Classical cross sections for relativistic spinning particles

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We propose a definition of classical differential cross sections for particles with essentially nonplanar orbits, such as spinning ones. We give also a method for its computation. The calculations are carried out explicitly for electromagnetic, gravitational, and short-range scalar interactions up to the linear terms in the slow-motion approximation. The contribution of the spin-spin terms is found to be at best 10^{-6} times the post-Newtonian ones for the gravitational interaction.

INTRODUCTION

There is a growing interest in classical models of relativistic particles possessing an internal structure. The main examples are extended bodies[1-4] and point particles with internal degrees of freedom such as spin,[5,6] Yang-Mills[7] fields, and even supersymmetric (Grassmann) particles.[8]

All those models share the property that the motion of the particles cannot be constrained to a plane in general. This is also the case for scalar particles provided that the interaction is not parity invariant, as occurs in charge-monopole systems.[9] As a consequence of this, the differential scattering cross sections may depend on the azimuthal angle, and one cannot simply use the standard formula

$$\frac{d\sigma}{d\Omega}=\frac{1}{2}\left|\frac{db^2}{d\cos\theta}\right|$$

without disregarding that dependence.

The main purpose of this paper is to propose a coherent definition of the differential scattering cross sections for those cases, together with an operative computation procedure. This is dealt with in the first section.

The next step is, as usual, to apply the ideas just developed to specific interactions. A crucial point in the method is the computation of the outgoing velocity of the incoming particles, where we have used the tools provided by predictive relativistic mechanics. The method has been applied to electromagnetic, gravitational, and short-range (scalar) interactions, the results being displayed in the second and third sections. A comparison with analogous results in the literature is also given.

I. CROSS SECTIONS FOR SPINNING PARTICLES

Let us recall the well-known concept of differential cross section for the scattering of classical (nonquantum) particles. In a classical scattering experiment, a beam of particles is sent toward the target and the flux scattered in a specific direction is counted. This means that the incident particles do not have a fixed impact parameter b nor a fixed azimuthal angle α (we take the incident direction as the z axis). Nevertheless, we shall assume that the other dynamical variables such as the energy, the spin, etc., have the same value so that all the particles in the incoming beam of transverse area

$$d\sigma=b\,db\,d\alpha$$

shall go into the specific infinitesimal outgoing cone of solid angle

$$d\Omega=d\phi\,d\cos\theta\ .$$

The angles θ and ϕ are connected with b and α via the dynamics of the interaction considered. This leads to the following expression for the conservation of the particle number:

$$Nb\,db\,d\alpha=N\frac{d\sigma}{d\Omega}d\Omega\ , \qquad (1.1)$$

where N is the incident flux and $d\sigma/d\Omega$ is, by definition, the differential cross section, that is, the number of outgoing particles per unit time and solid angle normalized to the incident flux. That is, according to (1.1)

$$\frac{d\sigma}{d\Omega}=\frac{1}{2}\left|\frac{\partial(b^2,\alpha)}{\partial(\cos\theta,\phi)}\right|=\frac{1}{2}\left|\frac{\partial(\cos\theta,\phi)}{\partial(b^2,\alpha)}\right|^{-1}, \tag{1.2}$$

where $\partial(,)/\partial(,)$ stands for the Jacobi determinant. This expression reduces to the standard one[10]

$$\frac{d\sigma}{d\Omega}=\frac{1}{2}\left|\frac{db^2}{d\cos\theta}\right| \tag{1.3}$$

if the incident azimuthal angle α coincides with the outgoing one ϕ. This coincidence always occurs in the nonrelativistic case for parity-invariant interactions between scalar particles. Arens[11] extended this result to the relativistic case whenever the interaction is described by means of a system of second-order differential equations, as the motion occurs then in a plane.

This result does not hold when one considers more general cases such as charge-monopole systems (which are not parity-invariant) or spinning particles. In those cases the full content of expression (1.2) must then be used and the expression (1.3) is recovered only if one averages (1.2) over ϕ, which amounts to disregarding the azimuthal dependence of the differential cross sections.[9]

One way to obtain the relationship between the "incoming coordinates" (b,α) and the "outgoing" ones (θ,ϕ) is to compute the outgoing velocity $\vec{v}_{\rm out}$ in terms of the incoming variables. To this end, we shall consider the following quantities: let $\vec{k}$ be the unit vector in the incident direction $\vec{k}=(0,0,1)$; let $\vec{b}=b(\cos\alpha,\sin\alpha,0)$ be the "impact vector." The identities

$$\vec{k}\cdot\vec{v}_{\rm out}=|v_{\rm out}|\cos\theta\ , \tag{1.4a}$$

$$\vec{b}\cdot\vec{v}_{\rm out}=b\,|v_{\rm out}|\sin\theta\cos(\phi-\alpha)\ , \tag{1.4b}$$

$$(\vec{b}\times\vec{k})\cdot\vec{v}_{\rm out}=-b\,|v_{\rm out}|\sin\theta\sin(\phi-\alpha)\ , \tag{1.4c}$$

provide us with the desired relationship.

