PHYSICAL REVIEW D VOLUME 28, NUMBER 2 15 JULY 1983

# Interaction among systems of finite size in predictive relativistic mechanics. III. Short-range interaction and second-order terms

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(Received 4 November 1982)

We compute up to and including all the $c^{-2}$ terms in the dynamical equations for extended bodies interacting through electromagnetic, gravitational, or short-range fields. We show that these equations can be reduced to those of point particles with intrinsic angular momentum assuming spherical symmetry.

## I. INTRODUCTION

The dynamical description of a many-particle system requires the use of many variables depending on the degrees of freedom of the system and the dynamical theory used. When a system of structureless particles is clumped into subsystems whose dimensions are much smaller than the distance among them (finite-size bodies) the motion of these bodies can be considered ignoring to some extent their internal structure and the number of variables is reduced drastically.

In this series of papers we have investigated what are appropriate dynamical variables for one such description, under which conditions it holds, and the dynamical equations governing such variables within the framework of predictive relativistic mechanics (PRM).

We can now take a closer look at the problems that have been considered and the ones that have been left.

In the previous papers[1,2] (hereafter referred to as I and II) we have shown that the dynamical description of interacting spherical bodies can be given by using the mass, center of mass, velocity, and the spin of bodies as variables. Our results within the slow-motion approximation were developed only for two long-range interactions (electromagnetic and gravitational) and neglecting nonlinear terms (second order in the coupling constant). In general, however, the static nonlinear terms are of order $c^{-2}$ and they must be considered in a slow-motion approximation scheme.

In this paper we shall take up the two problems just mentioned. First, we will apply the method to a short-range interaction such as the scalar short-range interaction. Second, we will include the nonlinear terms and will study whether their inclusion requires the use of new dynamical variables or not.

The plan and contents of the paper are as follows:

In Sec. II the scalar short-range interaction is considered including the two possible theories which are characterized by a parameter $\gamma$ ($\gamma=0,1$). We first deduce, in the framework of PRM, the acceleration for a system of pointlike particles interacting via a short-range field including all the $c^{-2}$ terms. They are the starting point for the evaluation of the dynamical equations of the finite-size subsystems. For the $\gamma=1$ theory the acceleration agrees with the acceleration that can be deduced from the Bopp Lagrangian[3] which does not include quadratic terms in the coupling constant. Our results include as a particular case those of Bel and Martin[4] for a two-particle system.

In Sec. III we make use of the results for pointlike particles to evaluate the quantities $H_i$, $\vec{P}_i$, $\vec{J}_i$, and $\vec{K}_i$ for a body $i$ ($i=1,2$) and the equivalent set of variables $M_i$, $\vec{X}_i$, $\vec{V}_i$, and $\vec{S}_i$ which were defined in paper I and have the physical meaning of the mass, center of mass, velocity, and spin of the bodies. The dynamical equations for these variables are then evaluated up to the order $c^{-2}$ in the linear approximation, assuming that the range of the field, $\mu^{-1}$, is larger than the dimension of the bodies. Some peculiarities of the interaction are that although the acceleration of the center of mass of a body has dependence on the spin $\vec{S}_i$, it has no direct dependence on the "magnetic" moment, $\vec{\mu}_i=\epsilon^a e_a \vec{r}_a \times \vec{v}_a$, and it contains no spin-spin terms. These are features not shared by the electromagnetic and gravitational interactions. This may be explained by the fact that being a scalar interaction the coupling with the spin is weaker than in vector or tensor interactions.

In Sec. IV a Lagrangian formulation is given which describes the short-range interaction of finite size bodies. The construction of the Lagrangian can be made if a coordinate transformation is performed from the "physical" coordinates mentioned above to some "canonical" coordinates. A similar coordinate transformation has been studied by Bel and Martin[5] for the electromagnetic interaction, by Barker and O'Connell[6] for the gravitational interaction, and also in paper II.

Finally, in the last section we consider the nonlinear terms in the coupling constant for the three interactions considered. The main difference from the linear terms is that they give rise to terms of self-energy and their derivatives which prevent a multipolar development. However, they can be properly included in our scheme without the need for new dynamical variables. Their role is somewhat different for each interaction.

For the electromagnetic interaction the self-energy terms cancel each other. The remaining nonlinear terms are identical to those for the interaction of pointlike particles.

For the gravitational interaction the situation is different since the self-energy terms do not cancel. However, "renormalizing" the mass of the bodies at the $c^{-2}$ order by including their self-energy, we obtain results for spinless bodies that are also identical with those for pointlike particles. This result is similar to that from the parametrized-post-Newtonian (PPN) formalism for extended fluid spheres.[7]

For the scalar short-range interaction, $\gamma=1$ theory, a similar result to the gravitational case is obtained renormalizing the scalar charge. However, for the $\gamma=0$ theory the renormalization is not possible and the dynamical variables proposed are not sufficient for the description of this interaction up to the nonlinear terms.

