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Radiation damping of color in classical SU(2) Yang-Mills theory

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A system of two pointlike isocharges in a classical non-Abelian Yang-Mills theory is studied by matched asymptotic expansions. It is assumed that the velocities are small compared with the speed of light and that the dimensionless quantity $\kappa \equiv |q| g^2$ is much smaller than unity (here g is the coupling constant and $|q|$ is the magnitude of either isocharge vector). The system is non-Abelian (and thus the Yang-Mills field equations are fully nonlinear), because the two isocharges are permitted to point in different directions in isospace. Radiation damping is shown to rotate the directions of the isocharge vectors while preserving their magnitudes—modulo higher-order effects. The charge vectors tend to rotate into commuting directions with respect to a basis that is parallel transported with respect to a background Abelian connection. In particular, bound systems approach an Abelian configuration after a time long compared with a characteristic time that depends on the magnitudes of the isocharges. This behavior suggests a classical analog of a "color-singlet" state in the quantized version of the theory.

I. INTRODUCTION

There is of course practically no evidence that classical Yang-Mills theories of the type to be treated below have any direct application in physics. Nevertheless, the study of classical gauge theories might at least give clues to some of the phenomena to be expected in quantum gauge theories, for which there exists incontrovertible evidence of physical relevance. As discussed in Ref. 1, classical solutions provide useful starting points for various semiclassical approximation procedures which make use of the path-integral approach to quantum fields.

General relativists have developed a collection of distinct techniques for calculating the effects of radiation reaction in gravitating systems. Unfortunately, because of the nonlinearities involved, there exist no exact methods for studying the sources of gravitational radiation in systems of practical interest, and no one has yet succeeded in obtaining rigorous estimates of the errors for the approximation methods. Controversies have thus arisen regarding the mathematical rigor and logical consistency of various "derivations" of radiation-reaction formulas.[2] Yang-Mills theories provide proponents of various techniques with a new arena to compare their methods.

The method of matched asymptotic expansions (see Ref. 3 for a geometrical description) has been applied to a number of radiation problems[4,5] and gives a *local* description of the effects of damping; such a description is advantageous when one seeks to describe the evolution of kinematic parameters which are not functionals of quantities whose rates of change can be calculated at "infinity." (For example, the period change due to gravitational radiation damping of a binary system cannot be inferred simply from a knowledge of the "energy" radiated at future null infinity.)

In non-Abelian gauge theories, one still has a conservation law relating the *energy* radiated at infinity to the kinematical energy of the system in question. However, color radiational modes are not necessarily associated with a change in the energy of the system.[6] Therefore, one must work either with the Yang-Mills field equations or with the covariant conservation law for color (or both). Trautman[7] introduces asymptotically null coordinates (u,r,θ,φ) and expands the vector potential in powers of $1/r$, (u,θ,φ) fixed. After using covariant current conservation, he concludes, "radiation of color is a truly nonlinear and non-Abelian phenomenon requiring at least two particles with noncommuting charges."

A local approach related to the classic "fast-motion" approximation in general relativity has been introduced by Drechsler, Havas, and Rosenblum.[8] Drechsler and Rosenblum[9] consider a single point charge in SU(2) Yang-Mills theory and evaluate the leading two orders in a weak-field (but not slow-motion) expansion. There seems to be no conflict with Trautman's conclusion if external forces vanish. However, it is not yet entirely clear from their complicated final expressions what effects (if any) damping causes when external fields are present or when the isocharge is accelerated by other means.

A generalization of the work of Drechsler and Rosenblum to the two-particle case has not yet been carried out. However, we can calculate the contribution of the lowest-order (linearized) fields to the equation of covariant current conservation for the two-particle case. Using their Eqs. (36), (38), and (42), we obtain an isocharge evolution equation for (say) particle 1 of the form

$$\dot{\vec{Q}}_1+\dot{\vec{Q}}_1\times\vec{Q}_1\sim\frac{u^\mu v_\mu}{(z_1^\mu-z_2^\mu)v_\mu}(\vec{Q}_2\times\vec{Q}_1) + (\text{nonlinear terms of same order}) , \quad (1.1)$$

where u^μ and v^μ are the velocities of particles 1 and 2, respectively, corrected for the lowest-order Coulomb-type accelerations. However, note that this expression does not yet include the—not *necessarily* smaller—corrections due to the first nonlinear order of the field equations.

