Quantum mechanics of the gravitational field in asymptotically flat space

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An expression for the propagation amplitude between two gravitational field configurations in asymptotically flat space is given. It depends on two three-geometries and on an element of the Poincaré group which specifies the relative location of two hyperplanes in the Minkowski space at infinity. The amplitude is obtained by path integrating over gravitational fields in the proper-time gauge which was previously used by the author for compact spaces. The causality condition, which is imposed by admitting in the path integral only positive proper-time separations between the initial and final surfaces, implies that the amplitude is not annihilated by the generator of normal deformations. It is argued that, as a consequence, it is not permissible to regard quantized gravitation theory in asymptotically flat space as an "ordinary gauge theory" even if one is only interested in asymptotic processes.

I. INTRODUCTION

The dynamics of the gravitational field present additional aspects of interest when spacetime is open instead of closed in the sense of cosmology. The extra features of the asymptotically flat case stem from the need to fix in the classical action principle not only the initial and final three-geometries as in the closed situation, but also the relative asymptotic location of the two three-dimensional spaces within the Minkowski spacetime at infinity.

As a consequence, in the quantum theory, the propagation amplitude between two field configurations depends on two three-geometries and, in addition, on an element of the Poincaré group specifying the relative location of two spacelike hyperplanes at infinity.

This paper is devoted to examining this further dependence of the transition amplitude proposed in Ref. 1 (hereafter referred to as I) for the compact case. In order to keep the discussion as concise as possible we shall rely heavily on the aspects dealt with in I. This paper should therefore be read as a sequel to I.

The presentation is organized as follows: Section II reviews the classical action principle for the gravitational field in asymptotically flat space. Appropriate surface integrals related to the Poincaré group at infinity are included in the Hamiltonian according to the discussion previously given in Ref. 2 (hereafter referred to as A). Next, the propagation amplitude obtained by summing over histories in the proper-time gauge the exponential of i times the action of Sec. II is given in Sec. III and its dependence on spatial translations and rotations is analyzed in Sec. IV. Finally, the requirement of causality (integration over positive proper times only) is implemented in Sec. V. As a consequence the amplitude vanishes unless the hypersurfaces become parallel and separated by a positive proper time at infinity. It is then shown that the amplitude is not annihilated by the generator of normal surface deformations approaching the identity at infinity and that, as a result (Sec. VI) it is not permissible to regard quantized gravitation theory in asymptotically flat space as an "ordinary gauge theory" even if one is only interested in asymptotic processes.

Just as in I the work remains formal throughout since no attempt is made to give a definite meaning to the functional integrals considered.

II. CLASSICAL ACTION PRINCIPLE

The Hamiltonian for the gravitational field in asymptotically flat space was studied in detail in A, where references to earlier work may be found. It takes the form

$$H=\int (N^{\perp}\mathscr{H}_{\perp}+N^{i}\mathscr{H}_{i})d^{3}x-\alpha^{\mu}P_{\mu}+\tfrac{1}{2}\beta^{\mu\nu}M_{\mu\nu}\ , \tag{2.1}$$

where $\mathscr{H}_{\perp}$ and $\mathscr{H}_{i}$ are the gravitational generators given by (4.2) and (4.3) of I and where P_{μ} and $M_{\mu\nu}$ are integrals over a remote two-dimensional surface. These integrals are related to the generators of the

Poincaré group in the manner discussed in A (and in Secs. IV and V below), and their explicit forms may be found in Eqs. (5.8)–(5.14) of A. Note that the generator $\mathscr{H}_\perp$ and the function $N^\perp$ in (2.1) are the $\mathscr{H}$ and N used in I, but differ by factors $g^{\pm 1/2}$ from the $\mathscr{H}$ and N found in A. Expression (2.1) remains nevertheless valid since $g^{1/2}$ approaches unity asymptotically.

The numbers α^μ and $\beta^{\mu\nu}=-\beta^{\nu\mu}$ appearing in (2.1), which may depend on the coordinate time τ, specify the asymptotic part of the lapse and shift functions $N^\perp$, N^i:

$$N^\perp \underset{r\to\infty}{\longrightarrow} \alpha^\perp+\beta^\perp{}_i x^i \ , \tag{2.2a}$$

$$N^i \underset{r\to\infty}{\longrightarrow} \alpha^i+\beta^{ij}x^j \ . \tag{2.2b}$$

Geometrically Eqs. (2.2) express the fact that the deformation connecting two surfaces separated by an amount $\delta\tau$ of coordinate time becomes asymptotically an infinitesimal Poincaré transformation, which consists of a translation by an amount $\alpha^\mu\delta\tau$ and of a Lorentz transformation of "angle" $\beta^{\mu\nu}\delta\tau$.