In conclusion, the ten dynamical variables describing the mass, center of mass, velocity, and spin of finite-size spherical bodies can be used to describe the dynamical interaction among bodies under the electromagnetic, the gravitational, and the scalar short-range ($\gamma=1$) interactions including all the effects in the slow-motion approximation (in the framework of PRM).

Similar formalisms for the electromagnetic interaction, although in a quantum context, have been studied by Krajcik and Foldy[8,9] (see also references in paper I). For the gravitational interaction we should mention the works by Barker and O'Connell[10] and references therein.[11,12]

## II. THE SHORT-RANGE SCALAR INTERACTION

The short-range scalar field is defined as the solution of the Klein-Gordon equation

$$(\Box-\mu^2)\Phi(x)=4\pi\rho(x) \ , \tag{2.1}$$

where $\Box\equiv\partial^\nu\partial_\nu$ and $\mu^{-1}$ is the interaction range. We use signature $+2$, and $c=1$.

A pointlike particle with scalar charge $e$ is the source of a scalar field whose charge density is given by

$$\rho(x)=e\int_{-\infty}^{+\infty}d\tau\delta^4[x^\alpha-\phi^\alpha(\tau)] \ ,$$

where $\phi^\alpha(\tau)$ is the world line of the charge. The Klein-Gordon equation can easily be solved with this source term and under suitable boundary conditions one finds

$$\Phi(x^\alpha;\epsilon)=e\left\{\epsilon\hat{r}^{-1}+\mu^2\int_{-\infty}^{+\infty}d\tau\theta[\epsilon(\tau-\hat{\tau})]\frac{J_1(\mu[-y^2(\tau)]^{1/2})}{[-y^2(\tau)]^{1/2}}\right\} \ , \tag{2.2}$$

where

$$y^\alpha(\tau)\equiv x^\alpha-\phi^\alpha(\tau) \ ,$$

$$\hat{r}\equiv-[x^\alpha-\phi^\alpha(\hat{\tau})]\dot{\phi}_\alpha(\hat{\tau}), \quad \dot{\phi}^\alpha(\tau)\equiv\frac{d\phi^\alpha}{d\tau} \ ,$$

$\theta$ is the Heaviside step function, $\epsilon$ takes only the values $-1$ or $+1$ for the retarded or the advanced solutions, respectively, $\hat{\tau}$ is the value of the parameter $\tau$ for which the world line of the particle intersects the light cone with vertex at $x^\alpha$, and $J_1$ is the Bessel function of the first kind and of the first order.

The motion of a particle of scalar charge $e$ and mass $m$ submitted to a scalar field $\Phi(x)$ can be described by means of the dynamical equation

$$\frac{du^\alpha}{d\tau}=-\frac{e}{m}[1+\gamma me\Phi(x)]^{-1}(\eta^{\alpha\sigma}+u^\alpha u^\sigma)\frac{\partial\Phi}{\partial x^\sigma} \ , \tag{2.3}$$

where $\gamma$ can take the values 0 or 1 only. There are therefore two different dynamical theories.

As the Klein-Gordon equation is linear the field produced by a swarm of particles is the linear superposition of the fields produced by each particle and Eqs. (2) and (3) are the only ingredients needed from classical field theory in order to build a predictive model for the scalar interaction.[4]

The technique required is standard[13] and will not be explained here. It is well known that if a formal series expansion in powers of the products of the charges is used for the acceleration $\xi_a^\alpha$ such as

$$\xi_a^\alpha=\sum_{n=1}^{\infty} g^n \xi_a^{\alpha(n)} \quad (g\equiv e_a e_{a'})$$

the problem posed has only one solution which in our case turns out to be at first order

$$\xi_a^{\alpha(1)}=-\frac{e_a}{m_a}\epsilon^{a'}e_{a'}\frac{1+\mu r_{aa'}}{r_{aa'}{}^3} \times e^{-\mu r_{aa'}}[x_{aa'}^\alpha+S_{aa'}u_a^\alpha+(x_{aa'}\cdot u_{a'})u_a^\alpha] \ , \tag{2.4}$$

where $\xi_a^{\alpha(1)}$ is the four-acceleration of the $a$ particle due to the action of the $a'$ particles ($a'\neq a$), $u_a$ is its four-velocity,

$$x_{aa'}^\alpha\equiv x_a^\alpha-x_{a'}^\alpha \ ,$$

$$r_{aa'}\equiv[x_{aa'}{}^2+(x_{aa'}\cdot u_{a'})^2]^{1/2} \ ,$$

$$S_{aa'}\equiv(x_{aa'}\cdot u_a)+(u_a\cdot u_{a'})(x_{aa'}\cdot u_{a'}) \ .$$

The dot represents a scalar product in Minkowski space and $\epsilon^{a'}$ indicates summation for the $a'$ index.

