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Construction of spherically symmetric monopoles using the Atiyah-Drinfeld-Hitchin-Manin-Nahm formalism

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Nahm's equations are solved for the cases corresponding to the spherically symmetric SU(N) monopoles of Wilkinson and Bais and their solutions reconstructed using his adaptation of the Atiyah-Drinfeld-Hitchin-Manin construction for self-dual gauge fields. The analysis of this class of solutions reveals the remarkably intricate structure of the construction.

I. INTRODUCTION

In recent years considerable effort has been expended towards understanding self-dual fields, that is, gauge fields $F_{\alpha\beta}$ satisfying

$$F_{\alpha\beta}={}^*F_{\alpha\beta}=\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F_{\gamma\delta}\ . \tag{1.1}$$

Originally, work concentrated on the search for instanton solutions, four-dimensional Euclidean self-dual fields with finite action. Later on, the methods developed to deal with instantons were extended to the construction of static monopoles in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit.[1,2] There the gauge theory is augmented by a scalar Higgs field in the adjoint representation of the gauge group and solutions to the classical equation sought when the scalar self-coupling vanishes. The Bogomol'nyi equations,

$$\tfrac{1}{2}\epsilon_{ijk}F_{jk}=B_i=D_i\Phi\ ,\quad i,j=1,2,3\ , \tag{1.2}$$

are locally equivalent to the self-duality equations (1.1). To see this an extra coordinate x_4 is introduced, on which nothing depends, and W_4 is defined to be Φ.

Globally, instantons and monopoles are very different. One of the differences is in the boundary conditions. For instantons the vector potential approaches a pure gauge as x^μ ($\mu=1,\ldots,4$) approaches infinity, but for monopoles

$$\mathrm{tr}\Phi^\dagger\Phi\rightarrow a^2 \quad\text{and}\quad D_i\Phi\rightarrow 0\ \text{ as } |\vec{x}|\rightarrow\infty\ . \tag{1.3}$$

Also, instantons have a finite action while BPS monopoles have a finite energy. They share the property of being local minima for the action or the energy, respectively.

A convenient, but not entirely explicit, description of instantons has been given by Atiyah, Hitchin, Drinfeld, and Manin (ADHM) (Ref. 3) and recently extended to the monopole situation by Nahm.[4] However, little work has been done using the ADHMN construction to build explicit monopole solutions. In this paper we investigate the construction of spherically symmetric monopoles for the gauge group SU(N). Although these solutions have already been constructed in a different way by Wilkinson and Bais[5] our purpose here is to understand the mechanisms of Nahm's procedure, and the spherically symmetric monopoles are a good vehicle for this. They demonstrate admirably both the ingredients and the power of the formulation.

In Sec. II we shall review briefly the main features of the ADHM formalism that we need and introduce Nahm's adaptation of it. In Sec. III the requirement of spherical symmetry and how to implement it in the ADHMN construction is discussed. At the heart of Nahm's procedure is the set of equations

$$\frac{dT_l}{dz}=i\epsilon_{lmn}T_mT_n,\quad l,m,n=1,2,3 \tag{1.4}$$

which are of considerable interest in their own right. In the spherically symmetric situation T turns out to be a block tridiagonal matrix with a single unknown function for each block. Thus, Eq. (1.4) reduces to a coupled set of differential equations for those functions. The equations and their solutions are described in some detail in Sec. IV. Finally, the necessary steps using the matrices T to construct the monopoles and the details of some special cases are the subject of the final sections.

II. THE ADHM AND ADHMN CONSTRUCTIONS

A. ADHM

The basic ideas of the ADHM construction have been extensively reviewed.[6–8] Here, we shall confine ourselves to stating the main steps and establishing notation. For SU(N) the gauge potentials W_α, $\alpha=1,\ldots,4$, are given in terms of an $(N+2k)\times N$ matrix function of x, $v(x)$, by the expression

$$W_\alpha = v^\dagger \partial_\alpha v, \quad v^\dagger v = 1_N \ . \tag{2.1}$$

The matrix v is not arbitrary but constructed to satisfy

$$\Delta^\dagger_{\alpha i A} v_{ij} = 0, \quad A=1,2, \quad \alpha = 1,\ldots,k \ ,$$
$$i=1,\ldots,N+2k, \; j=1,\ldots,N \tag{2.2}$$

for the choice

$$\Delta_{\alpha i A} = a_{\alpha i A} + b_{\alpha i A'} x_{A'A} \ , \tag{2.3}$$

where the constant matrices a,b are each $(N+2k)\times 2k$, and x is defined to be

$$x = x_4 + i\vec{x}\cdot\vec{\sigma} \equiv x_\alpha e_\alpha \ . \tag{2.4}$$

A straightforward calculation of the field strength $F_{\mu\nu}$ in terms of the basic ingredients a,b yields

$$F_{\alpha\beta} = v^\dagger b e_{[\alpha} (\Delta^\dagger \Delta)^{-1} e^\dagger_{\beta]} b^\dagger v \ , \tag{2.5}$$

which is sensible and self-dual provided $(\Delta^\dagger\Delta)_{A\alpha\beta A'}$, a $2k\times 2k$ matrix, is invertible and proportional to the tensor product of a $k\times k$ Hermitian matrix and the unit 2×2 matrix. The action functional computed for the gauge potentials derived as above is $8\pi^2 k$.

Noting that Δ' defined by

$$\Delta' = Q\Delta R, \quad Q\in \mathrm{SU}(n+2k), \; R\in\mathrm{SU}(k) \tag{2.6}$$

where Q and R are constant matrices, yields the same vector potential as Δ, up to a gauge transformation, we can write a useful canonical form for a and b. We may choose

$$a_{\alpha i A} = \lambda_{\alpha i A}, \quad b_{\alpha i A} = 0 \ ,$$
$$\alpha = 1,\ldots,k, \; A=1,2, \; i=1,\ldots,N$$
$$a_{\alpha i A} = \mu_{\alpha i A}, \quad b_{\alpha i A} = (1_{2k})_{\alpha i A} \ ,$$
$$\alpha=1,\ldots,k, \; A=1,2, \; i=N+1,\ldots,N+2k \ , \tag{2.7}$$

in which case the conditions on $\Delta^\dagger\Delta$ may be rephrased:

$$\lambda^\dagger\lambda + \mu^\dagger\mu = (\text{Hermitian})_{k\times k}\otimes 1_2 \tag{2.8}$$

and $\mu = \mu_\alpha e_\alpha$, with

$$\mu^\dagger_\alpha = \mu_\alpha, \quad \alpha=1,\ldots,4 \ . \tag{2.9}$$

One of the strengths of the ADHM construction is that various Green's functions and solutions to Dirac equations can be written down explicitly in terms of the basic ingredients. Thus, for the fundamental representation of the gauge group, for example, we have

$$G(x,y) = \frac{v^\dagger(x) v(y)}{4\pi^2 |x-y|^2} \tag{2.10}$$

for the scalar Green's function and

$$\tilde{\psi} = v^\dagger b (\Delta^\dagger\Delta)^{-1}, \quad \tilde{\psi} = \psi^T \epsilon, \quad \epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.11}$$

is a complete set of k solutions to the massless Dirac equations. [In Eq. (2.11) the transpose only refers to the spinor indices of ψ.]

On the other hand, given a complete set of k-independent solutions to the Dirac equation in the background k-instanton field (with asymptotic behavior $W_\alpha \sim g^{-1}\partial_\alpha g$ as $|x|\to\infty$), normalized so that

$$\int d^4x \, \psi^\dagger \psi = \pi^2 1_k \ , \tag{2.12}$$

we can compute λ and μ_α.[4,8] They are given by the asymptotic behavior of $\tilde{\psi}$, and the ψ-expectation value of x_α, respectively. That is,

$$\tilde{\psi} \sim -g \frac{\lambda x^\dagger}{|x|^4} \quad \text{as } |x|\to\infty \tag{2.13}$$

and

$$\int d^4x \, \psi^\dagger x_\alpha \psi = -\pi^2 \mu_\alpha \ . \tag{2.14}$$

Using these expressions and the completeness of the Dirac zero modes we can directly verify the conditions (2.8) and (2.9) and work backwards to reconstruct the W_α.[8] This demonstrates the completeness of the ADHM construction and indicates the central role played by the Dirac equation.

B. ADHMN

Nahm's extension of the formalism to encompass monopoles[4] also uses the Dirac equation as the cornerstone for the construction. In this case, in a monopole background the Dirac equation reads

$$e^\dagger_\alpha D_\alpha \psi = 0 \quad (\text{or } D_\alpha \tilde{\psi} e_\alpha = 0) \ , \tag{2.15}$$

where $D_\alpha = \partial_\alpha + W_\alpha$, and $W_4 = \Phi$. The vector potentials and Φ are independent of x_4. Thus we may take ψ to be combinations of solutions of the form

$$\psi = e^{ix_4 z} \psi(\vec{x}, z) \ , \tag{2.16}$$

and, setting $\Phi = -i\phi$, with $\phi^\dagger = \phi$, Eq. (2.15) reads

$$(\vec{\sigma}\cdot\vec{\mathrm{D}} + \phi - z)\psi = 0 \ . \tag{2.17}$$

It is convenient to write the asymptotic behavior of ϕ in terms of the projection operators (which are angular functions) P_l on to its various eigenspaces. Thus

$$\phi \sim \sum_{l=1}^{N} P_l \left[z_l + \frac{k_l}{2|\vec{x}|} \right] \quad \text{as } |\vec{x}|\to\infty \ , \tag{2.18}$$

where the z's and k's are not necessarily distinct. In terms of these quantities the energy of the monopole is given by

$$E=4\pi\sum_{l=1}^{N} z_l k_l \;, \tag{2.19}$$

and the number of normalizable solutions to the Dirac equation can be counted using a theorem of Callias.[9] Their number is crucially dependent on the value of z and may be written

$$k(z)=\sum_{l=1}^{N} k_l\theta(z-z_l) \;. \tag{2.20}$$

In particular, there are no normalizable solutions at all unless z lies between the greatest and least eigenvalue of the asymptotic Higgs field. The number of solutions jumps, typically, as z moves past an eigenvalue of ϕ, not doing so only if one of the "charges" k_l happens to vanish. For maximally embedded spherically symmetric monopoles in SU(N) the charges are the set

$$N-1, N-3, \ldots, -N+1 \;. \tag{2.21}$$

Hence, if N is odd there is precisely one zero "jumping point," but not otherwise. The k's are ordered so that $\sum_1^r k_l \geq 0$, for $r=1,\ldots,N$.