The action principle associated with (2.1) is the statement that the classical histories of the system (spacetimes which obey Einstein's equations) are those which make the action functional

$$S=\int_{\tau_1}^{\tau_2} d\tau\, dx^3(\pi^{ij}\dot{g}_{ij}-H) \tag{2.3}$$

have an extremum with respect to variations of its arguments, $g_{ij},\pi^{ij},N^\perp,N^i$, obeying the following conditions.

(a) The three-dimensional metric $g_{ij}(x,\tau)$ is fixed up to a change of spatial coordinates which becomes the identity at infinity, both at $\tau=\tau_1$ and $\tau=\tau_2$; or what is the same, the three-geometry is fixed at τ_1 and τ_2.

(b) The functions $N^\perp$ and N^i are varied keeping fixed the net amount of Poincaré transformation at infinity [which is obtained by the composition of the infinitesimal transformations with parameters $\alpha^\mu(\tau)$, $\beta^{\mu\nu}(\tau)$ for the whole interval $\tau_1<\tau<\tau_2$].

Conditions (a) and (b) amount to fixing only the gauge-invariant boundaries of the competing histories in the action principle. Indeed, the action (2.1) is invariant under the gauge transformation discussed in Sec. IV A of I provided the functions $\epsilon^i(x)$ appearing in it vanish at infinity. Such a gauge transformation leaves invariant both the three-geometries at the end points and the net Poincaré transformation at infinity.

III. PATH INTEGRAL

The propagation amplitude is obtained by summing the exponential of i times the action over all histories which have in common the quantities held fixed in the classical action principle. Those quantities become then the arguments of the amplitude.

Accordingly, in the compact case the amplitude turns out to depend just on the initial and final three-geometries G_1,G_2. However, as stated in the previous section, there is an additional quantity held fixed in the asymptotically flat case, namely, a Poincaré transformation Π_∞, which determines the relative asymptotic location of the initial and final surfaces within the Minkowski space at infinity. The amplitude with its arguments displayed will then read

$$K[G_2,G_1;\Pi_\infty] \ . \tag{3.1}$$

In order to obtain an expression for (3.1) in terms of a path integral it is necessary to fix the gauge and determine the correct measure. This may be done along the same lines as for the compact case treated in I provided one takes due care of the behavior at spacelike infinity.

Thus, if to begin with we impose the proper-time gauge conditions

$$N^i=0 \ , \tag{3.2a}$$

$$\dot{N}^\perp=0 \ , \tag{3.2b}$$

then the analysis performed in Ref. 3 (hereafter referred to as II) leading to the effective action given by (4.23) and (4.24) of I remains unchanged and one arrives at the auxiliary amplitude (4.25) of I. That amplitude may be written as

$$\tilde{K}[2,1,T(x)]=\langle 2\,|\,\underline{\tilde{K}}[T]\,|\,1\rangle \tag{3.3}$$

with the operator $\underline{\tilde{K}}$ given by

$$\underline{\tilde{K}}=\exp\left[-i\int T(x)\mathscr{H}_\perp^{\text{eff}}(x)d^3x\right] \ , \tag{3.4}$$

and where we have denoted

$$T^\perp(x)=N^\perp(x)(\tau_2-\tau_1) \ . \tag{3.5}$$

Note that the amplitude $\underline{\tilde{K}}$ depends through $T^\perp(x)$ on the time translation and boost parameters $\alpha^\perp,\beta^\perp{}_i$ [which are now time independent due to (3.2b)]. Indeed we have

$$T^\perp(x) \underset{r\to\infty}{\longrightarrow} A^\perp+B^\perp{}_i x^i \ , \tag{3.6}$$

with

$$A^\perp=\alpha^\perp(\tau_2-\tau_1) \ , \tag{3.7a}$$

$$B^\perp{}_i=\beta^\perp{}_i(\tau_2-\tau_1) \ . \tag{3.7b}$$

In the compact case the final amplitude $K[G_2,G_1]$ is obtained from $\tilde{K}$ after two steps: One integrates over all positive $T(x)$ and one averages over all changes of spatial coordinates at one end point. [This averaging procedure which renders the amplitude coordinate invariant in both $g_{ij}(2)$ and $g_{ij}(1)$ was not explicitly mentioned in I but it is equivalent to folding the amplitude with coordinate-invariant wave functionals in the manner indicated in Sec. IV G of that paper.] These steps must be modified as follows in the asymptotically flat case:

(a) One integrates over all positive $T^{\perp}(x)$ with fixed asymptotic behavior [determined by $A^{\perp}$ and $B_r^{\perp}$ in (3.6)].