This expression is, as one expects at this order, a linear superposition of two-body expressions like the one given by Bel and Martin.[4] Restoring $c$ in this expression and keeping terms only up to order $c^{-2}$, we find for the three-acceleration

$$a_a^{i(1)}=-\frac{e_a}{m_a}\epsilon^{a'}e_{a'}\frac{e^{-\mu x_{aa'}}}{x_{aa'}{}^3}\left\{\left[(1+\mu x_{aa'})\left[1-\frac{v_a{}^2}{c^2}\right]-(3+3\mu x_{aa'}+\mu^2 x_{aa'}{}^2)\frac{(\vec{x}_{aa'}\cdot\vec{v}_{a'})^2}{2c^2 x_{aa'}{}^2}\right]x_{aa'}^i - \frac{1}{c^2}(1+\mu x_{aa'})(\vec{x}_{aa'}\cdot\vec{v}_{a'})(v_a^i-v_{a'}^i)\right\} \ , \tag{2.5}$$

$$x_{aa'}\equiv|\vec{x}_{aa'}| \ .$$

Using elementary dimensional analysis the following two results can easily be proved:

(a) Only the $g$ and $g^2$ terms can contribute to order $c^{-2}$.

(b) The contribution of the $g^2$ terms does not depend on the velocities of the particles.

Therefore the complete expression for the three-acceleration up to terms of order $c^{-2}$ can be found using only the next-order term ($g^2$) in the series expansion. The calculation of the $g^2c^{-2}$ term can be easily done if we take the limit $\vec{v}_a\to 0, \vec{v}_{a'}\to 0$ before the integrations are performed. This is allowed by the second result stated above.

Using the standard procedure a straightforward but somewhat tedious calculation gives[14]

$$a_a^{i(2)}=-\frac{1}{2c^2}\frac{e_a}{m_a}\epsilon^{a'}\epsilon^{a''}e_{a'}e_{a''}\frac{e^{-\mu x_{aa'}}}{x_{aa'}{}^3}\left\{2\gamma e_a\frac{1+\mu x_{aa'}}{x_{aa''}}\frac{e^{-\mu x_{aa''}}}{m_a}x_{aa'}^i+e_{a'}\frac{1+\mu x_{a'a''}}{x_{a'a''}{}^3} \times\frac{e^{-\mu x_{a'a''}}}{m_{a'}}[(1+\mu x_{aa'})(\vec{x}_{aa'}\cdot\vec{x}_{a'a''})x_{aa'}^i-x_{aa'}{}^2x_{a'a''}^i]\right\} \ . \tag{2.6}$$

The values of $a'$ and $a''$ are restricted by the obvious condition that the right-hand side of this equation cannot be singular.

We find here the two-body terms, already found by Bel and Martin[4] and three-body terms as was expected owing to the nonlinearity of the equations of predictive relativistic mechanics.

Equation (2.6) when added to (2.5) gives the complete acceleration up to $c^{-2}$ for a system of interacting particles.

## III. EQUATIONS OF MOTION

Using the general procedure given in I we can calculate $H_i$, $\vec{P}_i$, $\vec{J}_i$, and $\vec{K}_i$ up to terms of order $c^{-2}$. The result is ($i=1,2$)

$$H_i=\epsilon^a m_a c^2+\epsilon^a\left[\tfrac{1}{2}m_a v_a{}^2+\frac{3}{8c^2}m_a v_a{}^4\right]-\tfrac{1}{2}\epsilon^a\epsilon^{a'}e_a e_{a'}\frac{e^{-\mu x_{aa'}}}{x_{aa'}}$$
$$-\frac{1}{4c^2}\epsilon^a\epsilon^{a'}e_a e_{a'}\frac{e^{-\mu x_{aa'}}}{x_{aa'}}\left[v_a{}^2+v_{a'}{}^2-\vec{v}_a\cdot\vec{v}_{a'}+\frac{1+\mu x_{aa'}}{x_{aa'}^2}(\vec{x}_{aa'}\cdot\vec{v}_a)(\vec{x}_{aa'}\cdot\vec{v}_{a'})\right]$$
$$+(1-\gamma)\frac{1}{c^2}\int_0^{-\infty}d\lambda\, R(\lambda)(\epsilon^a\vec{\Pi}_a\cdot\vec{V}_a)\ , \tag{3.1}$$

