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Multimonopoles in arbitrary gauge groups and the complete SU(2) two-monopole system

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We show how to extract exact expressions for the multimonopole gauge fields and Higgs field from their integral representation in the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction, for arbitrary gauge groups. In particular, this allows us to construct the complete two-monopole solution for the SU(2) gauge group.

I. INTRODUCTION

Recently, a new method was formulated for obtaining multimonopole solutions for arbitrary gauge groups in the Prasad-Sommerfield limit. This method, originally invented by Atiyah, Drinfeld, Hitchin, and Manin for instanton solutions was adopted by Nahm to the monopole case.[1]

Even though the relation between this method and a previous one by Atiyah and Ward[2,3] was elucidated at an abstract level,[4] the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction seems to present several practical advantages, namely, (a) the regularity of solutions is automatic, (b) it generalizes easily to gauge groups beyond SU(2), and (c) the construction of Green's functions for propagation in the monopole background is immediate.

In this paper, we start by working out in detail the procedure leading to the gauge fields and Higgs field at all points in space of the system of two separated SU(2) monopoles. Our results generalize earlier work on this subject,[5] where the Higgs field was found on the axis connecting the two monopoles. This is also the most general configuration for the two-monopole SU(2) case, i.e., the $(4k-1)$-dimensional parameter space is exhausted (where, for $k=2$, the distance between the monopoles is the only nontrivial parameter).

At the final stage in this procedure, we work in complete generality: We show that the integrals over the auxiliary variable z of the ADHMN construction, required for extracting the gauge and Higgs fields, can all be evaluated exactly for any gauge group and any number of monopoles, and we give compact expressions for them. These expressions, together with known exact solutions[6] to the differential equations arising in the ADHMN framework, in principle generate exact solutions to systems with more separated monopoles.

The results of this paper are relevant to examination of monopole dynamics[7] and of the motion of particles in a background of monopoles (thanks to the simple expression for Green's functions in terms of objects defined in the ADHMN construction).

II. MAGNETIC MONOPOLES AND THE ADHMN CONSTRUCTION

Consider a Yang-Mills-Higgs Lagrangian in Minkowski space of the form

$$-\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\tfrac{1}{2}D_\mu\phi^a D^\mu\phi^a \ , \tag{1}$$

where, as usual

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu$$

and ϕ^a is a scalar field in the adjoint representation of the gauge group. We are interested in static configurations with $A^a_0=0$ which give finite values (local extremes) for the energy of the system. These solutions will satisfy

$$D_i\phi^a=\pm\tfrac{1}{2}\epsilon_{ijk}F^a_{jk} \tag{2}$$

(the Bogomol'ny equation), together with boundary conditions for the eigenvalues of $\phi(\infty)$ [given here for SU(2), but easily generalized to other groups[1]]

$$|\phi(r)| \underset{r\to\infty}{\sim} c-\frac{r}{2k}+O(r^{-2}) \ ,$$
$$r=(x_1{}^2+x_2{}^2+x_3{}^2)^{1/2} \tag{3}$$

where $c>0$ is some constant and k is a positive integer called the topological charge. These solutions correspond to configurations of monopoles with total magnetic charge equal to k in appropriate units.

In the ADHMN approach, the construction of monopole solutions to the Bogomol'ny equation with topological charge k is translated to the following problem [shown here for SU(n), similar for Sp(n), O(n)] (Ref. 8).

Consider the following matrix differential operator:

$$\Delta=i\partial_z\otimes\mathbb{1}_{2k}+x_ie_i\otimes\mathbb{1}_k+ie_i\otimes T_i(z) \ , \tag{4}$$

where z is an auxiliary real variable (not related to the coordinates), e_i's are quaternions ($e_i=-i\sigma_i$), and the $T_i(z)$'s are $k\times k$ matrices. Requiring that $\Delta^\dagger\Delta$ commute with quaternions forces $T_i(z)$ to be anti-Hermitian and to obey the nonlinear equations

$$\frac{d}{dz}T_i(z)=\epsilon_{ijk}T_j(z)T_k(z), \quad i=1,2,3 \ . \tag{5}$$

Particular cases of these equations have been studied extensively in mathematical literature as Toda lattice equations or Lax pairs. Given such a set of T_i's, we are looking for solutions to $\Delta^\dagger v=0$, normalizable in the sense that $\int_{z_-}^{z_+} v^\dagger v\, dz<\infty$; here the end points of the z integration are adjusted (for a more detailed exposition, see Ref. 1), so

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that we obtain n orthonormal solutions v_α: $\int_{z_-}^{z_+} v_\alpha^\dagger v_\beta dz = \delta_{\alpha\beta}$.