In this paper, a simple, explicit, approximate expression for the damping of a system of two pointlike isocharges in SU(2) Yang-Mills theory will be obtained. The velocities are assumed to scale with the parameter $\epsilon \ll 1$ (in units such that $c=1$), and the magnitudes of the charges are assumed small compared with g^{-2}, where g is the coupling constant. That is, the currents are taken proportional to a dimensionless parameter $\kappa \ll 1$. The leading effects of radiation reaction on the charged mass points will be given by the expression

$$\frac{d}{dt}(\vec{Q}_1)=\tfrac{2}{3}(\vec{Q}_1\times\vec{Q}_2)(\dddot{R}^i_1 R^i_2) + \text{total time derivative} ,$$
$$\vec{Q}_1,\vec{Q}_2=O(\kappa\epsilon),\quad \epsilon \ll 1,\ \kappa \ll 1 , \tag{1.2}$$

where R^i_1 and R^i_2 are the positions of particles 1 and 2, respectively, $\vec{Q}_1$ and $\vec{Q}_2$ are their respective isocharge vectors at time t, and an overdot means d/dt. The corresponding expression for particle 2 is obtained by interchanging the subscripts.

In fact, one is really studying a two-parameter family of systems,[3] and the criterion of "validity" is not convergence, but rather that the errors be smaller than some function (in this case a power) of κ and ϵ. Assuming the existence of asymptotic expansions with respect to these two small parameters, one searches for the leading contribution to the isocharge evolution equations whose sign reverses according to whether the (approximate) radiation field is taken as outgoing or incoming. In a nonlinear theory, this "outgoing-radiation" requirement does not necessarily imply a condition for the absence of incoming radiation at past null infinity. However, such a condition will be satisfied at the leading order within our expansion method, to be described below.

II. EXPANSION PROCEDURE

Let us begin by considering the SU(2) Yang-Mills field equations in the form

$$\vec{F}_{\mu\nu}{}^{,\mu}+\vec{A}^{\mu}\times\vec{F}_{\mu\nu}=-4\pi\vec{j}_{\nu} , \tag{2.1}$$

$$\vec{F}_{\mu\nu}\equiv\vec{A}_{\nu,\mu}-\vec{A}_{\mu,\nu}+\vec{A}_{\mu}\times\vec{A}_{\nu} . \tag{2.2}$$

(Indices are raised and lowered with the Minkowski metric diag$[-1,1,1,1]$.) The arrows and cross products indicate isovectors and iso-cross-products. Exterior differentiation yields as an identity the "covariant" conservation of current

$$\vec{j}^{\,\nu}{}_{,\nu}+\vec{A}^{\nu}\times\vec{j}_{\nu}=0 . \tag{2.3}$$

It will be convenient to impose the gauge condition

$$\vec{A}^{\mu}{}_{,\mu}=0 . \tag{2.4}$$

The field equations (2.1) and (2.2) then take the simplified form

$$\begin{aligned}\Box\vec{A}_{\nu}&\equiv\vec{A}_{\nu,\mu}{}^{\mu}\\ &=-4\pi\vec{j}_{\nu}-2\vec{A}^{\mu}\times\vec{A}_{\nu,\mu}+\vec{A}^{\mu}\times\vec{A}_{\mu,\nu}\\ &\quad+\vec{A}^{\mu}\times(\vec{A}_{\mu}\times\vec{A}_{\nu}) .\end{aligned} \tag{2.5}$$

Note that the coupling constant g has been taken to be equal to unity. The isocharge thus is dimensionless. (Solutions for other values of the coupling constant can be recovered by a simple rescaling of the fields and currents.)