In the absence of zero jumping point the solutions to the Dirac equation fall off sufficiently rapidly that there is no contribution from terms like λ in Eqs. (2.13) and (2.8). The other terms do exist, however, and Nahm defines

$$(T_i)_{rs}=\int d^3x\, x_i \psi_r^\dagger(\vec{x},z)\psi_s(\vec{x},z) \;, \qquad i=1,2,3;\;\; r,s=1,\ldots,k(z) \tag{2.22}$$

and $\Delta(z)$ to be the differential operator

$$\Delta(z)=i\frac{d}{dz}+i(\vec{T}+\vec{x})\cdot\vec{\sigma} \;. \tag{2.23}$$

T_4 is chosen to be zero by suitable arrangement of the Q,R transformations [Eq. (2.6)] which are now z dependent. This is done for convenience in subsequent computations. v is now also a function of z satisfying

$$0=\Delta^\dagger v=i\left[-\frac{d}{dz}+(\vec{T}+\vec{x})\cdot\vec{\sigma}\right]v \;. \tag{2.24}$$

The conditions (2.8) and (2.9) reduce to a set of differential equations for the Hermitian matrices T_i:

$$\frac{dT_i}{dz}=i\epsilon_{lmn}T_mT_n \;. \tag{2.25}$$

When Eq. (2.25) is satisfied $\Delta^\dagger\Delta$ is computed to be

$$\Delta^\dagger\Delta=-\frac{d^2}{dz^2}+(\vec{x}+\vec{T})^\dagger\cdot(\vec{x}+\vec{T}) \;, \tag{2.26}$$

which is (usually) invertible.

The general solution to Eq. (2.25) for $N=2$, $k=2$ has been discussed by Brown, Panagopoulos, and Prasad.[10,11] One of the main tasks of this paper is to find the class of solutions to (2.25) corresponding to the maximally embedded spherical monopoles.

Given v, computed from (2.34), the vector potential and Higgs field are recovered from the analog of Eq. (2.1). Thus

$$W_i=\int dz\, v^\dagger\partial_i v \;, \tag{2.27}$$

$$\Phi=-i\int dz\, zv^\dagger v \;, \tag{2.28}$$

$$1=\int dz\, v^\dagger v \;, \tag{2.29}$$

where the integrations are to be performed over the range spanned by the minimum and maximum eigenvalues of the asymptotic form of Φ, conventionally $z_1 \leq z \leq z_N$.

Some care has to be taken to ensure that Eq. (2.24) has precisely N independent solutions [for SU(N)] and this consideration influences the acceptable singularities that are allowed to occur in T_i. Specifically, the singularities can be no worse than poles in z. These must be constrained, if we are solving Eq. (2.25) on the interval $[z_i, z_{i+1}]$, to occur either outside the interval or at z_i, z_{i+1} but not inside. Equation (2.29) cannot be satisfied if the v on a specific subinterval is singular and so singularities in T restrict the class of normalizable solutions to Eq. (2.24). How this works in particular cases will be explained below. On any given interval between jumping points $[z_1,z_{i+1}]$, say, the dimension of T is fixed, hence we concentrate our attention on one interval at a time and sew the pieces together afterwards. How this can be done for the cases we consider will be discussed below, in Sec. IV.

When one of the jumping points corresponds to $k_l=0$ for a particular l, the situation changes qualitatively and the Dirac solutions are no longer enough to specify completely the gauge potentials and Higgs field. In that case there are also solutions to the covariant scalar Laplace equations (in the fundamental representation)

$$D^2\tau=0 \;,$$

which tend to covariant constants asymptotically, and fail to be normalizable. These solutions exist only for $k_l=0$ and $z=z_l$.

Let the quantities a_P and α_P be defined by

$$(a_P)_r=\int d^3x\, \psi_r^\dagger(\vec{x},z_P)(\vec{\sigma}\cdot\vec{D}+z_P)\tau_P(\vec{x}) \tag{2.30}$$

and

$$\alpha_P=(\alpha_P)_\beta e_\beta=ia_P^\dagger a_P \tag{2.31}$$

(where P labels any of the zero jumping points). Then the equations for v and $\vec{T}$, Eqs. (2.24) and (2.25), are modified to

$$\Delta^\dagger v+\sum_P a_P^\dagger s_P\delta(z-z_P)=0 \tag{2.32}$$

and

$$\frac{dT_l}{dz}=i\epsilon_{lmn}T_mT_n+\sum_P(\alpha_P)_l\delta(z-z_P) \;, \tag{2.33}$$

respectively. We ought to remark that in Eq. (2.32) we have suppressed all indices for clarity, but it should be noted that s_P is a *row* vector of length N so that all the dimensions match up correctly. Finally, the definitions of

W_i and Φ are also altered to read

$$W_i=\int dz\, v^\dagger \partial_i v+\sum_P s_P^\dagger \partial_i s_P \ , \tag{2.34}$$

$$i\Phi=\int dz\, z v^\dagger v+\sum_P z_P s_P^\dagger s_P \ , \tag{2.35}$$

$$1=\int dz\, v^\dagger v+\sum_P s_P^\dagger s_P \ , \tag{2.36}$$

respectively.

We have chosen to use up part of the Q,R equivalence [Eq. (2.6)] and set $T_4=0$. There still remain z-independent transformations in $\mathrm{SU}(k(z))$, on any particular interval $[z_i,z_{i+1}]$, although these must be matched appropriately at the jumping points. Thus, using T', defined by

$$T'=Q^\dagger T Q,\ \ Q^\dagger Q=1,\ \ \frac{dQ}{dz}=0\ , \tag{2.37}$$

instead of T, leads to gauge-equivalent potentials.

III. SPHERICAL SYMMETRY

A monopole solution is spherically symmetric if the effect of a spatial rotation of coordinates on any field can be canceled out by performing a compensating gauge transformation. To construct the fields at $\vec{x}$ and at its image $R\vec{x}$ under a spatial rotation we must use Δ,Δ^R defined by

$$\begin{aligned}\Delta&=\frac{id}{dz}+i\vec{x}\cdot\vec{\sigma}+i\vec{T}\cdot\vec{\sigma}\ ,\\ \Delta^R&=i\frac{d}{dz}+iR\vec{x}\cdot\vec{\sigma}+i\vec{T}\cdot\vec{\sigma}\ ,\end{aligned} \tag{3.1}$$

respectively. However, for each $R\in\mathrm{SO}(3)$ we can select an element of SU(2), $g(R)$, so that

$$R\vec{x}\cdot\vec{\sigma}=g(R)\vec{x}\cdot\vec{\sigma}g(R^{-1})\ .$$

Thus Δ^R constructs gauge-equivalent fields to those constructed from

$$\Delta'=\frac{id}{dz}+i\vec{x}\cdot\vec{\sigma}+iR^{-1}\vec{T}\cdot\vec{\sigma}\ . \tag{3.2}$$

The fields will therefore be spherically symmetric provided we can pick an element $Q(R)$ of $\mathrm{SU}(k(z))$ satisfying

$$Q(R^{-1})T_iQ(R)=R_{ij}T_j\ . \tag{3.3}$$

Clearly, $Q(R)$ represents SO(3) within $\mathrm{SU}(k(z))$ but, it is not normally an irreducible representation. However, we can use the freedom (2.37) to organize $Q(R)$ into a direct sum of its irreducible components

$$Q=\bigoplus_j Q_j\ , \tag{3.4}$$

where j is the spin of the representation Q_j. Thus, if we divide $\vec{T}$ into blocks, $\vec{T}^{jj'}$, corresponding to the various spin components of Q, Eq. (3.3) will read

$$Q_j(R^{-1})\vec{T}^{jj'}Q_j(R)=R\vec{T}^{jj'}\ . \tag{3.5}$$

From Eq. (3.5) we deduce that $\vec{T}^{jj'}$ can be nonzero only if the tensor product of spin j with spin j' contains a spin 1 in its decomposition. In other words,

$$\vec{T}^{jj'}=0\ \ \text{if}\ |j-j'|\neq 0,1,\ j,j'\ \text{not both zero}\ . \tag{3.6}$$

Thus, if we agree to arrange $\vec{T}$ into blocks whose dimension increases along the main diagonal, then $\vec{T}$ is block tridiagonal.

For the maximal embedding in SU(N) that we wish to consider, corresponding to the magnetic "charges" listed previously, Eq. (2.21), the $\vec{T}$'s have the following composition on each interval $[z_i,z_{i+1}]$:

$$[z_1,z_2]:\ \text{spin}\ J\ \left[J=\frac{N-2}{2}\right],$$
$$[z_2,z_3]:\ \text{spin}\ J,J-1\ ,$$
$$\cdots$$
$$[z_i,z_{i+1}]:\ \text{spin}\ J,J-1,\ldots,J-i+1$$

until the middle intervals, where *either*

$$\begin{aligned}&[z_{N/2},Z_{N/2+1}]:\ \text{spin}\ J,J-1,\ldots,0\ ,\\ &[z_{N/2};z_{N/2+1}]:\ \text{spin}\ J,J-1,\ldots,0\ \ \text{if}\ N\ \text{is even}\\ &or\\ &\left.\begin{matrix}[z_{(N-1)/2},z_{(N+1)/2}]\\ [z_{(N+1)/2},z_{(N+3)/2}]\end{matrix}\right\}\text{spin}\ J,J-1,\ldots,\tfrac{1}{2}\ \ \text{if}\ N\ \text{is odd}\end{aligned} \tag{3.7}$$

and then, for the rest,

$$\cdots$$
$$[z_{N-2},z_{N-1}]:\ \text{spin}\ J,J-1\ ,$$
$$[z_{N-1},z_N]:\ \text{spin}\ J\ .$$

On the main diagonal we write

$$\vec{T}^{jj}=a_j(z)\vec{C}^j,\ \ j=J,J-1,\ldots \tag{3.8}$$

where $\vec{C}^j$ corresponds to the maximal embedding of SU(2) in $\mathrm{SU}(2j+1)$. For the other entries $\vec{T}^{jj+1}$ and $\vec{T}^{j+1j}$ we may write

$$\vec{T}^{jj+1}=b_{j+1}(z)\,^-\vec{C}^{j+1},\ \ \vec{T}^{j+1j}=b_{j+1}(z)\,^+\vec{C}^{j+1}\ ,\qquad j=0,\ldots,J-1 \tag{3.9}$$

where

$$Q_j(R^{-1})\,^-\vec{C}^{j+1}Q_{j+1}(R)=R\,^-\vec{C}^{j+1} \tag{3.10}$$

and

$$(^-\vec{C}^{j+1})^\dagger=\,^+\vec{C}^{j+1}\ . \tag{3.11}$$

It should be remembered that $^-\vec{C}^j$ is a $(2j-1)\times(2j+1)$ matrix. The main properties of $\vec{C}^j$ and $^+\vec{C}^j$ are collected together in Appendix A. Explicit forms can be found in Ref. 12. All other components of $\vec{T}$ are zero.