It is useful, however, to express the differential cross section in terms of the outgoing coordinates (θ,ϕ) rather than in terms of the incoming ones (b,α), as would result from (1.2) and (1.4). The inversion of (1.4) raises two different problems. The first is a conceptual one: it is clear that, (b,α) given, one must always obtain a well-defined result for (θ,ϕ) (unless the incoming particle is captured) but the converse is not true in general because particles which have been sent to the target with different impact vectors may emerge along lines with the same slope; this is a well-known fact in classical scattering theory and it is associated with phenomena such as orbiting, glory, and rainbow scattering.[10] The multiplicity of incident configurations associated with the same dispersion angles is due to the identification of the deflection angle θ with $\theta+2\pi n$ where n is the number of circular trips around the target. This problem may be overcome if we restrict ourselves to large enough impact parameters such that in any case $n=0$ and the relationship (1.4) between (θ,ϕ) and (b,α) is one to one.

The second problem is a technical one. It is illustrated by the short-range case treated below where transcendental functions appear in (1.4) so that the explicit inversion is not possible even in the case in which (1.4) is one to one. This forces us to make a choice between an explicit expression of $d\sigma/d\Omega$ in terms of the incoming variables or an implicit (or numerical) one in terms of the dispersion angles.

II. COMPUTATION OF THE OUTGOING VELOCITY

As in scattering problems only asymptotic configurations matter, it is useful to characterize them in a standard way. We will say that a generic configuration is an incoming (outgoing) one if the value of $\vec{R}\cdot\vec{V}$ approaches $-\infty$ $(+\infty)$ where $\vec{R}$ and $\vec{V}$ are the relative positions and velocities, respectively, of the particles. This suggests the definition of a "shift operator" $R(\lambda)$ as

$$R(\lambda)f(\vec{x}_a,\vec{v}_b,\vec{S}_c)=f(\vec{x}_a+\lambda\vec{v}_a,\vec{v}_b,\vec{S}_c)\ , \tag{2.1}$$

where a,b,c are particle indices. It has the properties

$$R(\lambda)R(\mu)=R(\lambda+\mu)\ , \tag{2.2}$$

$$R(\lambda)v_a^k\frac{\partial}{\partial x_a^k}=v_a^k\frac{\partial}{\partial x_a^k}R(\lambda)=\frac{d}{d\lambda}R(\lambda)\ . \tag{2.3}$$

Let us consider a function f of the dynamical variables $(\vec{R},\vec{v}_a,\vec{S}_b)$ corresponding to a generic configuration. One can easily obtain the asymptotic forms of f, valid for incoming (outgoing) configurations only:

$$f_{\pm}\equiv\lim_{\lambda\to\pm\infty}R(\lambda)f(\vec{R},\vec{v}_a,\vec{S}_b)\ . \tag{2.4}$$

We will search out a constant of the motion $\vec{K}$ such that $\vec{K}_+$ equals $\vec{v}$ in order to establish the numerical balance equation $K_+=K_-$. This method is inspired by the one proposed by Lapiedra *et al.*[12]

We shall restrict ourselves to relativistic interactions which can be described by the following system of equations[5]:

$$\frac{d^2\vec{x}_a}{dt^2}=\vec{a}_a(\vec{R},\vec{v}_b,\vec{S}_c)\ , \tag{2.5a}$$

$$\frac{d\vec{S}_a}{dt}=\vec{\beta}_a(\vec{R},\vec{v}_b,\vec{S}_c)\ , \tag{2.5b}$$

$$\lim_{\lambda\to\pm\infty} R(\lambda)(\lambda\,\vec{a}_a)=\lim_{\lambda\to\pm\infty} R(\lambda)(\lambda\,\vec{\beta}_a)=0\ . \tag{2.6}$$