(b) One averages over all changes of spatial coordinates (diffeomorphisms) which tend asymptotically to a given combination of translation and rotation.

Equivalently one may average over all diffeomorphisms which become the identity at infinity and subsequently apply any diffeomorphism which tends to the given combination of translation and rotation at infinity. The answer will be independent of the choice of this latter diffeomorphism since (i) any two diffeomorphisms which have the same asymptotic behavior are related by a third one that tends to the identity at infinity and (ii) the average is to be performed with an invariant measure.

It is through step (b) above that the dependence of the amplitude on spatial translations and rotations at infinity is introduced so that in the final answer a full-fledged ten-parameter element of the Poincaré group appears.

Now, the generator of spatial diffeomorphisms which tend to a given combination of translation and rotation at infinity is obtained by setting $N^{\perp}=0$ in the Hamiltonian (2.1). Furthermore, one may regard a finite diffeomorphism as achieved by a succession of identical infinitesimal steps characterized by a vector field $T^i(x)$ ("canonical group coordinate"). Then item (b) above is implemented by acting on $\underline{\tilde{K}}$ given by (3.4) with the operator

$$\exp\left[-i\left[\int T^i\mathscr{H}_i d^3x - A^iP_i+\tfrac{1}{2}B^{ij}M_{ij}\right]\right] \tag{3.8}$$

with

$$T^i(x)\underset{r\to\infty}{\longrightarrow} A^i+B^i{}_jx^j\ , \tag{3.9}$$

and subsequently averaging in a gauge-invariant way over all $T^i(x)$ which share the same $A^i,B^i{}_j$.

[If the infinitesimal steps leading to the final diffeomorphisms are labeled by a parameter, which we might call λ rather than τ to emphasize that the operation is performed entirely within the final surface, then—as discussed in II—using canonical group coordinates amounts to choosing $N^{\perp}=0$, $\partial N^i/\partial\lambda=0$ in (2.1) and writing $T^i=N^i(\lambda_2-\lambda_1)$, $A^i=\alpha^i(\lambda_2-\lambda_1)$, $B^{ij}=\beta^{ij}(\lambda_2-\lambda_1)$ in analogy with (3.2), (3.5), and (3.7).]

Summarizing we obtain the following representation for the propagation amplitude:

$$\begin{aligned}K[G_2,G_1;\Pi_\infty]=\int DT^iD[\ln T^{\perp}]\Big\langle 2\,\Big|\exp\left[-i\left[\int T^i\mathscr{H}_i d^3x-A^iP_i+\tfrac{1}{2}B^{ij}M_{ij}\right]\right]\\ \times\exp\left[-i\left[\int T^{\perp}\mathscr{H}_{\perp}^{\text{eff}}d^3x-A^{\perp}P_{\perp}+\tfrac{1}{2}B^{\perp i}M_{\perp i}\right]\right]\Big|1\Big\rangle\ .\end{aligned} \tag{3.10}$$

In (3.10) the symbol DT^i represents the invariant measure over the diffeomorphism—a formal object which need not be discussed here any further (see II for a representation in terms of ghost fields), whereas $D[\ln T^{\perp}]$ stands for the product of $dT^{\perp}(x)/T^{\perp}(x)$ over all points of three-dimensional space. The integrations over $T^{\perp}$ and T^i are to be performed keeping fixed A^{μ} and $B^{\mu\nu}$ in (3.6) and (3.9). In what regards the states $|1\rangle,|2\rangle$, they are eigenstates of the metric tensor with eigenvalues which may be taken to be any representative of the three-geometries G_2,G_1, respectively, and they are also eigenstates of the ghost fields $C,\bar{C}$, with eigenvalues zero (see I and II). Since $\bar{C}=C=0$ are reparametrization-invariant statements the tangential generator $\mathscr{H}_i$ in (3.7) may be taken to be just the usual gravitational one given by (4.3) of I, without a part acting on $\bar{C},C$.