We are now in a position to compute the Nahm equations (2.25) in terms of the a's and b's belonging to any particular interval in z. Thus,

$$\frac{da_j}{dz}\vec{C}^j = i(a_j{}^2\vec{C}^j \wedge \vec{C}^j + b_j{}^{2+}\vec{C}^j \wedge {}^-\vec{C}^j + b_{j+1}{}^{2-}\vec{C}^{j+1} \wedge {}^+\vec{C}^{j+1}) ,$$
$$\frac{db_j}{dz}{}^-\vec{C}^j = i(a_{j-1}b_j\vec{C}^{j-1} \wedge {}^-\vec{C}^j + a_jb_j{}^-\vec{C}^i \wedge \vec{C}^j) \tag{3.12}$$

and, using the properties of $\vec{C}^j$ and ${}^{\pm}\vec{C}^j$, we obtain

$$\frac{da_J}{dz} = -[a_J{}^2 + (2J-1)b_J{}^2], \quad J = \frac{N-2}{2}$$
$$\cdots$$
$$\frac{da_j}{dz} = -[a_j{}^2 + (2j-1)b_j{}^2 - (2j+3)b_{j+1}{}^2] , \tag{3.13}$$
$$\cdots$$

and

$$\frac{db_{j'}}{dz} = b_{j'}[(j'-1)a_{j'-1} - (j'+1)a_{j'}] . \tag{3.14}$$

Here, j runs over all the spins less than J which contribute to the $\vec{T}$ on a particular interval, and j' runs over all the spins greater than the lowest for that interval. If spin 0 is a possibility [as it is for the central interval for SU(N) and N even] then $a_0 \equiv 0$. This is, of course, because we cannot add spin 0 to itself to obtain spin 1. On the first and last interval, there is only one spin, and thus a single a_J satisfying

$$\frac{da_J}{dz} = -a_J{}^2 .$$

That is

$$a_J = \frac{1}{z - z_1} \text{ or } \frac{1}{z - z_N} .$$

The set of equations (3.13) and (3.14) are interesting in their own right and we shall explain their general solution in the next section. However, before doing so we need to say something about the way solutions on adjacent z intervals are patched together. Let us label the intervals as follows:

$$I_i = [z_i, z_{i+1}], \quad i = 1, \ldots, N-1 . \tag{3.15}$$

On the interval I_i the contributing spins are

$$j = J, J-1, \ldots, J-i+1, \quad i = 1, \ldots, \frac{N}{2} \text{ or } \frac{N-1}{2}$$
$$j = J, J-1, \ldots, J+i-N+1, \tag{3.16}$$
$$i = \frac{N}{2}+1 \text{ or } \frac{N+1}{2}, \ldots, N-1 .$$

For I_1, I_{N-1} there is a unique spin, the corresponding $a_J(z)$ is a single pole occurring at $z = z_1$, z_N, respectively. The residue of the pole is an irreducible representation of the SU(2) Lie algebra for spin J. For I_2, I_{N-2} there are two spins. For the larger J a_J is continuous at $z = z_2$, z_{N-1}, respectively, but for the other, $J-1$, a_{J-1} develops a pole at z_2 and z_{N-1}. The corresponding b_J's will go to zero. (Except when $J=1$ when this need not happen.) Thereafter, as i increases, the *new* spin contributions a_{J-i+1} have a pole at the boundary of adjacent intervals, the new b_{J-i+2} have a zero, but all other functions are continuous. The only exception to this rule occurs if N is odd. In that case $a_{1/2}(z)$, on the two central intervals $I_{(N-1)/2}$, $I_{(N+1)/2}$, is discontinuous at $z_{(N+1)/2}$, the "middle" eigenvalue. This discontinuity is allowed for by a corresponding delta-function contribution in the equations for $\vec{T}$ on these intervals [Eq. (2.33)]. The specific examples, discussed in Sec. V, should serve to clarify the rules.

IV. THE SOLUTION OF THE SPHERICALLY SYMMETRIC NAHM EQUATIONS

Equations (3.13) and (3.14) are quite a complicated set and it is rather remarkable that they can be solved in terms of a certain set of polynomials. We shall, first of all, describe how to solve the full set of equations for the interval $I_{N/2}$ if N is even, or $I_{(N-1)/2}$ if N is odd, these containing the maximum number of spins: $j = J, J-1, \ldots 0$ or $\frac{1}{2}$, respectively. The solutions to the equations on other intervals can be derived by truncating these.

The basic set of polynomials P_r, $r \geq 0$, is defined by the recurrence relation

$$\frac{dP_r}{dz} = rP_{r-1}, \quad r \geq 0 . \tag{4.1}$$

Although these polynomials are an infinite set, we shall need to use only the first $2J$ of them for our solutions.

Define the determinants E_n^r and F_n^r by

$$E_n^r = \det(E_n^r)_{ij}, \quad (E_n^r)_{ij} = P_{n-2(r+1)+i+j} , \quad i,j = 1, \ldots, r+1 \tag{4.2}$$

and

$$(n-2r)F_n^r = \frac{dE_n^r}{dz} . \tag{4.3}$$

It is quite easy to check that F_n^r differs from E_n^r only in its first column where each polynomial has its index lowered by one. Thus,

$$F_n^r = \det(F_n^r)_{ij}, \quad (F_n^r)_{ij} = (E_n^r)_{ij}, \quad j \neq 1, \; i = 1, \ldots, r+1$$
$$(F_n^r)_{i1} = P_{n-2(r+1)+i}, \; i = 1, \ldots, r+1 . \tag{4.4}$$

It is sometimes convenient to think of E_n^r and F_n^r as subdeterminants of the persymmetric matrix P defined by

$$P_{ij} = P_{i+j}, \quad i,j = 0,1,2, \ldots . \tag{4.5}$$

Let $q_{i_1, \ldots, i_r; j_1, \ldots, j_r}$ be the determinant of the $r \times r$ submatrix of P formed from the elements in the rows $i_{1, \ldots, i_r}$ and the columns $j_1 \cdots j_r$. Then

$$q_{i_1 \cdots i_r; j_1 \cdots j_r} = \sum_{\text{perm}\,\rho} (\text{sgn}\rho) \prod_{s=1}^{r} P_{i_s + j_{\rho(s)}} , \tag{4.6}$$

$$E_n^r = q_{01 \cdots r; n-2r, \ldots n-r} , \tag{4.7}$$

$$F_n^r = q_{01 \cdots r; n-2r-1, n-2r+1, \ldots n-r} . \tag{4.8}$$

The determinants q satisfy a generalization of persymmetry

$$q_{i_1+p,i_2+p,\ldots,i_r+p;\,j_1-p,j_2-p,\ldots,j_r-p}=q_{i_1\cdots i_r;\,j_1\cdots j_r}\,, \tag{4.9}$$

the differential relation

$$\frac{d}{dz}q_{i_1\cdots i_r;\,j_1\cdots j_r}=\sum_s i_s q_{i_1\cdots i_s-1,\cdots i_r;j_1\cdots j_r}+\sum_s j_s a_{i_1\cdots i_r;j_s\cdots,j_s-1,\cdots j_r}\,, \tag{4.10}$$

and are totally antisymmetric in the i or j indices separately, and symmetric under the interchange of all the i's and j's, i.e.,

$$q_{i_1\cdots i_r;\,j_1\cdots j_r}=q_{j_1\cdots j_r;\,i_1\cdots i_r}\,. \tag{4.11}$$

In terms of E_n^r, F_n^r the solution to the Nahm equations on the interval $I_{N/2}$ or $I_{(N-1)/2}$ is

$$a_{J-r}=\frac{F_{2J}^r}{E_{2J}^r}-\frac{F_{2J}^{r-1}}{E_{2J}^{r-1}}\,, \tag{4.12a}$$

$$(b_{J-r})^2=-\frac{E_{2J}^{r-1}E_{2J}^{r+1}}{(E_{2J}^r)^2},\quad r=0,1,\ldots,J-1 \text{ or } \frac{2J-1}{2} \tag{4.12b}$$

with the conventions

$$F_{2J}^{-1}=0,\quad E_{2J}^{-1}=1,\quad E_{2J}^0=P_{2J},\quad F_{2J}^0=P_{2J-1}\,.$$

Proving this assertion is straightforward for Eq. (3.14), less so for the others, Eqs. (3.13). In the case of (3.14) we can make direct use of the differential relation (4.3) to evaluate

$$\frac{1}{b_{J-r}}\frac{db_{J-r}}{dz}\,.$$

Explicitly, we have

$$\frac{1}{b_{J-r}}\frac{db_{J-r}}{dz}=\left[-(J-r+1)\left[\frac{F_{2J}^r}{E_{2J}^r}-\frac{F_{2J}^{r-1}}{E_{2J}^{r-1}}\right]+(J-r-1)\left[\frac{F_{2J}^{r+1}}{E_{2J}^{r+1}}-\frac{F_{2J}^r}{E_{2J}^r}\right]\right]$$

$$=-(J-r+1)a_{J-r}+(J-r-1)a_{J-r-1}\,, \tag{4.13}$$

which corresponds to Eq. (3.14) with $j'=J-r$.

To check the other equations we shall need an expression for dF_{2J}^r/dz and identities involving subdeterminants of a persymmetric matrix. Using Eq. (4.10) we have

$$\frac{dF_n^r}{dz}=(n-2r-1)H_n^r+(n-2r+1)K_n^r\,, \tag{4.14}$$

where

$$H_n^r=q_{0,1,\ldots,r;\,n-2r-2,n-2r+1,\ldots,n-r}$$

and

$$K_n^r=q_{0,1,\ldots,r;\,n-2r-1,n-2r,n-2r+2,\ldots,n-r}\,.$$

Alternatively, we could also write, using the persymmetry of F_n^r,

$$F_n^r=q_{1,2,\ldots,r+1;\,n-2r-2,n-2r,\ldots,n-r-1}\,,$$

in which case

$$\frac{dF_n^r}{dz}=G_n^r+(n-2r-2)H_n^r+(n-2r)K_n^r\,, \tag{4.15}$$

where

$$G_n^r=q_{0,2,\ldots,r+1;\,n-2r-2,n-2r,\ldots,n-r-1}\,.$$

Comparing (4.14) and (4.15) we derive the relation

$$G_n^r=H_n^r+K_n^r\,. \tag{4.16}$$

In Appendix B we use Jacobi's theorem on submatrices of an adjugate matrix to derive two useful identities amongst the various determinants introduced above:

$$E_n^rG_n^r-(F_n^r)^2=E_n^{r+1}E_n^{r-1} \tag{4.17}$$

and

$$H_n^{r-1}E_n^r+K_n^rE_n^{r-1}=F_n^{r-1}F_n^r\,. \tag{4.18}$$

Armed with these expressions and relations (4.14)–(4.18) it is now a matter of straightforward algebra to check the rest of Nahm's equations for the maximal number of spins.

So far we have discussed Nahm's equations on the central intervals $I_{N/2}$ or $I_{(N-1)/2}$ for N even or odd, respectively. On the other intervals, fewer spins enter but the same solution [(4.12a) and (4.12b)] will still work provided we arrange that $E_{2J}^{r+1}=0$ when $r=J-j_{\min}$ [where $j_{\min}=J-i+1$ on the interval I_i, Eq. (3.16)]. We can arrange this by setting

$$P_n=\sum_{k=1}^{r+1}\lambda_k(z-\alpha_n)^n,\quad n=2J \tag{4.19}$$

where λ_k and α_k are constants. Then, each of the $r+2$ columns of E_{2J}^{r+1} is a linear combination of columns with entries

$$(1,z-\alpha_i,(z-\alpha_i)^2,\ldots,(z-\alpha_i)^{r+1})$$

for $i=1,\ldots,r+1$. This construction yields the correct number of effective parameters for each interval, namely $2r+1=2(J-j_{\min})+1$.