For such systems, the two conditions required before are written

$$\frac{d}{dt}\vec{K}=v_a^i\frac{\partial\vec{K}}{\partial x_a^i}+a_b^j\frac{\partial\vec{K}}{\partial v_b^j}+\beta_c^k\frac{\partial\vec{K}}{\partial S_c^k}=0\ , \tag{2.7a}$$

$$\lim_{\lambda\to+\infty} R(\lambda)\vec{K}=K_+=\vec{v}\ , \tag{2.7b}$$

and are equivalent to the integral equation

$$\vec{K}(\vec{R},\vec{v}_a,\vec{S}_b)=\vec{v}+\int_0^{\infty}d\lambda\,R(\lambda)\times\left[\left[a_b^i\frac{\partial}{\partial v_b^i}+\beta_c^j\frac{\partial}{\partial S_c^j}\right]\vec{K}\right]\ , \tag{2.8}$$

as can easily be seen using (2.3) and (2.6). The balance equation then reads

$$\vec{v}_{\text{out}}=\left\{\vec{v}+\int_{-\infty}^{\infty}d\lambda\,R(\lambda)\times\left[\left[a_b^i\frac{\partial}{\partial v_b^i}+\beta_c^j\frac{\partial}{\partial S_c^j}\right]\vec{K}\right]\right\}_{\text{in}} \tag{2.9}$$

or, using (2.8),

$$\vec{v}_{\text{out}}=\vec{v}_{\text{in}}+\int_{-\infty}^{\infty}d\lambda\,R(\lambda)\,\vec{a}(\vec{R},\vec{v}_b,\vec{S}_c)+\int_{-\infty}^{\infty}d\lambda\,R(\lambda)\left[\left[a_b^i\frac{\partial}{\partial v_b^i}+\beta_c^j\frac{\partial}{\partial S_c^j}\right]\int_0^{\infty}d\lambda' R(\lambda')\vec{a}\right]+\cdots\ , \tag{2.10}$$

where the suffix "in" has been suppressed for simplicity in the integrals.

Let us take, for instance, the accelerations given in Ref. 4 for the electromagnetic and gravitational interactions,[13] that is

$$\begin{aligned}\left[\frac{d\vec{v}_1}{dt}\right]_{\text{EM}}=&\frac{Q_1Q_2}{m_1}\frac{\vec{R}}{R^3}+\frac{Q_1Q_2}{m_1c^2R^3}\left[\left[\frac{V^2}{2}-v_1{}^2-\frac{3}{2}\frac{(\vec{R}\cdot\vec{v}_2)^2}{R^2}\right]\vec{R}-(\vec{R}\cdot\vec{v}_1)\vec{V}\right]\\&+\frac{Q_1}{m_1c^2R^3}\left[\vec{V}\times\vec{\mu}_2-\frac{3}{2}\frac{(\vec{R}\cdot\vec{V})}{R^2}\vec{R}\times\vec{\mu}_2-\frac{3}{2}\frac{(\vec{R}\times\vec{V})\cdot\vec{\mu}_2}{R^2}\vec{R}\right]\\&+\frac{Q_2}{m_1c^2R^3}\left[\vec{V}\times\vec{\mu}_1-\frac{3}{2}\frac{(\vec{R}\cdot\vec{V})}{R^2}\vec{R}\times\vec{\mu}_1-\frac{3}{2}\frac{(\vec{R}\times\vec{V})\cdot\vec{\mu}_1}{R^2}\vec{R}\right]-\frac{Q_1Q_2}{m_1{}^2c^2R^3}\left[\vec{V}\times\vec{S}_1-3\frac{\vec{R}\cdot\vec{V}}{R^2}\vec{R}\times\vec{S}_1\right]\\&+\frac{3}{4m_1c^2R^5}\left[(\vec{R}\cdot\vec{\mu}_1)\vec{\mu}_2+(\vec{R}\cdot\vec{\mu}_2)\vec{\mu}_1+(\vec{\mu}_1\cdot\vec{\mu}_2)\vec{R}-5\frac{(\vec{R}\cdot\vec{\mu}_1)(\vec{R}\cdot\vec{\mu}_2)}{R^2}\vec{R}\right]\ ,\end{aligned} \tag{2.11a}$$