$$\vec{P}_1=\epsilon^a\left[m_a\vec{v}_a+\frac{1}{2c^2}m_a v_a{}^2\vec{v}_a\right]-\frac{1}{4c^2}\epsilon^a\epsilon^{a'}e_a e_{a'}\frac{e^{-\mu x_{aa'}}}{x_{aa'}}\left\{\vec{v}_a+\vec{v}_{a'}+\frac{1+\mu x_{aa'}}{x_{aa'}{}^2}[(\vec{x}_{aa'}\cdot\vec{v}_a)+(\vec{x}_{aa'}\cdot\vec{v}_{a'})]\vec{x}_{aa'}\right\}$$
$$+(1-\gamma)\frac{1}{c^2}\int_0^{-\infty}d\lambda\, R(\lambda)(\epsilon^a\vec{\Pi}_a)\ , \tag{3.2}$$

$$\vec{J}_1=\epsilon^a\left[\vec{x}_a\times\left[m_a\vec{v}_a+\frac{1}{2c^2}m_a v_a{}^2\vec{v}_a\right]\right]-\frac{1}{2c^2}\epsilon^a\epsilon^{a'}e_a e_{a'}\frac{e^{-\mu x_{aa'}}}{x_{aa'}}\vec{x}_a\times\left[2\vec{v}_a-\vec{v}_{a'}+\frac{1+\mu x_{aa'}}{x_{aa'}}(\vec{x}_{aa'}\cdot\vec{v}_{a'})\vec{x}_{aa'}\right]$$
$$+(1-\gamma)\frac{1}{c^2}\int_0^{-\infty}d\lambda\, R(\lambda)(\epsilon^a\vec{x}_a\times\vec{\Pi}_a)\ , \tag{3.3}$$

$$\vec{K}_1=\epsilon^a\left[m_a\left[1+\frac{1}{2c^2}v_a{}^2\right]\vec{x}_a\right]-\frac{1}{2c^2}\epsilon^a\epsilon^{a'}e_a e_{a'}\frac{e^{-\mu x_{aa'}}}{x_{aa'}}\vec{x}_a-(1-\gamma)\frac{1}{c^2}\int_0^{-\infty}d\mu\, R(\mu)\int_0^{-\infty}d\lambda\, R(\lambda)(\epsilon^a\vec{\Pi}_a)\ , \tag{3.4}$$

where

$$\vec{\Pi}_a\equiv\frac{e_a{}^2}{m_a}\epsilon^{a'}e_{a'}\left[\frac{e_{a'}e^{-\mu x_{aa'}}}{x_{aa'}}+\epsilon^{a''}\frac{e_{a''}e^{-\mu x_{aa''}}}{x_{aa''}}\right]\frac{1+\mu x_{aa'}}{x_{aa'}{}^3}e^{-\mu x_{aa'}}\vec{x}_{aa'} \tag{3.5}$$

and where the operator $R(\lambda)$ is defined by

$$R(\lambda)f(\vec{x},\vec{V})\equiv f(\vec{x}+\lambda\vec{V},\vec{V})\ .$$

The $\gamma=1$ theory contains only terms up to order $g/c^2$, and in this case the expressions can be derived from the Bopp Lagrangian.

These quantities can be used, as was explained in I, Sec. VIII, in order to define the mass of each subsystem

$$M_i\equiv\left[\frac{H_i{}^2}{c^4}-\frac{P_i{}^2}{c^2}\right]^{1/2}\quad(i=1,2)\ . \tag{3.6}$$

The center of mass is

$$\vec{X}_i\equiv\frac{1}{M_i c^2}\left[H_i\vec{K}_i-\vec{P}_i\times\vec{J}_i-\frac{(\vec{P}_i\cdot\vec{K}_i)\vec{P}_i}{H_i/c^2}\right]\quad(i=1,2) \tag{3.7}$$

and the spin is

$$\vec{S}_i\equiv\frac{H_i}{M_i c^2}\vec{J}_i-\frac{\vec{K}_i\times\vec{P}_i}{M_i}-\frac{(\vec{P}_i\cdot\vec{J}_i)\vec{P}_i}{M_i(M_i c^2+H_i)}\quad(i=1,2)\ . \tag{3.8}$$

The derivatives of these expressions can be easily calculated using the method described in II, Sec. III, or directly. Retaining only the terms linear in the coupling constant up to order $c^{-2}$ we find

$$\frac{dM_1}{dt}=0\ , \tag{3.9}$$

$$\frac{d\vec{S}_1}{dt}=-\frac{1}{c^2}\frac{e^{-\mu R}}{R^3}(1+\mu R)\left\{(\vec{R}\cdot\vec{V})Q_2\vec{\mu}_1+\frac{1}{2}\frac{Q_1Q_2}{M_1}(\vec{R}\times\vec{V}_1)\cdot\vec{S}_1\right\} , \tag{3.10}$$

$$\frac{d^2\vec{X}_1}{dt^2}=-\frac{Q_1Q_2}{M_1}\frac{e^{-\mu R}}{R^3}\left\{\left[(1+\mu R)\left[1-\frac{V_1^2}{c^2}\right]-(3+3\mu R+3\mu^2R^2)\frac{(\vec{R}\cdot\vec{V}_2)^2}{2c^2R^2}\right]\vec{R}-\frac{1}{c^2}(1+\mu R)(\vec{R}\cdot\vec{V}_2)\vec{V}\right\}$$
$$+\frac{1}{c^2}\frac{Q_1Q_2}{M_1^2}\frac{e^{-\mu R}}{R^3}\left\{(1+\mu R)\vec{V}\times\vec{S}_1-(3+3\mu R+\mu^2R^2)\frac{\vec{R}\cdot\vec{V}}{R^2}\vec{R}\times\vec{S}_1\right\} , \tag{3.11}$$