From the set of v_α's, the gauge potentials are now readily constructed:

$$[A_i]_{\alpha\beta} = \int_{z_-}^{z_+} v_\alpha^\dagger \frac{\partial}{\partial x_i} v_\beta dz , \quad \phi_{\alpha\beta} = i \int_{z_-}^{z_+} z v_\alpha^\dagger v_\beta dz \ . \tag{6}$$

Notice that performing an x-dependent change of basis for the v_α's, $v_\alpha \to v'_\alpha = v_{\alpha'} U_{\alpha'\alpha}$ $(U^\dagger = U^{-1})$ simply induces a gauge transformation on the potentials:

$$A_i \to A'_i = U^{-1} A_i U + U^{-1} \partial_i U , \quad \phi \to \phi' = U^{-1} \phi U \ . \tag{7}$$

III. EXPLICIT SOLUTION OF TWO SEPARATED MONOPOLES IN SU(2)

A. Preliminaries

In this case,[5] the T_i's are 2×2 matrices and the solution to Eq. (5), up to an overall orientation-fixing rotation, is

$$T_i(z) = -i f_i(z) \sigma_i(z)/2 , \tag{8}$$

where the functions $f_i(z)$, $i=1,2,3$ satisfy

$$\frac{d}{dz} f_1(z) = f_2(z) f_3(z) \quad (\text{cyc}) \ . \tag{9}$$

Note that Eq. (9) implies $f_i^2 - f_j^2 = c_{ij} = \text{const}$.

The equation $\Delta^\dagger v = 0$ takes the form

$$\left[\mathbb{1} \frac{d}{dz} + \mathscr{X} + \mathscr{T} \right] v = 0 , \quad \mathscr{X} = \begin{bmatrix} x_3 & x_1 - i x_2 & 0 & 0 \\ x_1 + i x_2 & -x_3 & 0 & 0 \\ 0 & 0 & x_3 & x_1 - i x_2 \\ 0 & 0 & x_1 + i x_2 & -x_3 \end{bmatrix} , \quad \mathscr{T} = \tfrac{1}{2} \begin{bmatrix} f_3 & 0 & 0 & f_1 - f_2 \\ 0 & -f_3 & f_1 + f_2 & 0 \\ 0 & f_1 + f_2 & -f_3 & 0 \\ f_1 - f_2 & 0 & 0 & f_3 \end{bmatrix} . \tag{10}$$

Of the four independent column vectors v_i solving this equation, we require that only two be normalizable and that the resulting [via Eq. (6)] Higgs field be traceless; these requirements force the limits of z integration to be symmetric about zero, $-z_s \le z \le +z_s$, and the functions $f_i(z)$ to have poles at $\pm z_s$.

Up to a scale transformation, the $f_i(z)$'s with appropriate poles are given in terms of elliptic functions by

$$f_1(z) = \frac{1}{\text{cn}(z/k',k)} , \quad f_2(z) = \frac{\text{dn}(z/k',k)}{k'\text{cn}(z/k',k)} ,$$
$$f_3 = \frac{\text{sn}(z/k',k)}{\text{cn}(z/k',k)} \quad [k' = (1-k^2)^{1/2}] \tag{11}$$

and $0 \le k \le 1$ is the only nontrivial parameter [it, or rather, $\delta = k/(1-k^2)^{1/2}$, is a measure of the distance between the two monopoles; $k=0$ reproduces the axisymmetric case]. We note that f_1, f_2, f_3 diverge at

$$z = \pm k' K \quad \left[K = \int_0^{\pi/2} \frac{dy}{(1-k^2 \sin^2 y)^{1/2}} \right] ,$$

as required.

B. Solving $\Delta^\dagger v = 0$

Once the operator Δ is known, the next task is to solve $\Delta^\dagger v = 0$.[9] Since $\mathscr{X}, \mathscr{T}$ are Hermitian, one can equally well solve $\Delta v = 0$ $[(d/dz - \mathscr{X} - \mathscr{T}) v = 0]$: If w is a 4×4 matrix of linearly independent solutions to $(d/dz - \mathscr{X} - \mathscr{T}) w = 0$, then $(w^\dagger)^{-1}$ satisfies $(d/dz + \mathscr{X} + \mathscr{T})(w^\dagger)^{-1} = 0$.