The Poincaré transformation Π_∞ appearing in (3.10) may be described in terms of the generators of the Poincaré group as

$$\begin{aligned}\Pi_\infty(A^{\mu},B^{\mu\nu})=\exp[-i(-A^iP_i+\tfrac{1}{2}B^{ij}M_{ij})]\\ \times\exp[-i(-A^{\perp}P_{\perp}+\tfrac{1}{2}B^{\perp i}M_{\perp i})]\ .\end{aligned} \tag{3.11}$$

It should be emphasized here that this last equation has been written just to display precisely how $A^{\mu},B^{\mu\nu}$ label an element of the Poincaré group. Thus (3.11) is self-contained without any mention of gravitation theory and the quantities $P_{\mu},M_{\mu\nu}$ appearing in it may be taken to be the generators of the Poincaré group in any representation. They *can-*

not, however, be thought of as being the surface integrals appearing in the Hamiltonian (2.1) since, as explained in A, those do not have well-defined commutators by themselves, i.e., separated from $\mathscr{H}_\perp$ and $\mathscr{H}_i$.

IV. SPATIAL TRANSLATIONS AND ROTATIONS AT INFINITY

The dependence of the amplitude (3.10) on the translation and rotation parameters at infinity deserves separate discussion both because it is of interest in its own right and also because it is quite different from the dependence of that same amplitude on time translations and boosts, which will be discussed in Sec. V below.

First we note that, as was already indicated, one may average first over the diffeomorphisms which tend to the identity at infinity and apply subsequently a fixed diffeomorphism which tends to a combination of translation and rotation at infinity. In that case the first exponential factor in (3.10) (the one containing T^i) is replaced itself by the product

$$\exp\left[-i\left[\int \xi^i\mathscr{H}_i d^3x - A^iP_i + \tfrac{1}{2}B^{ij}M_{ij}\right]\right] \times \exp\left[-i\int T^i\mathscr{H}_i d^3x\right] , \quad (4.1)$$

and the integration is carried over all $T^i(x)$ which vanish at infinity, whereas $\xi^i(x)$ is an arbitrary but fixed vector field which behaves asymptotically as

$$\xi^i(x) \underset{r\to\infty}{\longrightarrow} A^i + B^i{}_j x^j . \quad (4.2)$$

The decomposition of the Poincaré transformation corresponding to the splitting (4.1) is again (3.11).

Yet other representations also stemming from the basic gauge conditions (3.2) may be found. For example, one may rotate first and translate afterward. In that case one substitutes the product

$$\exp\left[-i\left[\int \eta^i\mathscr{H}_i d^3x - A'^iP_i\right]\right] \times \exp\left[-i\left[\int \rho^i\mathscr{H}_i d^3x - \tfrac{1}{2}B'^{ij}M_{ij}\right]\right] , \quad (4.3)$$

for the exponential factor containing ξ^i in (4.1). Here η^i and ρ^i are fixed but arbitrary vector fields obeying

$$\eta^i \underset{r\to\infty}{\longrightarrow} A'^i , \quad (4.4a)$$

$$\rho^i \underset{r\to\infty}{\longrightarrow} B'^i{}_j x^j . \quad (4.4b)$$

[Four factors altogether appear then in (3.11).] The associated Lorentz transformation is then written as

$$\Pi_\infty(A'^{\mu\nu}, B'^{\mu\nu}) = \exp[iA'^iP_i]\exp(-\tfrac{1}{2}iB'^{ij}M_{ij})\exp[-i(-A'^\perp P_\perp + \tfrac{1}{2}B'^{\perp i}M_{\perp i})] . \quad (4.5)$$

Here A'^i and B'^{ij} are certain functions of the A^i and B^{ij} appearing in (3.11) and vice versa whereas $A'^\perp = A^\perp$ and $B'^{\perp i} = B^{\perp i}$.