Where $Q_1$ and $Q_2$ are the total scalar "charges" of subsystems 1 and 2, respectively,

$$\vec{\mu}_1\equiv\epsilon^a e_a\vec{r}_a\times\vec{W}_a ,$$

$\vec{r}_a$ and $\vec{W}_a$ are the positions and velocities of the particles belonging to system 1 measured from its center of mass,

$$\vec{V}_1\equiv\frac{d\vec{X}_1}{dt}=\vec{P}_1\left[1-\frac{1}{2c^2}\frac{P_1^2}{M_1^2}\right]$$
$$-\frac{1}{c^2}\frac{e^{-\mu R}}{R^3}(1+\mu R)\vec{R}\times\left[\frac{1}{2}Q_2\vec{\mu}_1-\frac{Q_1Q_2}{M_1}\vec{S}_1\right],$$

$$\vec{R}\equiv\vec{X}_1-\vec{X}_2 ,$$

and

$$\vec{V}\equiv\vec{V}_1-\vec{V}_2 .$$

Exchange of labels 1 and 2 gives the equations for the mass, spin derivative, and acceleration of system 2.

The assumptions under which those equations are derived were explained in II, Sec.II and they are essentially quasispherical symmetry and rigidity for the subsystems as seen in its center-of-mass frame. The technical consequences of these assumptions were discussed in the above reference.

To get a closed system of dynamical equations, one more assumption is needed. Usually one assumes a linear relation between the "magnetic" moment and the spin such as

$$\vec{\mu}_1=g_1\frac{Q_1}{M_1}\vec{S}_1 . \tag{3.12}$$

Using the expression in (3.10), we can show that the system of equations (3.9)–(3.11) becomes a closed system of dynamical equations.

The acceleration (3.11) has spin-independent terms which are the same as for pointlike charges and several spin-dependent terms. However, it has no "magnetic"-moment-dependent terms nor spin-spin-dependent terms, a peculiarity not shared by the previously studied interactions.

In fact, the "magnetic" moment only appears in the spin equation.

## IV. LAGRANGIAN FORM OF THE EQUATIONS OF MOTION

As we did for the electromagnetic and gravitational interactions we have investigated the possibility of deriving Eq. (3.11) from a Lagrangian.

We find here a similar result. The spin-independent part of the acceleration can be derived from the Bopp Lagrangian[3] hereafter abbreviated as $\mathscr{L}_B$. Moreover, since Eq. (3.11) can be written in the form

$$\frac{d}{dt}\left[\vec{P}_{1B}-\frac{1}{c^2}\frac{Q_1Q_2}{M_1}\frac{e^{-\mu R}}{R^3}(1+\mu R)\vec{R}\times\vec{S}_1\right]=\vec{F}_{1B} , \tag{4.1}$$

where

$$\vec{P}_{1B}\equiv\frac{\partial\mathscr{L}_B}{\partial\vec{V}_1},\quad \vec{F}_{1B}\equiv\frac{\partial\mathscr{L}_B}{\partial\vec{X}_1} ,$$

it is clear that no Lagrangian can be constructed for this interaction using the $\vec{X}_i$ variables. However, as we previously found in II, using, instead of the center of mass $\vec{X}_i$, the new variable center of spin $\vec{Z}_i$,

$$\vec{Z}_i=\vec{X}_i+\frac{1}{2c^2}\frac{\vec{V}_i\times\vec{S}_i}{M_i} , \tag{4.2}$$

the resulting equations of motion can be derived from the following Lagrangian:

$$\mathscr{L}=\mathscr{L}_B+\frac{1}{2c^2}Q_1Q_2\frac{e^{-\mu R}}{R^3}(1+\mu R)\left[\frac{\vec{S}_1\cdot(\vec{R}\times\vec{V}_1)}{M_1}-\frac{\vec{S}_2\cdot(\vec{R}\times\vec{V}_2)}{M_2}\right] , \tag{4.3}$$

where now $\vec{R} \equiv \vec{Z}_1 - \vec{Z}_2$ and $\vec{V}_i \equiv d\vec{Z}_i/dt$.