To solve $\Delta v = 0$, we use the ansatz

$$v = (1 + i e_m u_m) \otimes \tilde{v} \ . \tag{12}$$

Here u_m is a z-independent unit vector, to be determined later, and $\tilde{v}$ is a two-element column. Note that this 4×2 ansatz, for any given u_m, represents one rather than two candidate solutions, since the two columns are linearly dependent. Thus, for the ansatz to give all four solutions, we expect to find four different unit vectors u.

The equation $\Delta v = 0$ now takes the form

$$\left[i \frac{d}{dz} + i e_m \otimes T_m + e_m x_m \otimes \mathbb{1} \right] [(1 + i e_m u_m) \otimes \tilde{v}] = 0 , \tag{13}$$

$$\Longrightarrow \left[i \frac{d}{dz} + i e_m \otimes T_m(z) + e_m x_m \otimes \mathbb{1} - e_m u_m \frac{d}{dz} + T_m u_m - \epsilon_{ijk} e_i T_j u_k - i x_m u_m + i \epsilon_{ijk} e_i x_j u_k \right] \tilde{v} = 0 ,$$

$$\Longrightarrow \begin{cases} \left[i \dfrac{d}{dz} + T_m u_m - i x_m u_m \right] \tilde{v} = 0 , & \text{(14a)} \\[2ex] \left[i T_m(z) - \epsilon_{mjk} T_j(z) u_k - u_m \dfrac{d}{dz} + x_m + i \epsilon_{mjk} x_j u_k \right] \tilde{v} = 0 \ . & \text{(14b)} \end{cases}$$

Defining vectors ξ, ξ', so that (u, ξ, ξ') constitutes a right-handed orthonormal set, we can replace Eq. (14b) by its projections on u, ξ', and ξ. This reproduces Eq. (14a) and an additional algebraic equation

$$[\xi_{+m}(T_m - ix_m \mathbb{1})]\tilde{v}=0, \quad \xi_{\pm m}=\xi_m \pm i\xi'_m \ . \tag{14c}$$

The remarkable fact which makes this ansatz work is that the equation $\det[\xi_{+m}(T_m - ix_m \mathbb{1})]=0$ is actually z independent:

$$\det[\xi_{+m}(T_m - ix_m \mathbb{1})] = -(\xi_+ \cdot x)^2 + \tfrac{1}{12}\left[\xi_{+3}{}^2\left[\frac{-1-k'^2}{k'^2}\right] + \xi_{+1}{}^2\left[\frac{-1+2k'^2}{k'^2}\right] + \xi_{+2}{}^2\left[\frac{2-k'^2}{k'^2}\right]\right] = 0 \ . \tag{15}$$

Using a null-vector parametrization of ξ_+,

$$\xi_+ = \alpha\left[\frac{1+\zeta^2}{2}, \frac{1-\zeta^2}{2i}, i\zeta\right],$$

Eq. (15) becomes

$$\zeta^4\left[\frac{x_+{}^2}{4}+\frac{\delta^2}{16}\right] + \zeta^3(ix_+x_3) + \zeta^2\left[\tfrac{1}{2}x_+x_- - x_3{}^2 - \frac{2+\delta^2}{8}\right] + \zeta(ix_-x_3) + \left[\frac{x_-{}^2}{4}+\frac{\delta^2}{16}\right] = 0 \ , \tag{16}$$

$$x_\pm = x_1 \pm ix_2 \ .$$

The same equation occurs also in the Atiyah-Ward construction[10] (notice, however, that differences of convention lead to a scale factor of $\frac{1}{2}$ there, plus $x_1 \rightleftarrows x_2$). This equation gives in general four different values of ζ, each of which corresponds to a unique vector u:

$$u = \frac{1}{|\zeta|^2+1}(i(\zeta-\zeta^*), -(\zeta+\zeta^*), |\zeta|^2-1) \ . \tag{17}$$

Thus, for each such u, we have an ansatz

$$v = \begin{bmatrix} 1+u_3 \\ u_1+iu_2 \end{bmatrix} \otimes \begin{bmatrix} i/2\xi_{+1}f_1 + \frac{1}{2}\xi_{+2}f_2 \\ -i\xi_+\cdot x - \frac{i}{2}\xi_{+3}f_3 \end{bmatrix} \phi_u \tag{18}$$

which solves $\Delta v = 0$, provided ϕ_u satisfies the first-order equation

$$\frac{d\phi_u}{dz} = \left[x\cdot u - \frac{1}{2}\frac{d}{dz}\ln\left[\frac{i}{2}\xi_{+1}f_1 + \tfrac{1}{2}\xi_{+2}f_2\right] - (\xi_+\cdot x)\left[\frac{f_1u_1 - if_2u_2}{f_1\xi_{+1} - if_2\xi_{+2}}\right]\right]\phi_u \ . \tag{19}$$