$$\begin{aligned}\left[\frac{d\vec{v}_1}{dt}\right]_G=&-\frac{Gm_2}{R^3}\vec{R}-\frac{Gm_2}{c^2R^3}\left[\left[2V^2-v_1{}^2-\frac{3}{2}\frac{(\vec{R}\cdot\vec{v}_2)^2}{R^2}\right]\vec{R}-(\vec{v}_1+3\vec{V})\cdot\vec{R}\vec{V}\right]\\&-\frac{G}{c^2R^3}\left[4\vec{V}\times\vec{S}_2-6\frac{(\vec{R}\cdot\vec{V})}{R^2}\vec{R}\times\vec{S}_2-6\frac{(\vec{R}\times\vec{V})\cdot\vec{S}_2}{R^2}\vec{R}\right]\\&-\frac{Gm_2}{m_1c^2R^3}\left[3\vec{V}\times\vec{S}_1-3\frac{(\vec{R}\cdot\vec{V})}{R^2}\vec{R}\times\vec{S}_1-6\frac{(\vec{R}\times\vec{V})\cdot\vec{S}_1}{R^2}\vec{R}\right]\\&-\frac{3G}{m_1c^2R^5}\left[(\vec{R}\cdot\vec{S}_1)\vec{S}_2+(\vec{R}\cdot\vec{S}_2)\vec{S}_1+(\vec{S}_1\cdot\vec{S}_2)\vec{R}-5\frac{(\vec{R}\cdot\vec{S}_1)(\vec{R}\cdot\vec{S}_2)}{R^2}\vec{R}\right]\ ,\end{aligned} \tag{2.11b}$$

where, for the sake of simplicity, only the terms linear in the coupling constants are considered, up to the post-Newtonian ($1/c^2$) approximation.

Substituting (2.11) into the right-hand side of (2.10), one obtains up to the first order in the coupling constants (after one elementary integration)

$$(\vec{v}_{\rm out})_{\rm EM}=\vec{v}_1+\frac{Q_1Q_2}{m_1V}\frac{1}{b^2}\left[2+\frac{1}{c^2}(V^2-2v_1{}^2-v_2{}^2)\right]\vec{b}+\frac{Q_1}{m_1c^2}\frac{1}{b^2}\left[(\vec{k}\times\vec{\mu}_2)-\frac{2}{b^2}(\vec{b}\times\vec{k})\cdot\vec{\mu}_2\vec{b}\right]$$
$$+\frac{Q_2}{m_1c^2}\frac{1}{b^2}\left[(\vec{k}\times\vec{\mu}_1)-\frac{2}{b^2}(\vec{b}\times\vec{k})\cdot\vec{\mu}_1\vec{b}\right]$$
$$+\frac{1}{m_1c^2V}\frac{1}{b^4}\left\{(\vec{\mu}_1\cdot\vec{b})[\vec{\mu}_2-(\vec{\mu}_2\cdot\vec{k})\vec{k}]+(\vec{\mu}_2\cdot\vec{b})[\vec{\mu}_1-(\vec{\mu}_1\cdot\vec{k})\vec{k}]\right.$$
$$\left.+\left[(\vec{\mu}_1\cdot\vec{\mu}_2)-(\vec{\mu}_1\cdot\vec{k})(\vec{\mu}_2\cdot\vec{k})-\frac{4}{b^2}(\vec{\mu}_1\cdot\vec{b})(\vec{\mu}_2\cdot\vec{b})\right]\vec{b}\right\}, \tag{2.12a}$$

$$(\vec{v}_{\rm out})_G=\vec{v}_1-\frac{Gm_2}{V}\frac{1}{b^2}\left[2+\frac{1}{c^2}(4V^2-2v_1{}^2-v_2{}^2)\right]\vec{b}-\frac{4G}{c^2}\frac{1}{b^2}\left[(\vec{k}\times\vec{S}_2)-\frac{2}{b^2}(\vec{b}\times\vec{k})\cdot\vec{S}_2\vec{b}\right]$$
$$-\frac{4Gm_2}{m_1c^2}\frac{1}{b^2}\left[(\vec{k}\times\vec{S}_1)-\frac{2}{b^2}(\vec{b}\times\vec{k})\cdot\vec{S}_1\vec{b}\right]$$
$$-\frac{4G}{m_1c^2V}\frac{1}{b^4}\left\{(\vec{S}_1\cdot\vec{b})[\vec{S}_2-(\vec{S}_2\cdot\vec{k})\vec{k}]+(\vec{S}_2\cdot\vec{b})[\vec{S}_1-(\vec{S}_1\cdot\vec{k})\vec{k}]\right.$$
$$\left.+\left[(\vec{S}_1\cdot\vec{S}_2)-(\vec{S}_1\cdot\vec{k})(\vec{S}_2\cdot\vec{k})-\frac{4}{b^2}(\vec{S}_1\cdot\vec{b})(\vec{S}_2\cdot\vec{b})\right]\vec{b}\right\}, \tag{2.12b}$$