It follows from the preceding discussion that the propagation amplitude with asymptotic translation and rotation may be obtained from the one without them by applying to this last amplitude the exponential factors in (4.3). In other words, the operators

$$-\int \eta^i{}_{(r)}\mathscr{H}_i d^3x + P_r , \quad (4.6a)$$

$$\int \rho^i{}_{(rs)}\mathscr{H}_i d^3x + M_{rs} , \quad (4.6b)$$

with η and ρ behaving asymptotically as

$$\eta^i{}_{(r)} \underset{r\to\infty}{\longrightarrow} \delta^i{}_r , \quad (4.7a)$$

$$\rho^i{}_{(rs)} \underset{r\to\infty}{\longrightarrow} \delta^i{}_r x^s - \delta^i{}_s x^r , \quad (4.7b)$$

generate, respectively, the action of asymptotic translations and rotations on the propagation amplitude and are therefore the corresponding components of the total momentum and angular momentum.

It should again be stressed here that, as discussed in A, one must always keep together the surface and volume integrals as in (4.6), since each piece separately is not well defined as an operator when the vector field does not vanish at infinity.

The effect of the operators (4.6) on the amplitude (3.10) is independent of the choice of η^i and ρ^i within the class defined by (4.7). This is so because on account of the invariant group average over the diffeomorphisms which tend to the identity at infinity in (3.10) one has

$$\left[\int \xi^i\mathscr{H}_i d^3x\right]\underline{K} = 0 \quad (4.8)$$

for

$$\xi^i \underset{r\to\infty}{\longrightarrow} 0 . \quad (4.9)$$

V. TIME TRANSLATIONS AND BOOSTS: CAUSALITY

The dependence of the propagation amplitude on asymptotic time translations and boosts turns out to be simple but conceptually quite interesting and strikingly different from its dependence on asymptotic translations and rotations.

The special features in question are a consequence of the basic demand of causality, which is incorporated at a fundamental level in the formalism, by only including in the path integral those histories for which the initial hypersurface lies in the past of the final surface. In the proper-time gauge (3.2) this requirement means that one must integrate in (3.10) only over positive proper-time separations $T^{\perp}(x)$.

Now, if $T^{\perp}(x)$ is to be positive for all x then the asymptotic boost coefficient $B^{\perp}{}_i$ in (3.6) must vanish since otherwise $T(x)$ would be negative for some x. Similarly the asymptotic time-translation coefficient $A^{\perp}$ must be positive. So we find that

$$K[G_2,G_1,A^{\mu},B^{\mu\nu}]=0 \quad \text{if } A^{\perp}<0 \text{ or } B^{\perp}{}_i\neq 0 \ . \tag{5.1}$$

Equation (5.1) is invariant under Poincaré transformations at infinity (excluding time reversal, of course) since it expresses the geometrical statement that the amplitude is zero unless, when "interpolated in flat spacetime," surface 2 lies entirely in the future of, or coincides with, surface 1.

Now, for simplicity in what follows we shall henceforth assume that $A^i=B^{ij}=0$. There is no real loss of generality in this assumption since one can always recover the amplitude with rotation and translation by applying an appropriate operator to the one with $A^i=B^{ij}=0$, as was explained in Sec. IV above. Thus we shall be dealing with an amplitude

$$K[G_2,G_1,T_{\infty}]=\theta(T_{\infty})\int DT^iD[\ln T]\left\langle 2\left|\exp\left[-i\left[\int T^{\perp}\mathscr{H}_{\perp}^{\text{eff}}d^3x-T^{\perp}_{\infty}P_{\perp}\right]\right]\right|1\right\rangle \ . \tag{5.2}$$

Here θ denotes the Heaviside step function and the integration is carried over all T^i vanishing at infinity and over all positive $T^{\perp}(x)$ which tend to a fixed proper-time separation T_{∞} at infinity:

$$T^{\perp}(x) \underset{r\to\infty}{\longrightarrow} T^{\perp}_{\infty} \ . \tag{5.3}$$

(We have denoted here $A^{\perp}=T^{\perp}_{\infty}$ to have a more descriptive notation in this simplified case.)

The amplitude (5.2) is radically different from that of, say, a Yang-Mills field. The reason is that (5.2) is not annihilated by the generator of normal deformations $\mathscr{H}_{\perp}^{\text{grav}}$, namely, one has

$$\left[\int \xi^{\perp}(x)\mathscr{H}_{\perp}^{\text{grav}}(x)d^3x\right]\underline{K}\neq 0 \tag{5.4a}$$

for

$$\xi^{\perp}(x) \underset{r\to\infty}{\longrightarrow} 0 \ . \tag{5.4b}$$

[This statement is also valid for the compact spaces treated in I provided one disregards (5.4b) which is meaningless in that case.]