We will be interested in the interactions of isocharged particles. For an isolated point charge at rest at the position $r^i(t)$,

$$\vec{\rho}\equiv\vec{j}^{\,t}=\vec{q}\,\delta(x^i-r^i),\quad \vec{j}^{\,i}=0 , \tag{2.6}$$

Eqs. (2.4) and (2.5) have the elementary Abelian Coulomb solution

$$\vec{A}^{t}=\vec{q}\,|x^i-r^i|^{-1},\quad \vec{A}^{j}=0 . \tag{2.7}$$

Suppose now that two particles are present, but that their isocharges point in different directions. Then the field equations no longer reduce to the Abelian or electromagnetic case and the solution cannot be obtained by superposition. Nevertheless, one can construct an approximate solution in the limit $\kappa\to 0$ as the dimensionless magnitudes of the isocharges approach zero.

It is instructive to note a fundamental difference between the Yang-Mills equations and the Einstein field equations: In general relativity, one can construct a "weak-field" expansion in the vicinity of two black holes of *any* mass, provided that their separation is much larger than either mass in units where $G=1$ (the immediate neighborhoods of the black holes need strong-field expansions, of course, even if they are widely separated). This procedure succeeds because the nonlinear terms in the Einstein field equations involve squares of connection coefficients and therefore die off relatively fast. In the Yang-Mills equations (2.5), the nonlinear terms involve undifferentiated or only once-differentiated combinations such as $\vec{A}^{\mu}\times(\vec{A}_{\mu}\times\vec{A}_{\nu})$, and thus if $\vec{A}^{\mu}$ has $1/r$ behavior, the nonlinearities have the same order of magnitude compared to the linear terms at any distance. Thus, the expansions used here, as in Refs. 9 and 10, apply only in the limit $|q|g^2 \ll 1$.

Let us introduce the additional assumption that the particles move slowly compared to $c=1$. Suppose the minimum distance between the particles during some interval of interest is L. The unit of distance in Eqs. (2.4) and (2.5) is taken as L. Let ϵ be the slow-motion parameter. Then the fields change on a time scale $\lambda=L/\epsilon$ or slower in the near zone, which is a region surrounding the sources whose size scales with L in the limit $\epsilon\to 0$. To lowest order in the charge parameter κ, the near-zone expansions of currents and fields can be given the following form:

$$\vec{\rho}\sim\kappa[\vec{q}_1\delta^3(\epsilon X^i-\epsilon R^i_1)+\vec{q}_2\delta^3(\epsilon X^i-\epsilon R^i_2)]+O(\kappa^2) , \tag{2.8a}$$

$$\vec{j}^{\,i}\sim\kappa\epsilon[\vec{q}_1\dot{R}^i_1\delta^3(\epsilon X^i-\epsilon R^i_1)+\vec{q}_2\dot{R}^i_2\delta^3(\epsilon X^i-\epsilon R^i_2)]+O(\kappa^2) \tag{2.8b}$$

$$\vec{\varphi} \equiv \vec{A}^t \sim \frac{\kappa}{\epsilon}(\vec{\alpha}+\epsilon^2\vec{\beta}+\cdots)+O(\kappa^2)\ , \tag{2.8c}$$

$$\vec{A}^i \sim \frac{\kappa}{\epsilon}(\epsilon\vec{C}^i+\epsilon^3\vec{D}^i+\cdots)+O(\kappa^2)\ . \tag{2.8d}$$

The currents and potentials $\vec{\alpha}$, $\vec{\beta}$, $\vec{C}^i, \ldots$ are permitted to depend functionally on the as-yet-unknown world lines $X^i=R_A^i(t)$ and on the near-zone coordinates (t,X^i), where t is measured in units of λ, and X^i is measured in units of L. The slow-motion assumption is incorporated automatically[5] by holding (t,X^i) fixed under the limit $\epsilon\to 0$.

Now, it is not the purpose of this paper to derive equations of motion, but rather to describe the evolution of the isocharge degrees of freedom. However, an expression for the *approximate* force between two slowly moving isocharges will be needed later; thus, a force law consistent with the approximation method to be used here is described below.