where we have omitted again the suffix "in" on the right-hand side. The incoming velocities of the two particles are supposed to be parallel, along the unit vector $\vec{k}\equiv\vec{V}/V$; we note that

$$\vec{b}=[\vec{R}-(\vec{R}\cdot\vec{k})\vec{k}]_{\rm in}\,.$$

III. ELECTROMAGNETIC AND GRAVITATIONAL CROSS SECTIONS

Allowing for the strong parallelism between Eqs. (2.12a) and (2.12b), we shall only compute the scattering cross section in detail for the electromagnetic case. If one substitutes (2.12) into (1.4), splitting the magnetic moment into its components,

$$\vec{\mu}_a\equiv\mu_a(\sin\lambda_a\cos\omega_a\,,\ \sin\lambda_a\sin\omega_a\,,\ \cos\lambda_a)\,, \tag{3.1}$$

one obtains up to the first order in the coupling constant in the post-Newtonian approximation

$$|\,v_{\rm out}\,|\,\cos\theta=v_1\,, \tag{3.2a}$$

$$|\,v_{\rm out}\,|\,\sin\theta\cos(\phi-\alpha)=\frac{1}{b}\frac{Q_1Q_2}{m_1V}\left[2+\frac{1}{c^2}(V^2-2v_1{}^2-v_2{}^2)\right]$$
$$+\frac{1}{b^2}\left[\frac{Q_1\mu_2\sin\lambda_2}{m_1c^2}\sin(\omega_2-\alpha)+\frac{Q_2\mu_1\sin\lambda_1}{m_1c^2}\sin(\omega_1-\alpha)\right]$$
$$-\frac{1}{b^3}\frac{(\mu_1\sin\lambda_1)(\mu_2\sin\lambda_2)}{m_1c^2V}\cos(\omega_1+\omega_2-2\alpha)\,, \tag{3.2b}$$

$$|v_{\text{out}}|\sin\theta\sin(\phi-\alpha)=\frac{1}{b^2}\left[\frac{Q_1\mu_2\sin\lambda_2}{m_1c^2}\cos(\omega_2-\alpha)+\frac{Q_2\mu_1\sin\lambda_1}{m_1c^2}\cos(\omega_1-\alpha)\right]$$
$$+\frac{1}{b^3}\frac{(\mu_1\sin\lambda_1)(\mu_2\sin\lambda_2)}{m_1c^2V}\sin(\omega_1+\omega_2-2\alpha)\ . \tag{3.2c}$$

Had we retained second-order terms, we would have obtained a $1/b^2$ scalar contribution, a $1/b^3$ spin-orbit contribution, and $1/b^4$ spin-spin contribution.

That is to say, (3.2) is an asymptotic development for large impact parameters (small dispersion angles) and we have retained only the leading contribution of every kind. This allows the explicit step-by-step inversion of (3.2). As we are only interested in the Newtonian and post-Newtonian contributions, the process is easier than in the general case; the Newtonian part is

$$\tan\theta=\frac{2}{b}\frac{|Q_1Q_2|}{m_1Vv_1}\ , \tag{3.3a}$$

$$\phi-\alpha=\eta(\epsilon)=\begin{cases}0\ , & \epsilon=1\\ \pi\ , & \epsilon=-1\ ,\end{cases} \tag{3.3b}$$

where $\epsilon\equiv Q_1Q_2/|Q_1Q_2|$. (Note that $0\le\theta\le\pi$.) At the post-Newtonian step we have

$$\frac{1}{b}=\frac{m_1v_1V}{2|Q_1Q_2|}\tan\theta\left\{1-\frac{1}{2c^2}(V^2-2v_1{}^2-v_2{}^2)-\frac{m_1v_1V^2}{4Q_1Q_2c^2}\tan\theta\right.$$
$$\times\left[\frac{g_2S_2}{m_2}\sin\lambda_2\sin(\omega_2-\phi)+\frac{g_1S_1}{m_1}\sin\lambda_1\sin(\omega_1-\phi)\right]$$
$$\left.+\frac{\epsilon}{2c^2}\left[\frac{m_1v_1V}{2Q_1Q_2}\tan\theta\right]^2\frac{g_1g_2S_1S_s}{m_1m_2}\sin\lambda_1\sin\lambda_2\cos(\omega_1+\omega_2-2\phi)\right\}\ , \tag{3.4a}$$