A Legendre transformation now gives the following Hamiltonian:

$$H = H_B - \frac{1}{2c^2} Q_1 Q_2 \frac{e^{-\mu R}}{R^3}(1+\mu R)\left[\frac{\vec{S}_1\cdot(\vec{R}\times\vec{P}_1)}{M_1{}^2} - \frac{\vec{S}_2\cdot(\vec{R}\times\vec{P}_2)}{M_2{}^2}\right] , \tag{4.4}$$

where

$$\vec{P}_i \equiv \frac{\partial \mathscr{L}}{\partial \vec{V}_i} .$$

## V. SECOND-ORDER TERMS

In order to complete our study of the interactions among extended bodies, including all the terms up to order $c^{-2}$, the contributions quadratic in the coupling constant must be taken into account.

Their computation needs the knowledge of the accelerations for structureless particles up to and including those terms. The expression for the scalar interaction has already been given in (2.6). Using the Darwin Lagrangian for the electromagnetic interaction and the Einstein-Infeld-Hoffmann Lagrangian for the gravitational interaction, we find

$$\vec{a}_a^{(2)} = -\frac{1}{2}\frac{e_a}{m_a}\epsilon^{a'}\epsilon^{a''}\frac{e_{a'}{}^2 e_{a''}}{c^2 m_{a'}}\left[\frac{\vec{x}_{aa'}\cdot\vec{x}_{a'a''}}{x_{aa'}{}^3 x_{a'a''}{}^3}\vec{x}_{aa'} + \frac{\vec{x}_{a'a''}}{x_{aa'}x_{a'a''}{}^3}\right] \tag{5.1}$$

and

$$\vec{a}_a^{(2)} = \frac{G^2}{c^2}\epsilon^{a'}\epsilon^{a''} m_{a'} m_{a''}\left[\frac{4\vec{x}_{aa'}}{x_{aa'}{}^3 x_{aa''}} + \left[1 - \frac{\vec{x}_{aa'}\cdot\vec{x}_{a'a''}}{2x_{a'a''}{}^2}\right]\frac{\vec{x}_{aa'}}{x_{a'a''}x_{aa'}{}^3} - \frac{7}{2}\frac{\vec{x}_{a'a''}}{x_{aa'}x_{a'a''}{}^3}\right] , \tag{5.2}$$

respectively. The values taken by $a'$ and $a''$ are only restricted by the obvious condition that the denominators cannot be singular.

Using the standard procedure given in I we can compute the complete expressions for the ten generating functions $H$, $\vec{P}$, $\vec{J}$, $\vec{K}$. The results for the scalar interaction have already been given in (3.1)—(3.4).

For the electromagnetic interaction the expressions (4.1)—(4.4) given in II for these functions need not be modified, because there are no $g^2/c^2$ contributions. For the gravitational interaction only the first of the expressions (5.1)—(5.4) given in II suffers a correction of order $G^2/c^2$:

$$H_1^{(2)} = \frac{G^2}{2c^2}\epsilon^a\epsilon^{a'}\epsilon^{a''}\frac{m_a m_{a'} m_{a''}}{x_{aa'}x_{aa''}} . \tag{5.3}$$

The complete expressions must now be introduced in the definitions of $\vec{X}_i$, $\vec{S}_i$, and $\vec{M}_i$, and the derivatives of these quantities must now be calculated taking into account the terms up to order $g^2/c^2$.

It can easily be seen that the expressions for $M_i$ and $S_i$ are not modified for the three interactions considered, therefore the Eqs. (3.9) and (3.10) of this paper, (4.17) and (4.18) of II, and (5.21) and (5.22) of II do contain already all the terms of order $c^{-2}$. Only the equations giving the accelerations are modified. Let us examine in detail how these modifications arise.

From

$$\frac{d\vec{X}_1}{dt} \equiv \vec{V}_1 = \frac{\vec{P}_1}{M_1}\left[1 - \frac{P_1{}^2}{2M_1{}^2c^2}\right] + a_a^i(1,2)\frac{\partial \vec{X}_1}{\partial v_a^i} \tag{5.4}$$

we find

$$\vec{A}_1 \equiv \frac{d\vec{V}_1}{dt} = \frac{1}{M_1}\left[1 - \frac{P_1{}^2}{2M_1{}^2c^2}\right]\frac{d\vec{P}_1}{dt} - \frac{1}{2c_1{}^2}\left[\frac{\vec{P}_1}{M_1{}^2}\frac{d\vec{P}_1}{dt}\right]\frac{\vec{P}_1}{M_1} + \frac{d}{dt}\left[a_a^i(1,2)\frac{\partial \vec{X}_1}{\partial v_a^i}\right] \tag{5.5}$$

and only the first term on the right-hand side can give contributions to the $g^2/c^2$ terms. These terms arise from

$$\frac{d\vec{P}_1}{dt} = \epsilon^a a_a^i(1,2)\left[m_a\delta_i^j + \frac{1}{2c^2}m_a v_a{}^2\delta_i^j + \frac{1}{c^2}m_a v_{ai}v_a^j\right] + \epsilon^a a_a^i(1,2)\frac{\partial \vec{P}_1(g/c^2)}{\partial v_a^i} . \tag{5.6}$$