The final issue to be settled in this section is the problem of matching the solutions on each interval I_i with solutions on the adjacent intervals $I_{i\pm1}$. We shall consider the two cases N even and odd separately. If N is even we can arrange the correct continuity, poles, and zeros on all the z intervals up to and including the central one by making the choice

$$P_{2J}=\lambda_1(z-z_1)^{2J}+\cdots+\lambda_m(z-z_m)^{2J} \tag{4.20}$$

on each of the intervals I_m, for $m=1,2,\ldots N/2$. Equally, to the right and including the central interval we can make the choice

$$P_{2J}=-[\lambda_{m+1}(z-z_{m+1})^{2J}+\cdots+\lambda_N(z-z_N)^{2J}] \quad (4.21)$$

on the intervals I_m for $m=N/2,\ldots,N-1$. Matching the solutions on the middle interval requires that (4.20) and (4.21) be identical for $z_{N/2}\le z\le z_{N/2+1}$. In other words,

$$\sum_{r=1}^{N}\lambda_r(z_r)^s=0 \quad \text{for } 0\le s\le 2J ,$$

which may be solved for λ_r, $r=1,\ldots N$, yielding

$$\lambda_r=\prod_{s=r}(z_r-z_s)^{-1} . \quad (4.22)$$

If N is odd, then to the left of the central jumping point $z_{(N+1)/2}$, we take (4.20) and to the right of it we take (4.21). For $z=z_{(N+1)/2}$ we must match the two expressions and also the expressions for $P_1,\ldots,P_{2J-1}$ derived from them. $a_{1/2}$ can be discontinuous at $z=z_{(N+1)/2}$ so P_0 need not be matched. These conditions lead to the same expression (4.22), for the λ's in terms of the jumping points $z_1\cdots z_N$.

In the next section we shall consider some special cases of these solutions to illustrate the general theory.

V. DETAILED EXAMPLES FOR $N=3,4$

For $N=2$, the $\vec{\mathrm{T}}$ matrices are all zero and the solution is the BPS monopole in the usual way.

For $N=3$ there are two intervals, $z_1\le z\le z_2$ and $z_2\le z\le z_3$, and correspondingly on each interval there is a single spin $J=\frac{1}{2}$, and the $\vec{\mathrm{T}}$'s may be written

$$\vec{\mathrm{T}}=\frac{\vec{\sigma}}{2(z-z_1)},\quad z_1\le z\le z_2$$
$$\vec{\mathrm{T}}=\frac{\vec{\sigma}}{2(z-z_3)},\quad z_2\le z\le z_3 . \quad (5.1)$$

Each is a simple pole at one of the extreme jumping points whose residue is an irreducible representation of the SU(2) Lie algebra of dimension 2. The discontinuity $\vec{\alpha}$ is computed to be

$$\vec{\alpha}=-\tfrac{1}{2}\vec{\sigma}\frac{(z_3-z_1)}{(z_2-z_1)(z_3-z_2)} , \quad (5.2)$$

and α_4 must be chosen suitably. To see how to choose α_4 we need to ensure that $\alpha_\mu e_\mu$ is rank one so that we can identify a row vector a and write $\alpha_\mu e_\mu=ia^\dagger a$. Thus

$$\alpha_4=i1_2\frac{(z_3-z_1)}{(z_2-z_1)(z_3-z_2)}$$

and

$$a=\left[\frac{(z_3-z_1)}{(z_2-z_1)(z_3-z_2)}\right]^{1/2}(0,1,-1,0) . \quad (5.3)$$

For $N=4$ we have three intervals and the solution may be conveniently described in terms of the polynomials P_0, P_1, P_2. Indeed, because of the relationship equation (4.1) we need only display the polynomials P_2 on each interval. Thus, for

$$z_1\le z\le z_2:\quad P_2=\lambda_1(z-z_1)^2,\quad a_1=\frac{1}{(z-z_1)} , \quad (5.4)$$

$$z_2\le z\le z_3:\quad P_2=\lambda_1(z-z_1)^2+\lambda_2(z-z_2)^2$$
$$=-\lambda_3(z-z_3)^2-\lambda_4(z-z_4)^2 ,$$
$$a_0=0,\quad (b_1)^2=\frac{(P_1)^2-P_2P_2}{(P_2)^2},\quad a_1=\frac{P_1}{P_2} , \quad (5.5)$$

$$z_3\le z\le z_4:\quad P_2=-\lambda_4(z-z_4)^2,\quad a_1=\frac{1}{(z-z_4)} . \quad (5.6)$$

Observe that the a_1 functions are continuous across each jumping point and that b_1 does not have a zero at z_2 or z_3 (as it need not do according to Nahm's rules). The central interval I_2 contains two spins (1,0), but a_0 is always zero.

To complete the analysis the other part of the story has to be considered. We shall need to solve Eq. (2.32) for the v's (and, when appropriate, the s's). This is the subject of the remainder of the paper.

VI. ANALYSIS OF THE EQUATION FOR v

The strategy we shall adopt to determine v is the following, again following Nahm.[4] It is a little simpler first to solve

$$\Delta w=0 . \quad (6.1)$$

Then, if v solves $\Delta^\dagger v=0$, v and w satisfy $(d/dz)(v^\dagger w)=0$ or

$$v^\dagger w=\text{constant} , \quad (6.2)$$

and v may be obtained by inverting the w matrix.

We shall have to solve Eq. (6.1) on each of the z intervals $I_1,\ldots,I_{N-1}$, but as before, to begin with, we shall consider the central intervals containing the maximal number of spins contributing to $\vec{\mathrm{T}}$. Recalling that Δ has the form

$$\Delta=i\left[\frac{d}{dz}+(\vec{\mathrm{x}}+\vec{\mathrm{T}})\cdot\vec{\sigma}\right] , \quad (6.3)$$

a suitable basis in which to express w consists of the tensor product of spin $\frac{1}{2}$ with each of the constituent spins in $\vec{\mathrm{T}}$. Thus we may write

$$w(z)=\sum_{j=0\text{ or }\frac{1}{2}}^{J}\left[\sum_{m=-j-\frac{1}{2}}^{j+\frac{1}{2}}w_{j,m}^{+}(z)\,|\,j,\tfrac{1}{2};j+\tfrac{1}{2},m\,\rangle+\sum_{m=-j+\frac{1}{2}}^{j-\frac{1}{2}}w_{jm}^{-}(z)\,|\,j,\tfrac{1}{2};j-\tfrac{1}{2},m\,\rangle\right] . \quad (6.4)$$

Substituting (6.4) into (6.1) using the expressions for $\vec{\mathrm{T}}$ and properties of the $\vec{\mathrm{C}}$, ${}^{\pm}\vec{\mathrm{C}}$ (Appendix A), we obtain a set of coupled differential equations for the coefficients $w_{jm}^{\pm}(z)$:

$$\frac{dw_{jm}^{+}}{dz}+ja_j w_{jm}^{+}+[(2j+1)(2j+3)]^{1/2}b_{j+1}w_{j+1\,m}^{-}+x_3\frac{2m}{2j+1}w_{jm}^{+}-2x_3\frac{[(j+\frac{1}{2})^2-m^2]^{1/2}}{2j+1}w_{jm}^{-}=0 \quad \text{for } |m|\le j+\tfrac{1}{2}\,, \tag{6.5a}$$

$$\frac{dw_{jm}^{-}}{dz}-(j+1)a_j w_{jm}^{-}+[(2j-1)(2j+1)]^{1/2}b_j w_{j-1\,m}^{+}-x_3\frac{2m}{2j+1}w_{jm}^{-}-2x_3\frac{[(j+\frac{1}{2})^2-m^2]^{1/2}}{2j+1}w_{jm}^{+}=0 \quad \text{for } |m|\le j-\tfrac{1}{2},\ j=J,J-1,\ldots,0 \text{ or } \tfrac{1}{2}\,. \tag{6.5b}$$

In evaluating Eq. (6.1) we have worked entirely on the x_3 axis. Hereafter, we shall relabel x_3 as r. The solutions for general $\vec{x}$ are obtainable from these by performing a spatial rotation. Notice that the equations do not mix differing values of m and, further, solutions for m negative can be obtained from those with positive m since

$$w_{j-m}^{+}(z,r)=(-)^j w_{jm}^{+}(z,-r)\,,$$
$$w_{j-m}^{-}(z,r)=(-)^{j+1} w_{jm}^{-}(z,-r)\,.$$

We need only concentrate on $m\ge 0$.

Each value of m will occur a number of times in the various tensor products. Thus $m=J+\frac{1}{2}$ can occur just once, $J-\frac{1}{2}$ three times, $J-\frac{3}{2}$ five times, and so on, so that for $m=J+\frac{1}{2}-s$ we should generally find $2s+1$ solutions to the equations. The only exception to this occurs when m happens to be zero. In that case there will be just $2J+1(=2s)$ solutions. The case $m=0$ can only occur when N is odd and then we also have to take into account the delta-function terms in Eqs. (2.22) and (2.33). For this reason we defer a discussion of the case $m=0$ until later.

As before, in Sec. IV, we shall write the solutions and verify that they satisfy Eqs. (6.5a) and (6.5b). Unfortunately, the solutions are quite complicated and we shall again have to rely heavily on the properties of persymmetric matrices outlined in Appendix B. One of the $2(J+1-m)$ solutions for a given m is relatively straightforward; the others fall conveniently into two sets which we shall refer to as $L^{(\omega)}$ and $R^{(\omega)}$, $1\le\omega\le s=J-m+\frac{1}{2}$. Thus, for the odd one out we can write

$$w_{jm}^{+}=\lambda_{jm}e^{-rz}\beta_{j+1}E_{2J-1}^{J-j-1}(E_{2J}^{J-j-1})^{-1}\,,$$
$$w_{jm}^{-}=\mu_{jm}e^{-rz}\beta_{j+1}E_{2J-1}^{J-j-1}(E_{2J}^{J-j-1})^{-1}\,, \tag{6.6}$$

where

$$\beta_j=(P_{2J})^{-1/2}(b_J)^{-1}(b_{J-1})^{-1}\cdots(b_j)^{-1}\,,$$
$$\beta_{J+1}=(P_{2J})^{-1/2}\,, \tag{6.7}$$

$$\mu_{jm}=-\lambda_{jm}\left[\frac{(j+\frac{1}{2}-m)}{(j+\frac{1}{2}+m)}\right]^{1/2},$$
$$\mu_{j+1m}=\lambda_{jm}\left[\frac{(2j+1)}{(2j+3)}\right]^{1/2}. \tag{6.8}$$

Equation (6.8) defines a recurrence relation for λ_{jm} for a given m, and we notice that $\mu_{j,j+1/2}\equiv 0=\lambda_{j,j+3/2}$ as it should. Using the identities

$$E_{2J}^{J-j}E_{2J-1}^{J-j-2}=E_{2J}^{J-j-1}F_{2J-1}^{J-j-1}-E_{2J-1}^{J-j-1}F_{2J}^{J-j-1}$$

and

$$E_{2J}^{J-j-1}E_{2J-1}^{J-j}=E_{2J-1}^{J-j-1}F_{2J}^{J-j}-E_{2J}^{J-j}F_{2J-1}^{J-j-1}\,. \tag{6.9}$$

The expressions (6.6) can be shown to satisfy Eqs. (6.5a) and (6.5b) without difficulty.