$$\phi-\alpha=\eta(\epsilon)+\frac{m_1v_1V^2}{4Q_1Q_2c^2}\tan\theta\left[\frac{g_2S_2}{m_2}\sin\lambda_2\cos(\omega_2-\phi)+\frac{g_1S_1}{m_1}\sin\lambda_1\cos(\omega_1-\phi)\right]$$
$$+\frac{1}{2c^2}\left[\frac{m_1v_1V}{2Q_1Q_2}\tan\theta\right]^2\frac{g_1g_2S_1S_2}{m_1m_2}\sin\lambda_1\sin\lambda_2\sin(\omega_1+\omega_2-2\phi)\ , \tag{3.4b}$$

where we have used $\sin^4\theta/2\ll 1$ [allowing for the fact that (3.4) is valid for small angles only]. This result agrees with the one presented without proof in Ref. 14, where a second-order scalar contribution [behaving as $(\sin\theta/2)^{-3}$, see Ref. 12 for details] is also included.

The differential cross section for the electromagnetic interaction is then

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{EM}}=\frac{1}{2}\left|\frac{\partial(b^2,\alpha)}{\partial(\cos\theta,\phi)}\right|$$
$$=\left[\frac{Q_1Q_2}{2m_1v_1V}\right]^2\left[\sin\frac{\theta}{2}\right]^{-4}\left[1+\frac{1}{c^2}(V^2-2v_1{}^2-v_2{}^2)\right.$$
$$\left.+\left[\frac{m_1v_1V}{Q_1Q_2}\right]^2\left[\sin\frac{\theta}{2}\right]^2\frac{g_1g_2S_1S_2}{m_1m_2c^2}\sin\lambda_1\sin\lambda_2\cos(\omega_1+\omega_2-2\phi)\right]\ , \tag{3.5}$$

where we have used $\sin^4\theta/2\ll 1$ [allowing for the fact that (3.4) is valid for small angles only]. This result agrees with the one presented without proof in Ref. 14, where a second-order scalar contribution [behaving as $(\sin\theta/2)^{-3}$, see Ref. 12 for details] is also included.

In the gravitational case, the same method applied to (2.12b) leads to

$$\left[\frac{d\sigma}{d\Omega}\right]_G=\left[\frac{Gm_2}{2v_1V}\right]^2\left[\sin\frac{\theta}{2}\right]^{-4}\left[1+\frac{1}{c^2}(4V^2-2v_1{}^2-v_2{}^2)+\left[\frac{2v_1V}{Gm_2}\right]^2\left[\sin\frac{\theta}{2}\right]^2\frac{S_1S_2}{m_1m_2c^2}\sin\lambda_1\sin\lambda_2\cos(\omega_1+\omega_2-2\phi)\right]. \tag{3.6}$$

In Ref. 15 the gravitational scattering between a scalar particle and a spinning one is considered in the "fast motion" approach. No explicit expression for the cross section is given, but the balance equation for the outgoing momenta agrees with our formula (2.12b). Let us note, however, that the case we consider is more general because both particles are allowed to carry spin.

Note also that in both (3.5) and (3.6) the first significant deviation from the scalar case comes from the spin-spin terms[16] and that this contribution vanishes if we average over the orientations of the incoming spins.

Let us estimate the order of magnitude of those spin terms versus the post-Newtonian scalar contribution. First of all, we recall that our results are valid for small angles in the slow-motion approximation, that is, according to (3.3)

$$\frac{v^2}{c^2}\ll 1\ ,\quad \left[\sin\frac{\theta}{2}\right]_G\sim\frac{Gm}{bv^2}\ll 1\ ,$$
$$\left[\sin\frac{\theta}{2}\right]_{\rm EM}\sim\frac{|Q_1Q_2|}{bmv^2}\ll 1\ , \tag{3.7}$$

where it is clear that $\sin\theta/2\sim 1$ is the transition between bound and dispersion states and we have excluded this zone from the beginning (see the discussion at the end of Sec. I). The contributions in (3.5) and (3.6) of the post-Newtonian terms are of the order of v^2/c^2 times the Newtonian one; the spin terms contribute in both cases with $(\rho/b)^2$ times the Newtonian part, where we have noted

$$\rho_{\rm EM}\equiv\frac{gS}{mc}\ ,\quad \rho_G\equiv\frac{2S}{mc}\ . \tag{3.8}$$