Suppose first that the motion of particle 1 in a *given* external field $F^{\mu\nu}_{\rm ext}$ satisfies the (generalized) Lorentz force law

$$\frac{d}{d\tau_1}\left[m_1\frac{dR_1^\mu}{d\tau_1}\right]=\vec{q}_1\cdot\vec{F}^{\mu}_{{\rm ext}\ \nu}\frac{dR_A^\nu}{d\tau_1},\quad A=1,2 \tag{2.9a}$$

$$\tau_1 \equiv \text{proper time of particle 1}\ . \tag{2.9b}$$

The divergences that would have arisen in a derivation of this law from conservation of the stress energy

$$T^{\mu\nu}=t^{\mu\nu}_{\text{point masses}}+\frac{1}{4\pi}(\vec{F}^{\mu\alpha}\cdot\vec{F}^{\nu}{}_{\alpha}-\tfrac{1}{4}\eta^{\mu\nu}\vec{F}_{\alpha\beta}\cdot\vec{F}^{\alpha\beta}) \tag{2.10}$$

are thus assumed, as in the work of Mathisson,[11] to reside in the physical mass κm_1. Now, the introduction of a second particle causes the total field to differ by an amount

$$\vec{F}^{\mu\nu}_{\rm ext}=\vec{F}^{\mu\nu}-\vec{F}^{\mu\nu}_{\rm Coul} \tag{2.11}$$

from the value $\vec{F}^{\mu\nu}_{\rm Coul}$ associated with the potential (2.7) [rewritten in (t,X^i) coordinates]. Our prescription for calculating perturbations in the equations of motion associated with radiation reaction would then be to substitute at each stage the current approximation for $\vec{F}^{\mu\nu}_{\rm ext}$ into (2.9).

In the electromagnetic case, this splitting of the fields gives rise to the usual radiation-reaction forces when the analog of (2.9) and Maxwell's equations are evaluated in the slow-motion approximation.[12] The only part of the self-field that is assumed to reside in the mass κm_1 is the static Coulomb field. In any case, the only consequence of (2.9) to be used in what follows is the approximate Coulomb attraction of two slowly moving particles. The force law (2.9) has been derived in the context of a coupled theory of Yang-Mills fields and fermions by Wong.[13]

As we shall shortly see, the force between two particles at distance L leads to corrections to $R_A^i(t)$ proportional to κ/ϵ^2, and therefore the potentials, beginning with $\vec{\alpha}$, will implicitly contain corrections of relative order κ/ϵ^2 to the expressions that would be obtained for straight-line motion. [For example, $\vec{\alpha}$ is an $O(\kappa/\epsilon)$ term in $\vec{A}^t$ and will contain corrections at $O(\kappa^2/\epsilon^2)$.] These corrections will lead to the leading radiation reaction via matching. Explicit corrections to expansion (2.8) proportional to κ^2 will arise from products of the linearized potentials in the field equations (2.5). As we shall shortly see, the first such corrections occur at $O(\kappa^2)$ in $\vec{\phi}$. A refined form of the near-zone expansion for the potentials is thus

$$\vec{\phi}\sim\frac{\kappa}{\epsilon}(\vec{\alpha}+\epsilon^2\vec{\beta}+\cdots)+\kappa^2(\vec{a}+\cdots)+O(\kappa^3)\ , \tag{2.12a}$$

$$\vec{A}^i\sim\frac{\kappa}{\epsilon}(\epsilon\vec{C}^i+\epsilon^3\vec{D}^i+\cdots)+\kappa^2(\epsilon\vec{H}^i+\cdots)+O(\kappa^3)\ . \tag{2.12b}$$

We will later be forced by the requirements of matching to include *time-odd* terms at $O(\kappa^2)$ in $\vec{\phi}$ and at $O(\kappa^2/\epsilon)$ in $\vec{A}^i$, i.e., terms whose signs depend on the choice of outgoing (rather than incoming) radiation. See Eqs. (4.5) and (4.6) below. (Nothing prevents one from revising asymptotic expansions later due to the requirements of matching. For example, terms proportional to $\epsilon^n\ln\epsilon$ have been shown[14] to arise in an analogous gravitational problem.)