The construction of the other solutions is more intricate and some preliminary definitions are necessary. Let $\{M_p\}$ be another set of polynomials satisfying

$$\frac{dM_p}{dz}=pM_{p-1}\,, \tag{6.10}$$

but unrelated to the P's introduced before (4.1). In terms of these we define other sets of polynomials, one for each of ω as follows:

$$D_q^{(\omega)}=\begin{vmatrix} M_0 & \cdots & M_{\omega-2} & P_a \\ \vdots & & \vdots & \vdots \\ M_{\omega-1} & \cdots & M_{2\omega-3} & P_{q+\omega-1}\end{vmatrix},\quad \omega=2,\ldots,\ s(=J+\tfrac{1}{2}-m),\ m\le J-\tfrac{3}{2}$$
$$D_q^{(1)}=P_q,\quad m=J-\tfrac{1}{2}\,. \tag{6.11}$$

For each ω, $D_q^{(\omega)}$ is of degree q and again satisfies $dD_q^{(\omega)}/dz=qD_{q-1}^{(\omega)}$ properties easily checked using (4.1) and (6.10). We also need to define the quantities $A_i(m,j,\omega,z)$:

$$A_i(m,j,\omega,z)=\int_a^z dt_1\int_a^{t_1}dt_2\cdots\int_a^{t_{j-(1/2)-m+i}}dt_{j+1/2-m+i}\left[\sum_{\kappa=0}^{i-1}\binom{i-1}{\kappa}\frac{(2j+1+i)!}{(2j+\kappa+2)!}(2r)^{\kappa}D_{j+m+1/2+\kappa}^{(\omega)}\right]e^{2rt_{j+1/2-m+i}}\,. \tag{6.12}$$

If $\alpha=z_1$ then the A_i will be used to construct solutions in the set $L^{(\omega)}$: for $\alpha=z_N$ they will give solutions in the other set, $R^{(\omega)}$. In formula (6.12) the integrals actually span more than one interval. It is to be understood that the polynomials M_p are defined to satisfy Eq. (6.10) and for all $z_1 \le z \le z_N$, but that for each subinterval we use the appropriate set of polynomials P_r [Eqs. (4.19) and (4.20)], for that interval to define the quantities D_r. These match on the boundaries of adjacent intervals and vanish at z_1 and z_N.

Using these ingredients, the solutions are

$$w_{jm}^{+}(\omega)=\sigma_{jm}e^{-rz}\beta_{j+1}\Omega_{jm}^{(\omega)}(2r)^{j-J}(E_{2J}^{J-j-1})^{-1}\,, \qquad (6.13a)$$

$$w_{jm}^{-}(\omega)=\tau_{jm}e^{-rz}\beta_{j+1}\Theta_{jm}^{(\omega)}(2r)^{j-J-1}(E_{2J}^{J-j-1})^{-1}\,, \qquad (6.13b)$$

where

$$\sigma_{jm}=\left[\frac{2J+1}{2j+1}\right]^{1/2}\prod_{\kappa=j+1}^{J}\left[\frac{\kappa+\frac{1}{2}+m}{\kappa+\frac{1}{2}-m}\right]^{1/2},$$
$$\tau_{jm}=\left[\frac{2J+1}{2j+1}\right]^{1/2}\prod_{\kappa=j}^{J}\left[\frac{\kappa+\frac{1}{2}+m}{\kappa+\frac{1}{2}-m}\right]^{1/2}, \qquad (6.14)$$

and Ω, Θ are given in terms of A_i, $i=1\ldots J-j+2$, by the expressions

$$\Omega_{jm}^{(\omega)}=\begin{vmatrix} A_1(m,j,\omega) & A_2(m,j,\omega) & \cdots & A_{J-j+1}(m,j,\omega) \\ P_{2j+1} & & & P_{J+j+1} \\ \vdots & & & \vdots \\ P_{J+j} & & \cdots & P_{2J} \end{vmatrix}, \qquad (6.15)$$

$$\Theta_{jm}^{(\omega)}=\begin{vmatrix} A_2(m,j-1,\omega) & A_3(m,j-1,\omega) & \cdots & A_{J-j+2}(m,j-1,\omega) \\ P_{2j+1} & & & P_{J+j+1} \\ \vdots & & & \vdots \\ P_{J+j} & & \cdots & P_{2J} \end{vmatrix} \qquad (6.16)$$

for each choice of ω, and $\alpha=z_1$ or z_N.

Note, that when j happens to be $m-\frac{1}{2}$ we can compute the A_i exactly. They are

$$A_i=e^{2rz}D_{2m-2+i}\,.$$

Substituting these expressions into Eq. (6.16) yields $\Theta_{m-m/2}^{(\omega)}=0$. This is not strictly necessary, but desirable in view of the way we have chosen to write Eqs. (6.5a) with an extra term when $j=m-\frac{1}{2}$. A sketch of the way in which Eqs. (6.5a) and (6.5b) can be verified is given in Appendix C.

So far, in the preceding paragraphs, we have constructed a set of solutions valid for the central intervals in z, for which the maximum number of spins contribute to $\vec{T}$. On the other intervals there are fewer spins contributing. We can say that each value of m turns up at least twice because every interval contains spin J but as we move toward the center, from z_1 or z_N, one extra spin occurs at each interval. This means that as we move from left to right, for example, $m=J+\frac{1}{2}$ always occurs just once, $m=J-\frac{1}{2}$ occurs twice on the first interval, three times on the next one, and three times thereafter. $m=J-\frac{3}{2}$ occurs twice on the first interval, four times on the next, five times on the third interval, which is then its multiplicity thereafter, and so on. In other words, as we move from the central interval(s) towards z_1 and z_N the number of solutions drops and we might expect that the solutions we have already found will still work, except that some of them will fail to be independent (just as was the case for the a's and b's of Sec. IV). That this happens is guaranteed by the form of the polynomials P_r, Eqs. (4.19) or (4.20), and the way in which they are used to construct the polynomials $D_\sigma^{(\omega)}$. It is easy to see that when P_q is truncated then fewer of the polynomials $D_q^{(\omega)}$ are linearly independent on each interval. In fact, on the outside intervals I_1 or I_{N-1}, there is just one D polynomial of degree q and, moving in toward the center the number of independent polynomials increases by one at each jumping point. It is rather complicated to see how everything works in detail but we can look into some examples.

For a given m value, $L^{(\omega)}$ and $R^{(\omega)}$ are clearly solutions on those intervals for which $j_{\min} \le m-\frac{1}{2}$. This is because these are the intervals for which this value of m (or larger) achieves its maximum multiplicity. For the other intervals where $j_{\min} > m-\frac{1}{2}$ this is not so and we must do further calculations to establish the $L^{(\omega)}$ as solutions on the left-hand intervals and the $R^{(\omega)}$ as solutions on the intervals right of center.

Thus, on the intervals left of center we can show

$$A_i(m,j,\omega)=e^{2rz}\sum_{L=0}^{\infty}(-2r)^L D_{2j+L+i+1}\frac{(L+1)(L+2)\cdots(L+j+\frac{1}{2}-m)}{(j+m+\frac{3}{2})\cdots(2j+2+L)(j+\frac{1}{2}-m)!}\,, \qquad (6.17)$$

so that the determinants in the definition of $w_{j_{\min}m}^{\pm}$ via Eqs. (6.13)–(6.16) vanish on those intervals for which $j_{\min} > m-\frac{1}{2}$ for a given m. To show Eq. (6.17) we first note that

$$K_k^n(z)=\int_{z_1}^{z}dt_1\cdots\int_{z_1}^{t_{n-1}}dt_n D_k(t_n)e^{2rtn}=e^{2rz}\sum_{L=0}^{\infty}D_{n+n+L}(z)(-2r)^L\frac{(L+1)\cdots(L+n-1)}{(k+1)\cdots(k+n+L)(n-1)!}\ ,$$

in which case $A_i(m,j,\omega)=K_{j+m+1/2}^{j+m+3/2}$ agrees with Eq. (6.17) without further ado. For the others, the argument is more complicated. For example, the terms contributing to A_3,

$$A_3(m,j,\omega)=(2j+3)(2j+4)K_{j+m+1/2}^{j+(1/2)-m}+4r(2j+4)K_{j+m+3/2}^{j+1/2-m}+(4r)^2K_{j+m+5/2}^{j+1/2-m}\ ,$$

collect together on noting the identity

$$(2j+3)(2j+4)(L+j-\tfrac{3}{2}-m)(L+j+\tfrac{5}{2}-m)-2L(2j+4)(j+m+\tfrac{3}{2})(L+j+\tfrac{3}{2}-m)$$
$$+L(L-1)(j+m+\tfrac{3}{2})(j+m+\tfrac{5}{2})=(2j+L+3)(2j+L+4)(j+\tfrac{3}{2}-m)(j+\tfrac{5}{2}-m)\ .$$

For $i>3$ there are other useful identities, for example,

$$\sum_{k=0}^{i-1}(-)^k\begin{bmatrix}i-1\\k\end{bmatrix}(n_1-k)(n_2-k)\cdots(n_{i-2}-k)=0\ ,$$

$$\sum_{k=0}^{i-1}(-)^k\begin{bmatrix}i-1\\k\end{bmatrix}(n_1+k+1-i)(n_2-k)\cdots(n_{i-2}-k)=0\ .$$

The next step of the construction, given the set of w's, is to construct the quantity $v(z)$. Referring to the isolated solution, Eq. (6.6), as w_0 we must have

$$v^+(z)w_0(z)=C_1\ ,$$
$$v^+(z)L^{(\omega)}(z)=C_L^{(\omega)}\ , \tag{6.18}$$
$$v^+(z)R^{(\omega)}(z)=C_R^{(\omega)}$$

for each of the allowable values of w on each interval. In Eq. (6.18) the quantities C_1, $C_L^{(\omega)}$, $C_R^{(\omega)}$ are constants. Since $v(z)$ is supposed to be normalizable and its components that survive through a jumping point are continuous at the jumping point, the C's must be universal having the same value on each interval. On the other hand, $L^{(\omega)}$ and $R^{(\omega)}$ clearly vanish at $z=z_1$ or z_N, respectively [because of the definitions of the A_1, Eq. (6.12)] and so $C_L^{(\omega)}=0=C_R^{(\omega)}$. C_1 does not vanish and we may take $C_1=1$. If we decompose $v(z)$ in the same way as $w(z)$, Eq. (6.4), then Eqs. (6.18) are a set of linear equations for the components $v_{jm}^{\pm}(z)$. Since the original equations for $w_{jm}^{\pm}$ [6.5a) and (6.5b)] did not mix different values of m and neither do the differential equations for the components of v, we can arrange a basis of solutions labeled by m, w_m, where

$$w_m=\sum_{j=m-\frac{1}{2}}^{J}w_{jm}^{+}\,|j,\tfrac{1}{2};j+\tfrac{1}{2},m\rangle+\sum_{j=m+\frac{1}{2}}^{J}w_{jm}^{-}\,|j,\tfrac{1}{2};j-\tfrac{1}{2},m\rangle \tag{6.19}$$

for any of the solution sets ω_0, $L^{(\omega)}$, $R^{(\omega)}$, $\omega=1\cdots J-m+\frac{1}{2}$. Consequently, we can concentrate on the set of equations (6.18) for a fixed m, using this basis, and compute the $v_{jm}^{\pm}$ contributing to the analogous expression [to (6.19)] for v_m in the decomposition of v. We know of no general way of writing the solution to these equations other than straightforwardly as ratios of determinants.