This yields

$$\phi_u = \left[\frac{i}{2}\xi_{+1}f_1 + \tfrac{1}{2}\xi_{+2}f_2\right]^{-1/2} e^{z(x\cdot u)}\exp\left[-(\xi^+\cdot x)\int^z \frac{f_1u_1 - if_2u_2}{f_1\xi_{+1} - if_2\xi_{+2}}dz'\right] C(x,\xi_+) \ . \tag{20}$$

Here, $C(x,\xi_+)$ is a normalization constant to be fixed by requiring

$$\int_{-z_s}^{z_s} v^\dagger v \, dz = 1 \ .$$

The integral in Eq. (20) can be evaluated in terms of standard Θ functions H, H_1, and Jacobi's Z function (their implied dependence on the parameter k is suppressed):

$$\int^z \frac{f_1u_1 - if_2u_2}{f_1\xi_{+1} - if_2\xi_{+2}}dz' = \frac{k'u_1 + iu_2 - ik}{k'\xi_{+1} + i\xi_{+2}}z + \frac{ik'\mathrm{dn}\alpha}{k(k'\xi_{+1} + i\xi_{+2})\mathrm{sn}\alpha\,\mathrm{cn}\alpha}\left[\ln\frac{H(z/2k'-\alpha)H_1(z/2k'-\alpha)}{H(z/2k'+\alpha)H_1(z/2k'+\alpha)} + \frac{2z}{k'}Z(\alpha)\right], \tag{21}$$

where

$$\mathrm{sn}^2\alpha \equiv \frac{i(k\xi_{+2} + k'\xi_{+3})}{k(k'\xi_{+1} + i\xi_{+2})}$$

is a convenient parametrization.

We have thus found a 4×4 matrix w of linearly independent solutions to $\Delta v=0$. We note in passing that these are non-normalizable solutions, as expected by construction in the ADHMN approach, in which

$$\Delta^\dagger\Delta = -\frac{\partial^2}{\partial z^2} + (T_i - ix_i)^\dagger(T_i - ix_i) \tag{22}$$

is a positive-definite operator. The object of interest, however, is $(w^\dagger)^{-1}$, which indeed has two normalizable solutions as expected. To extract these columns, one must find the residues of all poles of $(w^\dagger)^{-1}$ at $z=\pm z_s$, and form linear combinations which make the residues vanish, a straightforward but not illuminating task.

Already at this level, the results have been checked against previous calculations,[5] done for the special case of the axis on which the monopoles lie, $x_1 = x_3 = 0$, and indeed the results agree.

IV. EXPRESSIONS FOR THE FIELDS

The final stage, once the solution to $\Delta^\dagger v=0$ has been obtained, is to find $A^i_{\alpha\beta}$ and $\phi_{\alpha\beta}$ via Eq. (6). We would like to extract analytic expressions for the potentials in our nonaxisymmetric case, by exactly performing the required integrations over the auxiliary variable.

In fact, we obtain compact results examining the most general case in the ADHMN construction; thus, these results hold for general gauge groups and arbitrary monopole numbers.

In particular, we are interested in evaluating the indefinite integrals

$$\int v'^\dagger v\,dz,\quad \int v'^\dagger vz\,dz,\quad \int v'^\dagger \frac{\partial}{\partial x_i} v\,dz\ . \tag{23}$$

Here v and v' are solutions to $\Delta^\dagger v=0$, with Δ any operator satisfying Eqs. (4) and (5). Again, we abbreviate

$$\mathscr{X}\equiv\vec{x}\cdot\vec{\sigma}\otimes\mathbb{1},\quad \mathscr{T}\equiv -e_i\otimes T_i(z)\ . \tag{24}$$