We shall not make here the detailed calculations of these expressions, which are rather cumbersome, but we shall indicate how these terms can be handled. We find two general types of terms, those that can directly be written in terms of the variables we have been using as, for instance,

$$\frac{G^2}{c^2}\epsilon^a\epsilon^{A'}\epsilon^{A''}\frac{m_{A'}m_{A''}m_a\vec{X}_{aA}}{x_{aA'}{}^3x_{aA''}}\to\frac{G^2}{G^2}M_1M_2{}^2\frac{\vec{R}^2}{R^4}$$

and those that cannot be analyzed through a multipolar development as

$$\frac{G^2}{c^2}\epsilon^a\epsilon^A\frac{m_am_A}{r_{aA}{}^3}\epsilon^{a'}\frac{m_{a'}}{r_{aa'}}\vec{r}_{aA} \quad . \tag{5.7}$$

We have borrowed our examples from the gravitational interaction but the same problem arises in the electromagnetic and scalar cases and the terms are formally analogous.

What we have proved is that assuming spherical symmetry and uniform charge distribution all the terms belonging to the second class can be written as functions of $M_i$, $\vec{R}_i$, and $U_i$, were $U_i$ are the self-energies of the extended bodies (see the Appendix for more details). It must be noted that these assumptions are by no means additional assumptions.

We shall now analyze the results for each interaction, the expression for the second-class terms as functions of $M_i$, $\vec{R}_i$, and $U_i$ being given in the Appendix.

### A. Electromagnetic interaction

Once we have added all the $g^2/c^2$ terms, the terms containing self-energies cancel each other and we find

$$\vec{A}_1^{(2)}=\frac{1}{c^2}\frac{Q_1{}^2Q_2{}^2}{M_1M_2}\frac{\vec{R}}{R^4} \quad . \tag{5.8}$$

The complete electromagnetic acceleration up to order $c^{-2}$ among extended bodies is thus given, adding (5.8) to (4.19) in II. In fact (5.8) is identical with the $g^2/c^2$ term giving the acceleration for pointlike scalar particles as can be seen from (5.1).[15]

### B. Gravitational interaction

In this case there is no cancellation and we find

$$M_1\vec{A}_1^{(2)}=\frac{G^2}{c^2}(5M_2M_1{}^2+4M_1M_2{}^2)\frac{\vec{R}}{R^4} + \frac{G}{c^2}(M_2U_1+M_1U_2)\frac{\vec{R}}{R^3} \quad . \tag{5.9}$$

However, the terms containing the self-energies can be reabsorbed in the Newtonian term, defining the effective masses as

$$\widetilde{M}_i\equiv M_i\left[1+\frac{U_i}{M_ic^2}\right] \quad ,$$

where

$$U_i\equiv\tfrac{3}{5}G\frac{M_i{}^2}{r_i} \quad ,$$

$$r_i=\text{radius of mass } M_i \quad . \tag{5.10}$$

We are left with the contributions

$$\vec{A}_1^{(2)}=\frac{G^2}{c^2}(5\widetilde{M}_2\widetilde{M}_1+4\widetilde{M}_2{}^2)\frac{\vec{R}}{R^4} \tag{5.11}$$

which have to be added to (2.23) in II to get the complete accelerations up to order $G^2/c^2$, substituting everywhere $M_i$ by the "renormalized" masses $\widetilde{M}_i$.[15]

Therefore the dynamical variables proposed can handle the problem. We need only shift the value of $M_i$ to $\widetilde{M}_i$ which will be the physical mass and to the expression we gave in II, there is only a contribution which is the same as for pointlike particles.

### C. Short-range scalar interaction

*Case* $\gamma=0$. This dynamical theory differs from all the previously studied cases in that the variables $M_i,\vec{X}_i,\vec{S}_i$ are not sufficient to describe the dynamics of extended bodies. The terms containing the self-energies of the bodies do not cancel and their lack of symmetry prevents their absorption in the Newtonian terms. In this theory the self-energies must be introduced as new dynamical variables and the scheme proposed breaks down.