We should also bear in mind that the number of components of $v(z)$ varies on each interval according to the number of participating spins on that interval. On the central interval(s) where the number of participating spins is a maximum we can write all the components of v as ratios of determinants. Formally, this solution set works on all the other intervals also, for the following reason. Wherever a particular component of v should have gone to zero, because the relevant spin has disappeared from the interval in question, a convenient zero will appear in the ratio of determinants. If a component of v survives then cancellations take place, effectively reducing the sizes of the determinants appearing in the ratio to that appropriate for the current interval, and its set of equations (6.18).

For example, if $m=J+\frac{1}{2}$ then on any interval the only equation to be solved is

$$(v_{JJ-1/2}^{+})^{*}w_{0JJ-1/2}^{+}=1\ . \tag{6.20}$$

If $m=J-\frac{1}{2}$ then on all but the first or last interval we have

$$(v_{JJ-1/2}^{+})^{*}w_{0JJ-1/2}^{+}+(v_{J-1,J-1/2}^{+})^{*}w_{0J-1,J-1/2}^{+}+(v_{JJ-1/2}^{-})^{*}w_{0JJ-1/2}^{-}=1\ , \tag{6.21a}$$

$$(v_{JJ-1/2}^{+})^{*}L_{JJ-1/2}^{+(1)}+(v_{J-1,J-1/2}^{+})^{*}L_{J-1J-1/2}^{+(1)}+(v_{JJ-1/2}^{-})^{*}L_{JJ-1/2}^{-(1)}=0\ , \tag{6.21b}$$

$$(v_{JJ-1/2}^{+})^{*}R_{JJ-1/2}^{+(1)}+(v_{J-1,J-1/2}^{+})^{*}R_{J-1J-1/2}^{+(1)}+(v_{JJ-1/2}^{-})^{*}R_{JJ-1/2}^{+(1)}=0\ . \tag{6.21c}$$

On the first and last intervals the middle terms in each equation will be missing. Hence both $R^{(1)}$ and $L^{(1)}$ cannot be solutions to the w equation on these intervals. We argued above that $L^{(1)}$ was a solution on the first interval, $R^{(1)}$ on the last. The other solution on these two intervals is a mixture of w_0 and $L^{(1)}$ or $R^{(1)}$, respectively. In other words, although Eq. (6.21b) survives into the first or last interval, a combination of (6.21a) and (6.21c) replaces that pair of equations. Alternatively, as mentioned above, solving (6.21a)–(6.21c) on all the intervals formally, and using the same expression on the first and last intervals, actually works since in the expressions for $v_{JJ-1/2}^{\pm}$ a cancellation occurs, and in the expression for $v_{J-1J-1/2}^{+}$ the denominator becomes infinite providing a zero. For other values of m and on intervals with less than maximum multiplicity we can always arrange, by adjusting the polynomials M, Eq. (6.10), that $R^{(\omega)}$ is a solution left of center for $\omega=2,\ldots,s$ (though not necessarily all independent) and, similarly, $L^{(\omega)}$, $\omega=2,\ldots,s$, right of center. The remaining solution is a mixture of w_0 and $R^{(1)}$ or $L^{(1)}$, respectively. A convenient choice of the polynomials M which is appropriate to the calculation of $v^\dagger v$ is given below [Eq. (7.3)]. Indeed, changing the constants in the polynomials M induces a linear transformation on the sets $L^{(\omega)}$ and $R^{(\omega)}$ but does not affect v at all. The result of calculating $v^\dagger v$, etc., will always involve just the special quantities $\hat{M}$ described below.

When N is odd the central pair of intervals must be treated carefully, in view of the discontinuity at $z_{J+3/2}$, and we have put off a discussion of the $m=0$ components of v until now. For the case $m=0$ we can find $v(z)$ in much the same way as when $m\neq 0$ and the lowest spin disappears. That is, it can be considered as a formal ratio of $(2J+1)$-dimensional determinants. $v(z)$ must be discontinuous at $z_{J+3/2}$ and, in general, C_1 need not be 1 on both sides of $z_{J+3/2}$. We have been unable to prove, but conjecture, that taking $C_1=1$ everywhere does in fact give the correct discontinuity in $v(z)$ (it is true for $J=\frac{1}{2},\frac{3}{2}$, see below). To be able to construct the Higgs field we need to know s. To find s we must first calculate a for the discontinuity in $a_{1/2}$ [Eq. (4.12a)]. Now

$$a_{1/2}=-\frac{F_{2J}^{J-3/2}}{E_{2J}^{J-3/2}}+\frac{F_{2J}^{J-1/2}}{E_{2J}^{J-1/2}} \tag{6.22}$$

and the discontinuity comes from the second term since it is the only one containing a P_0. α_i will only be nonzero in its $j=\frac{1}{2}$ part, that is the intersection of its first two rows and columns. From Eq. (6.22) this is

$$\frac{\lambda_{J+3/2}}{2}\frac{E_{2J}^{J-3/2}(z_{J+3/2})}{E_{2J}^{J-1/2}(z_{J+3/2})}\sigma_i(\equiv\epsilon\sigma_i)$$

leading to the following expression for a:

$$a=(0,\sqrt{2\epsilon},-\sqrt{2\epsilon},0,\ldots,0)\ .$$

Matching at $z_{J+3/2}$ in Eq. (2.32) then gives the following. Only $v_{\bar{1}/2,0}$ is discontinuous there and the components of s are computed from its discontinuity and the expression for a. We obtain

$$s_{1i}=\tfrac{1}{2}[v_{\bar{1}/2,0}]_{z_{J+3/2}^-}^{z_{J+3/2}^+}(\epsilon)^{-1/2}\delta_{i,J+3/2}\ . \tag{6.23}$$

The row vector s has only one nonzero component, using orthonormality. Since we do not know of any simple expression for $v_{\bar{1}/2,0}$ we cannot proceed further in evaluating s other than the implicit formula in terms of $v_{\bar{1}/2,0}$ obtained above.

VII. EVALUATION OF THE HIGGS FIELD

In this section we should like to use the machinery we have evolved in the preceding chapters to make contact with the previously known solutions of Bais and Wilkinson for maximally embedded spherically symmetric monopoles. Specifically, we can compute the Higgs field using the expression for v developed in Sec. VI and the formulas (2.28) and (2.29). Unfortunately, there are technical difficulties involved in this calculation which we have so far been unable to overcome (a recent paper of Panagopoulos[11] may be helpful in this respect). In a gauge for which the Higgs field is diagonal on the x_3 axis each value of m corresponds to a specific diagonal component. We can compute these straightforwardly for $m=J\pm\frac{1}{2}$ to obtain the result of Bais and Wilkinson.[5] For other values of $m\neq 0$ we are unable to complete the calculation without making a reasonable conjecture at an intermediate stage. The case $m=0$ we have been unable to deal with at all.

When $m=J+\frac{1}{2}$ we have, using Eqs. (6.20), (6.6), (6.7), and (6.8),

$$v^\dagger v=e^{2rz}P_{2J}(-)^{2J-1}\ .$$

Hence

$$\frac{\int_{z_1}^{z_N}dz\,zv^\dagger v}{\int_{z_1}^{z_N}dz\,v^\dagger v}=\frac{1}{2}\frac{d}{dr}\ln\int_{z_1}^{z_N}v^\dagger v\,dz\ ,$$

which we can evaluate using the expressions (4.19) and (4.20) for the polynomials P_{2J} on each of the intervals in z. We find

$$\phi_{NN}=\frac{1}{2}\frac{d}{dr}\left[\ln\sum_1^N\lambda_i e^{2rz_i}\right]-\frac{2J+1}{2}\frac{1}{r} \tag{7.1}$$

and, asymptotically,

$$\phi_{NN}\sim z_N-(J+\tfrac{1}{2})\frac{1}{r},\quad r\to\infty \tag{7.2}$$

as it should.

When $m=J-\frac{1}{2}$ we will have

$$(-)^{2J-1}v^\dagger v=\frac{2J+1}{4r^2}\frac{d^2}{dz^2}(v_{J,J-1/2}^\dagger)^2 -\frac{4J^2-1}{2r^2}e^{2rz}\{\hat{M}_0{}^2P_{2J}-2\hat{M}_1\hat{M}_0P_{2J-1} +\hat{M}_1{}^2P_{2J-2}\}\ ,$$

where

$$\hat{M}_k=\sum_{i=1}^N\lambda_i e^{2rz_i}(z-z_i)^k \tag{7.3}$$

are the particularly convenient choice of the polynomials M_k [Eq. (6.10)] alluded to previously. Defining

$$Q_{N-\alpha}(r)=\sum_{i,\ldots,i_\alpha=1}^N\lambda_{i_1}\cdots\lambda_{i_\alpha}e^{2r(z_{i_1}+\cdots+z_{i_\alpha})} \times\prod_{\beta<\gamma}(z_{i_\beta}-z_{i_\gamma})^2\ , \tag{7.4}$$

we find the next entry in the Higgs field to be

$$\phi_{N-1,N-1}=\frac{1}{2}\frac{d}{dr}(\ln Q_{N-2}-\ln Q_{N-1})-\frac{2J-1}{2}\frac{1}{r}\ . \tag{7.5}$$

It has the asymptotic behavior

$$\phi_{N-1,N-1}\sim z_{N-1}-(J-\tfrac{1}{2})r,\quad r\to\infty\ .$$

For $m=J+\frac{1}{2}-\alpha$ (>0) we are unable to perform the computation completely. However, suppose we conjecture a form for the quantity $v^\dagger v$, agreeing with the above two cases, as follows:

$$v^\dagger v=\frac{d^2X}{dz^2}+\gamma\sum_{i=0}^{2\alpha}e^{2rz}Y_iP_{2J-i}(-)^i\ . \tag{7.6}$$

In Eq. (7.6) X is a quantity vanishing quadratically at z_1 and z_N so, as far as computing the Higgs field is concerned it is irrelevant. γ is a function of J alone (independent of r,z) vanishing at $J=\alpha-\frac{1}{2}$. Y_i is defined in terms of the $\hat{M}$ and P polynomials:

$$Y_i=\sum_{\substack{j,k\geq 0,\\ j+k=i}}\tilde{M}_j\tilde{M}_k\ ,$$

where $\tilde{M}_j$ is the cofactor of P_{2J-j} in the determinant

$$\begin{vmatrix}\hat{M}_0 & \cdots & \hat{M}_{\alpha-1} & P_{2J-\alpha}\\ \vdots & & & \vdots\\ \hat{M}_\alpha & \cdots & \hat{M}_{2\alpha-1} & P_{2J}\end{vmatrix}\ .$$

Then,

$$\phi_{N-\alpha,N-\alpha}=\frac{1}{2}\frac{d}{dr}(\ln Q_{N-1-\alpha}-\ln Q_{N-\alpha})-\left[\frac{2J+1-2\alpha}{2}\right]\frac{1}{r}\ , \tag{7.7}$$

which, using the definition of the Q's, has the correct asymptotic behavior for large r. That Eq. (7.6) implies (7.7) is indicated in Appendix D.