For any of these integrands, generically represented as $v'^\dagger A v$, we expect that $\int v'^\dagger A v\,dz$ has the form $v'^\dagger B v$, where B is some matrix operator. We must then have

$$\begin{aligned} v'^\dagger A v &= \frac{d}{dz}(v'^\dagger B v)\\ &= v'^\dagger(-\mathscr{X}-\mathscr{T})Bv + v'^\dagger\frac{dB}{dz}v + v'^\dagger B(-\mathscr{X}-\mathscr{T})v\\ &= v'^\dagger\left[\frac{dB}{dz}-(\mathscr{X}+\mathscr{T})B-B(\mathscr{X}+\mathscr{T})\right]v\ . \end{aligned} \tag{25}$$

We thus look for matrix operators B satisfying

$$\mathscr{D}(B)\equiv\frac{dB}{dz}-(\mathscr{X}+\mathscr{T})B-B(\mathscr{X}+\mathscr{T})=A \tag{26}$$

for A being $\mathbb{1}$, $z\cdot\mathbb{1}$, $\mathbb{1}\cdot\partial_i$. To find these, we first note the following fact [using Eq. (2)]:

$$\begin{aligned} \mathscr{T}^2 &= (e_ie_j)\otimes T_iT_j = -\mathbb{1}_2\otimes T_iT_i + \epsilon_{ijk}\epsilon_k\otimes T_iT_j\\ &= -\mathbb{1}_2\otimes T_iT_i - \frac{d\mathscr{T}}{dz} \end{aligned} \tag{27a}$$

and, therefore

$$\left[\mathscr{T}^2+\frac{d\mathscr{T}}{dz},\mathscr{X}\right]=0\ . \tag{27b}$$

Generalizing from the expressions for $\int v'^\dagger v\,dz$ in earlier examples, one finds in the general case

$$\int v'^\dagger v\,dz = v'^\dagger\left[\frac{1}{r^2}\mathscr{X}\mathscr{T}\mathscr{X}-\mathscr{T}\right]^{-1} v \equiv v'^\dagger Q^{-1} v\ . \tag{28}$$

Indeed,

$$\begin{aligned} \mathscr{D}(Q^{-1}) &= Q^{-1}\left[-\frac{d}{dz}\left[\frac{1}{r^2}\mathscr{X}\mathscr{T}\mathscr{X}-\mathscr{T}\right]-(\mathscr{X}+\mathscr{T})\left[\frac{1}{r^2}\mathscr{X}\mathscr{T}\mathscr{X}-\mathscr{T}\right]-\left[\frac{1}{r^2}\mathscr{X}\mathscr{T}\mathscr{X}-\mathscr{T}\right](\mathscr{X}+\mathscr{T})\right]Q^{-1}\\ &= \frac{1}{r^2}Q^{-1}[\mathscr{X}\mathscr{T}^2\mathscr{X}-\mathscr{X}\mathscr{T}\mathscr{X}\mathscr{T}-\mathscr{T}\mathscr{X}\mathscr{T}\mathscr{X}+\mathscr{T}^2r^2]Q^{-1}=Q^{-1}Q^2Q^{-1}=\mathbb{1}\ , \end{aligned} \tag{29}$$

where we have made use of Eq. (27b) and of $(1/r^2)\mathscr{X}^2=\mathbb{1}$.

Working similarly for the remaining two integrals, we obtain (using also $[(1/r^2)\mathscr{X}\mathscr{T}\mathscr{X}-\mathscr{T},\mathscr{X}]_+=0$)

$$\int v'^\dagger vz\,dz = v'^\dagger Q^{-1}\left[z+2\mathscr{X}\frac{d}{d(r^2)}\right]v\ , \tag{30a}$$

$$\begin{aligned} &\int v'^\dagger\frac{\partial}{\partial x_i}v\,dz\\ &\quad= v'^\dagger Q^{-1}\left[\frac{\partial}{\partial x_i}+\mathscr{X}\frac{z}{r^2}x_i+\mathscr{X}\frac{1}{r^2}i(\vec{x}\times\vec{\nabla})_i\right]v\ . \end{aligned} \tag{30b}$$

Upon evaluating these integrals in our case, considerable simplifications occur. In particular, the Θ functions disappear, just as was the case on the monopole axis. Our answers have been checked against earlier evaluations of ϕ on the axis,[5] and against recent results in the Atiyah-Ward framework,[11] with agreement in both cases.

Concluding, we have seen that, given solutions to the differential equations in the ADHMN construction, one can always extract the fields exactly. In particular, the SU(2) two-monopole system has been exhausted. For systems of more monopoles and higher groups, the solutions to Eqs. (5) and (10) need to be studied further for the full parameter space. Also, exact expressions for the Green's functions must be worked out.

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