supplemented by a confining interaction between the quarks. This is usually taken as nonsingular at $r=0$, and has a characteristic scale of variation which is large compared to m^{-1}. For r smaller than this scale but larger than m^{-1}, the behavior of the wave function is determined primarily by the static Coulomb interaction, and $\psi(r)$ is of the form determined above, but with the overall normalization changed by a factor which can be related to the inverse density of states.[2,3] The normalization of the radiatively corrected wave function for $r\lesssim m^{-1}$ will be changed by the same factor. We have used this observation and known perturbative results for the gluonic radiative corrections in the process $e^+e^-\to q\bar{q}$ to estimate the radiative corrections to the leptonic widths of the $^3S_1$ states in charmonium and b-quarkonium.[2]

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[1]E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951); J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. 37, 452 (1951); 37, 455 (1951).

[2]B. Durand and L. Durand, Phys. Rev. D 25, 2312 (1982); Phys. Lett. 113B, 338 (1982).

[3]B. Durand and L. Durand, University of Wisconsin Report No, MAD/TH-62 (unpublished).

[4]This procedure is a generalization of Laplace's method for the solution of ordinary differential equations with coefficients linear in x; see, e.g., L. D. Landau and E. M. Lifschitz, *Quantum Mechanics* (Pergamon, New York, 1965), Appendix A, or E. Goursat, *Differential Equations* (Dover, New York, 1945), Sec. 46. BD has also used this method to solve Eq. (3) for a one-dimensional linear potential [B. Durand, University of Wisconsin Report No. MAD/TH-90 (unpublished)].

[5]See, e.g., M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972), Sec. 14.1.

[6]The general theory of singular integral equations is treated in N. I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, Holland, 1953). See also W. Pogorzelski, *Integral Equations and Their Applications* (Pergamon, New York, 1966). Unfortunately Eq. (40) is not of a standard type.

[7]This representation is quite general. See Muskhelishvili, Ref. 6, Secs. 26 and 27. For an extension to the representation of generalized functions (distributions), see H. J. Bremermann and L. Durand, J. Math. Phys. 2, 240 (1961).

[8]N. I. Muskhelishvili, Ref. 6, Sec. 17.