For negative values of m the corresponding component of the Higgs field is found by replacing r by $-r$ in the expressions for positive m. They have the correct asymptotic behavior also, remembering that $z_1\leq z_2\leq\cdots\leq z_N$ and $\sum_i^N z_i=0$.

All of the expressions derived above agree with the corresponding expressions obtained by Bais and Wilkinson but, as mentioned before, we have not been able to treat fully the case $m=0$. However, one can work through the case for $N=3$ explicitly, and, for completeness, we include the results of this calculation.

We have

$$v^+_{1/2,0}=e^{-rz}P_1^{-1/2}\int_{z_1}^{z}dt\begin{vmatrix}\hat{M}_0P_0\\ M_1P_1\end{vmatrix}e^{2rt}\ ,$$

$$v^-_{1/2,0}=-\frac{e^{-rz}}{r}P_1^{-1/2}\left[\begin{vmatrix}\hat{M}_0P_0\\ \hat{M}_1P_1\end{vmatrix}e^{2rz}+\left[\frac{P_0}{P_1}-1\right]\int_{z_1}^{z}dt\begin{vmatrix}\hat{M}_0P_0\\ \hat{M}_1P_1\end{vmatrix}e^{2rt}\right]\ ,$$

$$s_{1i}=-\frac{e^{-rz}}{r}\left[\frac{\lambda_2}{2}\right]^{1/2}\left[-\hat{M}_1(z_2)e^{2rz_2}-\frac{1}{2r}\frac{\hat{M}_1(z_1)}{(z_2-z_1)}(e^{2rz_2}-e^{2rz_1})\right]\delta_{i,2}\ ,$$

$$v^\dagger v=\frac{1}{2r^2}\frac{d^2}{dz^2}(v^+_{1/2,0})^2\ .$$

Using these expressions, we find

$$\int_{z_1}^{z_3}dz\,v^\dagger v+s^\dagger s=-\frac{1}{2r^2}\sum_{i=1}^{3}\lambda_ie^{2rz_i}[\hat{M}_1(z_1)]^2=-\frac{1}{2r^2}\sum_i\lambda_ie^{2rz_i}\sum_j\lambda_je^{-2rz_j}\ ,$$

where a subtle cancellation of terms of order $1/r^2$ and $1/r^4$ has occurred. This is the same normalization factor as obtained for $m=J-\frac{1}{2}$, $J>\frac{1}{2}$. Finally,

$$\frac{\int_{z_1}^{z_3}dz\,zv^\dagger v+z_2s^\dagger s}{\int_{z_1}^{z_3}dz\,v^\dagger v+s^\dagger s}=\frac{\sum_i\lambda_iz_ie^{2rz_i}[\hat{M}_1(z_i)]^2}{\sum_i\lambda_ie^{2rz_i}\hat{M}_1(z_i)}=\frac{1}{2}\frac{d}{dr}\left[\ln\sum_i\lambda_ie^{-2rz_i}-\ln\sum_i\lambda_ie^{2rz_i}\right]\ .$$

That this is the correct result can be checked by taking the expressions for ϕ_{33} [Eq. (7.1)] and ϕ_{11} and insisting that ϕ be traceless.

VIII. CONCLUSION

We have seen (contrary to our own expectations) that it is not a simple exercise to carry through Nahm's construction even in the case of spherically symmetric monopoles. In fact, this task has taken us through a number of interesting mathematical problems whose elegant solutions would have come as a complete surprise had they been posed out of the present context. Among these, perhaps the most interesting is the solution to the equations for $\vec{T}$ described in Secs. III and IV, where we were able to obtain a complete solution in terms of ratios of polynomials.

Thus we conclude that Nahm's construction does not facilitate the description of spherically symmetric monopoles, but it may well still provide a way of understanding their parameter spaces and quantum numbers.

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APPENDIX A

In this appendix we collect together various properties of the quantities $\vec{C}^j$, ${}^{\pm}\vec{C}^j$ which we have used in Secs. III–VI. $\vec{C}^j$ is a spin-j representation of SU(2) so

$$\vec{C}^j \wedge \vec{C}^j = i\vec{C}^j . \tag{A1}$$

The quantities ${}^{\pm}\vec{C}^j$ have to satisfy Eq. (3.10) which implies, considering an infinitesimal rotation, that

$$-C_k^{j+1}C_l^{j+1} - C_l^j\, {}^-C_k^{j+1} = i\epsilon_{klm}\, {}^-C_m^{j+1} . \tag{A2}$$

(A1) and (A2) together imply that

$$\vec{C}^j\, {}^-\vec{C}^{j+1} \equiv 0 = {}^-\vec{C}^{j+1}\vec{C}^{j+1} . \tag{A3}$$

Clearly

$$\vec{C}^j \wedge {}^-\vec{C}^{j+1} = i\alpha\, {}^-\vec{C}^{j+1} . \tag{A4}$$

So, taking the cross product of (A4) with $\vec{C}^j$ on the left and the cross product of (A1) with ${}^-\vec{C}^{j+1}$ on the right we obtain

$$C_l^j\vec{C}^j\, {}^-C_l^{j+1} - j(j+1)\, {}^-\vec{C}^{j+1} = i\alpha\vec{C}^j \wedge {}^-\vec{C}^{j+1}$$
$$= -\alpha^2\, {}^-\vec{C}^{j+1} ,$$
$$C_l^j\vec{C}^j\, {}^-C_l^{j+1} - \vec{C}^j(\vec{C}^j\, {}^-\vec{C}^{j+1}) = i\vec{C}^j \wedge {}^-\vec{C}^{j+1}$$
$$= -\alpha\, {}^-\vec{C}^{j+1} .$$

Subtracting, and using (A3), we find $\alpha = j+1$ or $-j$. Also

$${}^-\vec{C}^{j+1} \wedge \vec{C}^{j+1} = i\beta\, {}^-\vec{C}^{j+1} , \tag{A5}$$

and a similar argument leads to $\beta = j+2$ or $-j-1$. However, using (A2) twice to reorder the left-hand side of (A5) and comparing with (A4) we have $\alpha = 2-\beta$. Hence $\alpha = -j$, $\beta = j+2$.

Similar manipulation leads to

$${}^+\vec{C}^j \wedge {}^-\vec{C}^j = i(2j-1)\vec{C}^j , \tag{A6}$$

$${}^-\vec{C}^{j+1} \wedge {}^+\vec{C}^{j+1} = -i(2j+3)\vec{C}^j \tag{A7}$$

on assuming a specific normalization for ${}^+\vec{C}^j \cdot {}^-\vec{C}^j$, viz.,

$${}^+\vec{C}^j \cdot {}^-\vec{C}^j = j(2j-1) . \tag{A8}$$

Other identities are obtained by conjugation.

In order to obtain the equations satisfied by components of w [Eqs. (6.5a) and (6.5b)] we made use of the following properties:

$$\vec{C}^j \cdot \vec{\sigma}\, | j, \tfrac{1}{2}; j+\tfrac{1}{2}, m \rangle = j\, | j, \tfrac{1}{2}; j+\tfrac{1}{2}, m \rangle ,$$

$$\vec{C}^j \cdot \vec{\sigma}\, | j, \tfrac{1}{2}; j-\tfrac{1}{2}, m \rangle = -(j+1)\, | j, \tfrac{1}{2}; j-\tfrac{1}{2}, m \rangle ,$$

$${}^{\pm}\vec{C}^j \cdot \vec{\sigma}\, | m, \tfrac{1}{2}; j\mp\tfrac{1}{2}, m \rangle = 0 ,$$

$${}^+\vec{C}^j \cdot \vec{\sigma}\, | j, \tfrac{1}{2}; j+\tfrac{1}{2}, m \rangle$$
$$= [(2j+1)(2j+3)]^{1/2}\, | j+1, \tfrac{1}{2}; j+\tfrac{1}{2}, m \rangle ,$$

$${}^-\vec{C}^j \cdot \vec{\sigma}\, | j, \tfrac{1}{2}; j-\tfrac{1}{2}, m \rangle$$
$$= [(2j-1)(2j+1)]^{1/2}\, | j-1, \tfrac{1}{2}; j-\tfrac{1}{2}, m \rangle ,$$

and

$$\sigma_3\, | j, \tfrac{1}{2}; j\pm\tfrac{1}{2}, m \rangle = \pm\frac{2m}{2j+1}\, | j, \tfrac{1}{2}; j\pm\tfrac{1}{2} m \rangle$$
$$- \frac{2}{2j+1}[(j+\tfrac{1}{2})^2 - m^2]^{1/2}$$
$$\times\, | j, \tfrac{1}{2}; j\mp\tfrac{1}{2}, m \rangle .$$

APPENDIX B

The basic tool used in deriving relations between determinants, such as (4.17) and (4.18), is a result of Jacobi which we may state as follows. Suppose that $\widetilde{M}^{(r)}$ is an $r\times r$ submatrix of the adjugate matrix $\widetilde{M}$ of the $N\times N$ matrix M, and $M^{(N-r)}$ denotes the $(N-r)\times(N-r)$ submatrix of M obtained from M by striking out the rows and columns similarly placed to the rows and columns of M used to define the elements of $\widetilde{M}^{(r)}$, then

$$\det\widetilde{M}^{(r)} = (\det M)^{r-1}\det M^{(N-r)} . \tag{B1}$$

To prove Eq. (4.17) we note that each of E_n^{r-1}, E_n^r, G_n^r, and F_n^r may be regarded as submatrices of E_n^{r+1}. Thus E_n^r is obtained on striking out the first row and column, E_n^{r-1} by striking out the first two rows and columns, F_n^r the first row and second column, and G_n^r by removing the second row and second column. Thus, a straightforward application of (B1) yields (4.17).