*Case* $\gamma=1$. This dynamical theory is similar in its behavior to the gravitational interaction because the terms containing the self-energies appear symmetrically and can be reabsorbed in the Newtonian terms, if we redefine the charges in the following way:

$$\widetilde{Q}_i\equiv Q_i\left[1+\frac{U_i}{M_ic^2}\right],\quad U_i\equiv\frac{3}{5}\frac{Q_i{}^2}{r_i} \quad . \tag{5.12}$$

This procedure gives the following $g^2/c^2$ terms:

$$\vec{A}_a^{(2)}=\frac{1}{2c^2}\frac{\widetilde{Q}_1{}^2\widetilde{Q}_2{}^2}{m_1}\frac{e^{-2\mu R}}{R^3}(1+\mu R)\left[\frac{\mu}{M_2}-\frac{2}{M_1R}\right]\vec{R} \tag{5.13}$$

which must be added to (3.11) in order to get the complete $c^{-2}$ expression substituting everywhere $Q_i$ by the effective charges $\widetilde{Q}_i$.

### D. Order-of-magnitude estimates

When the $g^2/c^2$ terms we have just calculated are added to the post-Newtonian terms given in (4.19) and (5.22) of Ref. 2, or (3.11) of this paper, we can claim that we know all the $c^{-2}$ terms of the acceleration. At this point it may be interesting to make some estimates about (i) the validity of the expansion, (ii) the relative order of magnitude of the different terms.

In spite of the fact that we have been using quite independently coupling-constant expansions and low-velocity expansions, a close look at the expressions we have found reveals that they can be considered as expansions in powers of the following dimensionless parameters:

$$\frac{Q^2}{mc^2R} \text{ (electromagnetic and scalar interactions)} ,$$

$$\frac{GM}{c^2R} \text{ (gravitational interaction)} .$$

The quantity $\rho = Q^2/mc^2$ can be safely bounded for ordinary matter and electromagnetic interaction by $10^3 r_e$ where $r_e$ is the classical electron radius whereas $GM/c^2 = 2r_S$, where $r_S$ is the Schwarzschild radius. Therefore the expressions we are using can be safely used in the classical domain as far as $10^3 r_e/R$ or $r_S/R$ are small quantities, which covers a wide variety of situations.

The relative order of magnitude of the different terms appearing in the accelerations can be estimated factoring out the factor $Q_1Q_2/M$ $(-GM)$ from the whole expression. Putting aside the Newtonian term, which is of course the most relevant one, we have to weight terms which are purely $\beta^2 = v^2/c^2$ dependent, spin-orbit terms containing the product $\beta(g\sigma/cR)$, where $\sigma$ is the angular momentum per unit mass, spin-spin terms containing the product $(g\sigma/cR)^2$, and the terms originated by three-body forces characterized by the factor $Q_1Q_2/Mc^2R$ $(GM/c^2R)$.

The relative weight of the first and the last can be evaluated for bound states via the virial theorem. In this case they are the same order of magnitude

$$\beta^2 \sim \frac{Q_1Q_2}{Mc^2R} \quad \left[\frac{GM}{c^2R}\right] .$$

Therefore the effect of three-body forces cannot be neglected at this order $(c^{-2})$.

The spin-orbit terms are usually smaller, and elementary calculations prove that a necessary condition for those terms to be of the same order as the $\beta^2$ terms for gravitationally bound bodies is that $r \sim R$ which is out of the range of our approximation. More information about this point can be found in Refs. 10 and 16.

## ACKNOWLEDGMENT

This work was partially supported by research projects CAICYT Nos. 3809/79 and 534.

## APPENDIX

We give here a list of the result that the spherical-symmetry assumption provides for the gravitational interaction $G^2$ terms:

$$\epsilon^a \epsilon^A \frac{m_a m_A}{r_{aA}} \epsilon^{a'} \frac{m_{a'}}{r_{aa'}{}^3} \vec{r}_{aa'} \rightarrow -\frac{1}{3} M_2 U_1 \frac{\vec{R}}{R^3} , \tag{A1}$$

$$\epsilon^a \epsilon^A \frac{m_a m_A}{r_{aA}{}^3} \epsilon^{a'} \frac{m_{a'}}{r_{aa'}} \vec{r}_{aA} \rightarrow 2 M_2 U_1 \frac{\vec{R}}{R^3} , \tag{A2}$$

$$\epsilon^a \epsilon^A \frac{m_a m_A}{r_{aA}{}^3} \left[\epsilon^{a'} \frac{m_{a'}}{r_{aa'}{}^3} \vec{r}_{aa'} \cdot \vec{r}_{aA}\right] \vec{r}_{aA} \rightarrow \frac{1}{3} M_2 U_1 \frac{\vec{R}}{R^3} , \tag{A3}$$

$$\epsilon^a \epsilon^A \frac{m_a m_A}{r_{aA}{}^3} \epsilon^{a'} m_{a'} \frac{r^i_{aa'} r_{aa'j}}{r_{aa'}{}^3} r^j_{aA} \rightarrow \frac{2}{3} M_2 U_1 \frac{\vec{R}}{R^3} . \tag{A4}$$

Similar results can be proved for the electromagnetic and short-range scalar interactions.

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