For elementary particles, we have that $\rho_{\rm EM}$ has the order of magnitude of the Compton wavelength $\lambda\!\!\!^{-}$ so that the contribution of the spin terms is irrelevant at the classical level. It is interesting, however, to consider extended objects, where the gravitational case may be relevant; Eqs. (3.7) give the following bounds for v^2/c^2:

$$\frac{Gm}{c^2}\frac{1}{b}\ll\frac{v^2}{c^2}\ll 1\ . \tag{3.9}$$

Let us consider, for instance, the "superfast" pulsar 1937 + 214 (Ref. 17) whose rotation rate is 642 Hz, near the upper limit for a neutron star; the estimated moment of inertia in $\sim 10^{45}$ g cm^2 for standard density (1 $M_\odot$ with 10-km radius): so that $\rho\sim 2\times 0^{-1}$ km. On the other hand, Eq. (3.9) gives

$$\frac{1.5\ \text{km}}{b}\ll\frac{v^2}{c^2}\ll 1\ ,$$

so that the closest approach may be $b\sim 2\times 10^3$ km for $v/c\sim 10^{-1}$ in order to stay in the dispersion zone. In this case (the most favorable one) the contributions of the spin-spin terms would be at best 10^{-6} times the post-Newtonian one.

In the electromagnetic case, we can also consider extended bodies where $\mu_a\neq Q_a g_a S_a/m_a$. If we consider, for instance, neutral bodies with magnetic moment, the Newtonian contribution vanishes so that Eqs. (3.3)—(3.5) are no longer valid. In that case, we have, instead of (3.3)

$$\tan\theta=\frac{1}{b^3}\frac{\mu_1\mu_2}{m_1v_1Vc^2}\sin\lambda_1\sin\lambda_2\ , \tag{3.10a}$$

$$\phi-\alpha=\pi-(\omega_1+\omega_2-2\alpha)\ . \tag{3.10b}$$

The differential cross section in this case (scattering of neutral magnets) for small angles ($\sin^2\theta\ll 1$) is

$$\frac{d\sigma}{d\Omega}=\frac{1}{2}\frac{\partial(b^2,\alpha)}{\partial(\cos\theta,\phi)}=\frac{1}{9}\left[\frac{\mu_1\mu_2}{m_1v_1Vc^2}\frac{\sin\lambda_1\sin\lambda_2}{\sin^4\theta}\right]^{2/3}, \tag{3.11}$$

where we must note that $0\leq\lambda\leq\pi$ from (3.1). This expression does not vanish when one averages over the incoming orientations of the magnetic momenta, so this is a case in which the spin-spin terms are the dominant ones.

IV. THE SHORT-RANGE INTERACTION

Let us consider now the short-range interaction. The accelerations in this case are, up to the first order in the coupling constant in the post-Newtonian approximation,[18]

$$\left[\frac{d\vec{v}_1}{dt}\right]_{SR} = -\frac{Q_1Q_2}{m_1}\frac{e^{-\mu R}}{R^3}\left\{(1+\mu R)\vec{R} - \frac{(1+\mu R)}{c^2}[v_1{}^2\vec{R} + (\vec{R}\cdot\vec{v}_2)\vec{V}] - \frac{3+3\mu R+\mu^2R^2}{2c^2}\frac{(\vec{R}\cdot\vec{v}_2)^2}{R^2}\vec{R} - \frac{1+\mu R}{m_1c^2}(\vec{V}\times\vec{S}_1) + \frac{3+3\mu R+\mu^2R^2}{m_1c^2}\frac{(\vec{R}\cdot\vec{V})}{R^2}(\vec{R}\times\vec{S}_1)\right\}. \quad (4.1)$$