For Eq. (4.18) we have to be more subtle. We could regard H_n^{r-1}, E_n^r, K_n^r, E_n^{r-1}, F_n^{r-1}, and F_n^r as submatrices of E_n^{r+1} but it is not so helpful. It is more useful to consider $\hat{E}_n^{r+1}$ obtained from E_n^{r+1} by deleting the first row and repeating the last row. Clearly $\det\hat{E}_n^{r+1} = 0$. Then the listed matrices are submatrices formed by striking out the following listed rows and columns:

F_n^{r-1}: first and last row, first and third column.
F_n^r: next to last row, second column.
E_n^{r-1}: first and last row, first and second column.
K_n^r: next to last row, third column.
H_n^{r-1}: first and last row, second and third column.
E_n^r: next to last row, first column.

Applying (B1) to the 3×3 matrix found by the intersections of the first three columns of $\hat{E}_n^{r+1}$ with the first row and the last two rows yields (4.18).

The pair of equations listed in Eqs. (6.9) are proved in a similar way

APPENDIX C

In order to show that Eqs. (6.5a) and (6.5b) are actually solved by the expressions (6.13a) and (6.13b) we shall need to check that

$$\frac{d\Omega_{jm}}{dz}-(2j+1)\left\{\frac{F_{2J}^{J-j-1}\Omega_{jm}-E_{2J}^{J-j}\Theta_{j+1m}}{E_{2J}^{J-j-1}}\right\}-\frac{r(2j+1-2m)}{2j+1}\Omega_{jm}=\frac{j+m+\frac{1}{2}}{2j+1}\Theta_{jm} \tag{C1}$$

and

$$\frac{d\Theta_{jm}}{dz}-(2j+1)\left\{\frac{F_{2J}^{J-j}\Theta_{jm}+E_{2J}^{J-j-1}\Omega_{j+1m}}{E_{2J}^{J-j}}\right\}-\frac{r(2j+1+2m)}{2j+1}\Theta_{jm}=\frac{j-m+\frac{1}{2}}{2j+1}(2r)^2\Omega_{jm} \tag{C2}$$

each arising from a direct substitution of the ansatz into the equations. It is convenient to organize the calculations as follows. Firstly, compute each of the expressions in curly brackets to be $\widetilde{\Omega}_{jm}$, $\widetilde{\Theta}_{jm}$, respectively, where in (C1)

$$\widetilde{\Omega}_{jm}=\begin{vmatrix} A_1(m,j,\omega) & A_2(m,j,\omega) & \cdots & A_{J-j+1}(m,j,\omega) \\ P_{2J} & P_{2j+1} & \cdots & P_{J+j} \\ P_{2j+2} & P_{2j+J} & \cdots & P_{J+j+1} \\ \vdots & \vdots & & \vdots \\ P_{J+j} & P_{J+j+1} & \cdots & P_{2J} \end{vmatrix} \tag{C3}$$

and, in (C2)

$$\widetilde{\Theta}_{jm}=\begin{vmatrix} A_1(m,j-1,\omega) & A_j(m,j-1,\omega) & \cdots & A_{J-j+2}(m,j-1,\omega) \\ P_{2j} & P_{2j+1} & \cdots & P_{J+j+1} \\ \vdots & \vdots & & \vdots \\ P_{J+j-1} & P_{J+j+1} & \cdots & P_{2J} \end{vmatrix} . \tag{C4}$$

That these expressions are correct can be verified for the coefficient of each A_r in turn using the techniques of Appendix B applied to $\Delta_{ik}=P_{2j+(i+k)-2}$, $i,k=1\cdots J-j+1$. For (C3) all we need is a straightforward application of Jacobi's theorem to Δ. For (C4) we apply it to $\hat{\Delta}$, a modification of Δ in which the first column and last row are repeated. Clearly, $\det\hat{\Delta}=0$, and we make use of the second trick outlined in Appendix B.

Next, we can collect together the first two terms of each equation:

$$\frac{d\Omega_{jm}}{dz}-(2j+1)\widetilde{\Omega}_{jm}=\widetilde{\Omega}_{jm} ,$$

$$\frac{d\Theta_{jm}}{dz}-(2j+1)\widetilde{\Theta}_{jm}=\widetilde{\Theta}_{jm} ,$$

where

$$(\widetilde{\Omega}_{jm})_{1k}=\frac{dA_k}{dz}(m,j,\omega)-(k-1)A_{k-1}(m,j,\omega) ,$$
$$(\widetilde{\Omega}_{jm})_{ik}=P_{2J+(i+k)-2} \tag{C5}$$

and

$$(\hat{\Theta}_{jm})_{1k}=\frac{dA_k}{dz}(m,j-1,\omega)-(2j+k)A_k(m,j-1,\omega) ,$$
$$(\hat{\Theta}_{jm})_{ik}=(\hat{\Omega}_{jm})_{ik} \tag{C6}$$

for $i=2,\ldots,J-j+1$, $k=1,\ldots,J-j+1$. These are again proved for each coefficient of A_k or its derivative in turn, using the tricks of Appendix B. Finally, all the remaining determinants entering Eqs. (C1) and (C2) differ only in their first rows. Hence, we collect them together using the identities

$$\frac{dA_i}{dz}(m,j,\omega)=\frac{2r}{2j+1}(j-m+\tfrac{1}{2})A_i(m,j,\omega)+\frac{1}{2j+1}(j+m+\tfrac{1}{2})A_{i+1}(m,j-1,\omega)+(i-1)A_{i-1}(m,j,\omega) \tag{C7}$$

and

$$\frac{dA_{i+1}}{dz}(m,j-1,\omega)=\frac{2r}{2j+1}(j+m+\tfrac{1}{2})A_{i+1}(m,j-1,\omega)+\frac{(2r)^2}{2j+1}(j-m+\tfrac{1}{2})A_i(m,j,\omega)+(2j+i)A_i(m,j-1,\omega) . \tag{C8}$$

These follow from straightforward manipulations of the integrals occurring in the definition of A_i [Eq. (6.12)].

APPENDIX D

To show Eq. (7.7) we first note that

$$\frac{d}{dz}\sum_i Y_i P_{2J-\kappa-i}(-)^i=\sum_i Y_i P_{2J-\kappa-i-1}(-)^i(2J-2\alpha-\kappa) .$$

Then

$$\frac{\int_{z_1}^{z_N} dz\, z v^\dagger v}{\int_{z_1}^{z_N} dz\, v^\dagger v} = \left[\frac{1}{2}\frac{d}{dr}\left[\sum_{i=1}^{N}\lambda_i e^{2rz_i}[\widetilde{M}_\alpha(z_i)]^2\right] - \tfrac{1}{2}\sum_{i=1}^{N}\lambda_i e^{2rz_i}\frac{d}{dr}[\widetilde{M}_\alpha(z_i)]^2\right]\left[\sum_{i=1}^{N}\lambda_i e^{2rz_i}[\widetilde{M}_\alpha(z_i)]^2\right]^{-1} - \frac{2J+1-2\alpha}{2r} \tag{D1}$$

and we must evaluate

$$\sum_1^N \lambda_i e^{2rz_i}[\widetilde{M}_\alpha(z_i)]^2 \text{ and } \sum_1^N \lambda_i e^{2rz_i}\frac{d}{dr}[\widetilde{M}_\alpha(z_i)]^2 .$$

For convenience we put

$$\Lambda_{i\cdots j} = \lambda_i \cdots \lambda_j e^{2r(z_i + \cdots + z_j)}$$

so that from the definition of $\widetilde{M}$ preceding Eq. (7.7)

$$\widetilde{M}_\alpha = \sum_{i_1\cdots i_\alpha} \Lambda_{i_1\cdots i_\alpha}(z-z_{i_1})\cdots(z-z_{i_\alpha})^\alpha \prod_{\gamma<\delta}(z_{i_\gamma}-z_{i_\delta}) . \tag{D2}$$

The expression for $\widetilde{M}_\alpha$ can be split into sums of terms each containing α distinct subscripts chosen from $1,\ldots,N$. Thus summing over all permutations of the α elements we have typically

$$\widetilde{M}'_\alpha = \sum_\sigma \epsilon(\sigma)(z-z_{\sigma(1)})(z-z_{\sigma(2)})^2\cdots(z-z_{\sigma(\alpha)})^\alpha \Lambda_{\sigma(1)\cdots\sigma(\alpha)}\prod_{i<j}(z_i-z_j) .$$

Obviously, $\Lambda_{\sigma(1)\cdots\sigma(\alpha)}$ is completely symmetric from its definition so we only need to consider

$$\sum_\sigma \epsilon(\sigma)(z-z_{\sigma(1)})(z-z_{\sigma(2)})^2\cdots(z-z_{\sigma(\alpha)})^\alpha .$$

It is a polynomial of degree α and one way to see this is to compute $(d^{\alpha+1}/dz^{\alpha+1})\widetilde{M}^\alpha$ verifying that it vanishes. It also vanishes, not only when $z=z_i$ for some i but also if any pair of the z's happen to be equal. Hence

$$\widetilde{M}'_\alpha \propto \Lambda_{1\cdots\alpha}\prod_{i<j}(z_i-z_j)^2\sum_\sigma(z-z_{\sigma(1)})(z-z_{\sigma(2)})\cdots(z-z_{\sigma(\alpha)})$$

and

$$\widetilde{M}_\alpha \propto \sum_{i_1\cdots i_\alpha}\Lambda_{i_1\cdots i_\alpha}(z-z_{i_1})\cdots(z-z_{i_\alpha})\prod_{\gamma<\delta}(z_{i_\gamma}-z_{i_\delta})^2 . \tag{D3}$$

Armed with (D3) we can compute the first of the quantities we need:

$$\sum_{i=1}^N \Lambda_i[\widetilde{M}_\alpha(z_i)]^2 \propto \sum_{\substack{i,i_1\cdots i_\alpha\\ j_1\cdots j_\alpha}} \Lambda_{ii_1\cdots i_\alpha j_1\cdots j_\alpha}(z_i-z_{i_1})\cdots(z_i-z_{i_\alpha})(z_i-z_{j_i})\cdots(z_i-z_{j_\alpha})\prod_{\gamma<\delta}(z_{i_\gamma}-z_{i_\delta})^2(z_{j_\gamma}-z_{j_\delta})^2 . \tag{D4}$$

Analyzing terms corresponding to a fixed set $j_1\cdots j_\alpha$ we see from arguments along the same lines as above that the term is actually independent of $z_{j_1}\cdots z_{j_\alpha}$, in which case a factor $Q_{N-\alpha}$ [Eq. (7.4)] appears naturally in (D4). The other factor is proportional to $Q_{N-1-\alpha}$.

The second quantity is similarly computed to be proportional to $2Q_{N-1-\alpha}\,(d/dr)Q_{N-\alpha}$, with the same constant of proportionality. Substituting these results into (D1) yields the quoted result, Eq. (7.7).

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