The drastic difference between the actions of $\mathscr{H}_{\perp}$ and $\mathscr{H}_i$ on the amplitude stems from the causality requirement which admits only positive proper-time separations $T^{\perp}(x)$ in (5.2) whereas all the tangential deformations (diffeomorphisms in three-space) are included.

In fact, if the left-hand side of (5.4a) were zero, it would amount to saying that the action of the exponential of

$$i\int \xi^{\perp}\mathscr{H}_{\perp}^{\text{grav}}d^3x$$

would have no effect on $\underline{K}$. However, such an action would effectively replace $T^{\perp}$ in the integrand of (5.2) by its composition with $\xi^{\perp}$, which we might denote by $T^{\perp}\circ\xi^{\perp}$. The point now is that if T runs over positive values only then $T^{\perp}\circ\xi^{\perp}$ does not have that same range of variation; for example, if $\xi^{\perp}$ is positive then $T^{\perp}_0\circ\xi^{\perp}$ does not start from zero but it spans rather only the "future of $\xi^{\perp}$." Therefore, if one would change the variable of integration from $T^{\perp}$ to $T^{\perp}_0\circ\xi^{\perp}$ the new variable would run over an interval different from that of the original one and the value of the integral would be changed. For tangential deformations on the other hand, all values of T^i are admitted so that the composition $(T^{\perp}\circ\xi^{\perp})^i$ of T^i and ξ^i runs over the same set of functions as T^i. Consequently one may establish (4.8) by acting on the amplitude with

$$\exp\left[i\int T^i\mathscr{H}_i\right]$$

and subsequently changing the variable of integration in (5.2) from T^i to $(T\circ\xi)^i$.

Incidentally, another consequence of the preceding discussion is that one cannot define generators of asymptotic translations and boosts that can meaningfully act on $\underline{K}$, in the same way as one defined the total momentum and angular momentum operators through (4.6): on account of (5.4) the corresponding definitions would not be independent of the choice of the auxiliary interpolating functions analogous to η and ρ in (4.6).

[The above reference to the composition of normal deformations permits one to capture the essence of the issue but must not be taken too literally. In fact, the metric g_{ij} appears explicitly in the commutator of two normal deformations, therefore deformations do not form a group (tangential ones do, however) and it is not possible to speak about their composition abstractly, without reference to g_{ij}. Thus in order to implement the argument in detail one should deal with the path integral leading to (5.2) as a whole, including the integration over $g_{ij}(x,\tau)$. It is, however, hardly necessary to go through all this labor since one may readily convince oneself, for example, of the validity of (5.4) in the particular case of zero signature ($\sigma=0$ in I) where deformations do form a group. Additional confirmation of (5.4) is provided by the analogy with the point particle given below. In any case the key point is that only half of the normal deformations are included in the integral over $T(x)$ in (5.2) and this basic feature is not undone by any other complications that the theory may have.]

What goes on in the preceding discussion may be recast in the following, perhaps more familiar, context: restricting the integration to positive $T^{\perp}(x)$ converts the amplitude in the functional counterpart of a Green's function ("Green's functional"), instead of making it a solution of the homogeneous equation that would arise by setting the left-hand side of (5.4) equal to zero.

The analogy with the point particle, so heavily relied upon in I, may be used to illuminate this fact. In that case

$$\underline{K}=\int_0^{\infty}\exp(-i\mathscr{H}T)dT \quad , \tag{5.5}$$

with

$$\mathscr{H}=p^2+m^2 \quad , \tag{5.6}$$

is a Green's function (the Feynman propagator), i.e.,

$$\mathscr{H}\underline{K}=\underline{1} \quad . \tag{5.7}$$

However, if we consider instead

$$\underline{\Delta}=\int_{-\infty}^{+\infty}\exp(-i\mathscr{H}T)dT=\delta(\mathscr{H}) \quad , \tag{5.8}$$

we have

$$\mathscr{H}\underline{\Delta}=0 \quad , \tag{5.9}$$

a solution of the homogeneous equation.

In the present case the analog of the identity operator on the right-hand side of (5.7) is not a simple object because of the coupling of the degrees of freedom of the gravitational field at different space points (see the discussion of perturbation theory in Sec. IV F of I). However, the key point, namely, that (5.4) is valid, has the same origin: the causality condition in proper time.