Let us substitute (4.1) into the right-hand side of (2.10). The integration is more involved than before, even at first order. Nevertheless, the result is easily expressed in terms of the modified Bessel functions K_ν (see the Appendix for details). If we suppose again that the incoming velocities of the two particles are parallel, the final result is

$$\left[\vec{v}_{out}\right]_{SR} = \vec{v}_1 - \frac{2\mu Q_1Q_2}{m_1 v}K_1(\mu b)\frac{\vec{b}}{b} \times \left[1 - \frac{v_1{}^2}{c^2} - \frac{v_2{}^2}{2c^2}\right], \quad (4.2)$$

where the contribution of the spin has disappeared. After taking the standard projections, we have

$$\phi - \alpha = \eta\left[-\frac{Q_1Q_2}{|Q_1Q_2|}\right], \quad (4.3a)$$

$$\tan\theta = \frac{2\mu\,|Q_1Q_2|}{m_1v_1V}K_1(\mu b)\left[1 - \frac{v_1{}^2}{c^2} - \frac{v_2{}^2}{2c^2}\right]. \quad (4.3b)$$

The asymptotic behavior of (4.2) for large impact parameters is that of K_1 (Ref. 19):

$$K_1(\mu b) \simeq \left[\frac{\pi}{2\mu b}\right]^{1/2} e^{-\mu b}\left[1 + \frac{3}{8\mu b} + \cdots\right]. \quad (4.4)$$

It follows from this that (4.2) is again a truncated asymptotic development: had we retained second-order terms in (4.1) (like those presented in Ref. 15), we would have obtained contributions behaving like $e^{-2\mu b}$ to (4.2).

Equation (4.3b) defines a one-to-one relationship between the impact parameter b and the dispersion angle θ in an implicit way. We shall state this as

$$b = \frac{1}{\mu}K_1{}^{-1}\left[\frac{m_1v_1V}{2\mu\,|Q_1Q_2|}\tan\theta\left[1 + \frac{v_1{}^2}{c^2} + \frac{v_2{}^2}{2c^2}\right]\right], \quad (4.5)$$

where it is clear that $K_1{}^{-1}$ stands for the inverse function of K_1.

The differential cross section can be computed, allowing for (4.3a), as in the scalar case:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left|\frac{db^2}{d\cos\theta}\right|,$$

where $b(\theta)$ is given by (4.5) in the small-angle limit. The result is the same as in the scalar case owing to the surprising fact that the outgoing velocity has no spin dependence although the acceleration has at the order considered.

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APPENDIX

We use the following integral expression for the functions $K_\nu(x)$ (Ref. 19):

$$K_\nu(x) = \frac{\pi^{1/2}}{(v-\frac{1}{2})!}\left[\frac{x}{2}\right]^\nu \int_1^\infty e^{-x\rho}(\rho^2-1)^{\nu-1/2}d\rho. \quad (A1)$$

The following relationships hold true[19]:

$$\frac{d}{dx}(x^{\pm\nu}K_\nu) = -x^{\pm\nu}K_{\nu\pm1}, \quad (A2)$$

$$K_{\nu+2} = K_\nu + 2\frac{\nu+1}{x}K_{\nu+1}. \quad (A3)$$

It is easy to show that

$$\int_{-\infty}^{\infty} R(\lambda)\left[\frac{e^{-\mu R}}{R}\right]d\lambda = \frac{2}{V}K_0(\mu b), \quad (A4)$$

where we have made

$$\rho\equiv\frac{1}{b}R(\lambda)(R)=\frac{1}{b}[b^2+(z+\lambda V)^2]^{1/2}\,,$$
$$z\equiv\vec{R}\cdot\vec{k}\,.$$

Let us consider the identities

$$\frac{1+\mu R}{R^3}e^{-\mu R}=-\frac{1}{b}\frac{d}{db}\left[\frac{e^{-\mu R}}{R}\right],$$
$$\frac{3+3\mu R+\mu^2R^2}{R^5}e^{-\mu R}=-\frac{1}{b}\frac{d}{db}\left[\frac{1+\mu R}{R^3}e^{-\mu R}\right].$$

We have then, allowing for (A2) and (A3), the results

$$\int_{-\infty}^{\infty}R(\lambda)\left[\frac{1+\mu R}{R^3}e^{-\mu R}\right]d\lambda=\frac{2\mu}{Vb}K_1(\mu b)\,, \tag{A5}$$

$$\int_{-\infty}^{\infty}R(\lambda)\left[\frac{3+3\mu R+\mu^2R^2}{R^5}e^{-\mu R}\right]d\lambda=\frac{2\mu^2}{Vb^2}K_2(\mu b)$$
$$=\frac{2\mu^2}{Vb^2}\left[K_0(\mu b)+\frac{2}{\mu b}K_1(\mu b)\right]. \tag{A6}$$

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