To finish this section we observe that the parallel just made poses an interesting question, namely, whether an analog of $\underline{\Delta}$ obeying (5.9) would be obtained by letting $T^{\perp}(x)$ in (3.10) run over both positive and negative values. This is indeed so. One may, in fact, show that such a $\underline{\Delta}$ obeys

$$\left[\int \xi^{\perp}\mathscr{H}_{\perp}^{\text{grav}}d^3x\right]\underline{\Delta}=0 \tag{5.10a}$$

and also, of course,

$$\left[\int \xi^{i}\mathscr{H}_{i}^{\text{grav}}d^3x\right]\underline{\Delta}=0 \tag{5.10b}$$

for ξ, ξ^i vanishing at infinity.

Equation (5.1) does not hold for the amplitude $\underline{\Delta}$ and one may define operators acting on it for all ten generators of the asymptotic Poincaré group in a manner analogous to that employed for spatial translations and rotations in (4.6). Nevertheless, as stressed in the next section, it appears that one should build the theory on $\underline{K}$ rather than on $\underline{\Delta}$.

The proof of (5.10) is achieved by using the techniques of II and the fact that the amplitude is independent of the choice of the gauge condition provided the Lagrange multipliers run from $-\infty$ to $+\infty$. The key point is to compare the amplitude derived in II with the one obtained from a different gauge condition. The new gauge condition is constructed by dividing the total time interval in three subintervals 1,2,3. Then in the first two intervals 1,2, one imposes the gauge conditions of II, whereas in interval 3 one requires $N^{\perp}=\xi^{\perp}$ and $N^{i}=0$. Equation (5.10a) then follows in the form

$$\exp\left[i\int \xi^{\perp}\mathscr{H}_{\perp}^{\text{grav}}\right]\underline{\Delta}=\underline{\Delta} \quad .$$

VI. CAUSALITY AND GAUGE INVARIANCE

The point of view is usually taken that, in asymptotically flat space, one may think of the quantized gravitational field as an ordinary field provided one is only interested in processes which are eventually observed at spacelike infinity. The group of four-dimensional diffeomorphisms which become the identity at infinity is then treated as an ordinary gauge group (noncompact, though) and its transformations are taken to have no effect on processes observable at infinity.

When translated into Hamiltonian language this view is consistent only if the propagation amplitude is annihilated by all the generators of the gauge group, as indeed occurs in Yang-Mills theory.

However, in the present case this property does not hold for the amplitude $\underline{K}$. Indeed, on account of (5.4), which itself stems from the causality condition, normal deformations generated by $\mathscr{H}_{\perp}$ cannot be considered as a gauge symmetry of the quantum theory. On the other hand, tangential deformations generated by $\mathscr{H}_i$ do qualify as a gauge symmetry thanks to (4.6).

It is to be noted here that the causality condition, which demands that only positive proper times should be admitted in the amplitude, is a statement that can only be formulated with the help of the spacetime metric. It effectively splits off one half of the four-dimensional diffeomorphism group in a way that cannot even be conceived in terms of the group by itself.

(The metric is automatically brought into the generators when passing to the Hamiltonian form of the classical theory because the normal deformations generated by $\mathscr{H}_\perp$ are the projections of the infinitesimal spacetime diffeomorphisms along the unit normal to a spacelike hypersurface—a metric concept.)

Incidentally, it should be emphasized here that the radically new feature [Eq. (5.4)] does not rely on the asymptotic part of the deformation. In fact (5.1) also holds for the causal transition amplitude between two field configurations in flat spacetime. It is, rather, the response of the amplitude to the action of the generator of normal deformations vanishing at infinity [Eq. (5.4b)] which is the key issue.

In a more descriptive language, (5.4) expresses the fact that the amplitude "remembers" that the transition from the initial to the final surface can occur only causally: in each history contributing to the amplitude the final surface must lie wholly (and not only asymptotically) in the future of the initial one. It is in this sense that what happens "inside" is not a physically irrelevant gauge freedom but leaves footprints in the propagation amplitude.

Lastly, we should remark that using the amplitude $\underline{\Delta}$ obeying (5.10) in place of $\underline{K}$ would correspond to the usual treatment in which the four-dimensional diffeomorphisms approaching the identity at infinity appear as an ordinary gauge group. However, it is this author's belief that the proper procedure, which correctly takes into account the fundamental requirement of causality, is to build up the theory using $\underline{K}$ rather than $\underline{\Delta}$ as the propagation amplitude.

We plan to examine in the future the more detailed implications of the causality condition.

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