The near-zone equations are now to be generated by substitution of these expansions into the field equations (2.5), gauge condition (2.4), and the force law (2.9). The near-zero field equations and gauge conditions resulting from this substitution are

$$\nabla_X{}^2\vec{\alpha}=-4\pi\vec{\rho}_I\ , \tag{2.13}$$

$$\nabla_X{}^2\vec{\beta}=\frac{\partial^2\vec{\alpha}}{\partial t^2}\ , \tag{2.14}$$

$$\nabla_X{}^2\vec{C}^i=-4\pi\vec{j}^{\,i}_I\ , \tag{2.15}$$

$$\nabla_X{}^2\vec{D}^i=\frac{\partial^2\vec{C}^i}{\partial t^2}\ , \tag{2.16}$$

$$\nabla_X{}^2\vec{a}=-\vec{\alpha}\times\frac{\partial\vec{\alpha}}{\partial t}\ , \tag{2.17}$$

$$\nabla_X{}^2\vec{H}^i=\vec{\alpha}\times\frac{\partial\vec{\alpha}}{\partial X^i}\ , \tag{2.18}$$

$$\frac{\partial\vec{\alpha}}{\partial t}+\frac{\partial\vec{C}^i}{\partial X^i}=0\ , \tag{2.19}$$

$$\frac{\partial\vec{H}^i}{\partial X^i}=0\ , \tag{2.20}$$

$$\frac{\partial\vec{\beta}}{\partial t}+\frac{\partial\vec{D}^i}{\partial X^i}=0\ , \tag{2.21}$$

where

$$\vec{\rho}_I\equiv\vec{q}_1\delta^3(X^i-R_1^i)+\vec{q}_2\delta^3(X^i-R_2^i)\ , \tag{2.22a}$$

$$\vec{j}^{\,i}_I\equiv\vec{q}_1\dot{R}^i_1\delta^3(X^i-R^i_1)+\vec{q}_2\dot{R}^i_2(X^i-R^i_2)\ , \tag{2.22b}$$

$$\nabla_X{}^2\equiv\frac{\partial}{\partial X^i}\frac{\partial}{\partial X^i}\ . \tag{2.23}$$

Let us choose the origin of coordinates as that unaccelerated line satisfying, at $t=0$,

$$R^i_1m_1+R^i_2m_2=0\ , \tag{2.24a}$$

$$\dot{R}^i_1 m_1 + \dot{R}^i_2 m_2 = 0 \ . \tag{2.24b}$$

Initial data are assumed such that the particular solutions of (2.13) and (2.15) take the Coulomb form

$$\vec{\alpha} = \frac{\vec{q}_1}{|X^i - R^i_1|} + \frac{\vec{q}_2}{|X^i - R^i_2|} \ , \tag{2.25}$$

$$\vec{C}^i \frac{\vec{q}_1 \dot{R}^i_1}{|X^i - R^i_1|} + \frac{\vec{q}_2 \dot{R}^i_2}{|X^i - R^i_2|} \ . \tag{2.26}$$

Using (2.25) in the force law (2.9), one finds that the deflection of, say, particle 1 is given by

$$\ddot{R}^j_1 = \frac{\kappa}{\epsilon^2} \frac{1}{m_1} \vec{q}_1 \cdot \vec{q}_2 |R^i_1 - R^i_2|^{-3} (R^j_1 - R^j_2) \ . \tag{2.27}$$

Thus, as mentioned above, $\ddot{R}^j_A$ is of $O(\kappa/\epsilon^2)$, because m_A and $\vec{q}_A$ remain finite in the limits $\kappa \to 0$, $\epsilon \to 0$.

It will be convenient to postpone the solution of the remaining equations until after the wave-zone expansion has been introduced. (In fact, the detailed form of $\vec{a}$ and $\vec{H}^i$ will not affect the radiation reaction to lowest order.)

III. MATCHING OUT TO WAVE-ZONE EXPANSION

At distances of about L/ϵ or greater from the sources, spatial and time derivatives of the fields will be of the same order of magnitude. A weak-field, but *not* slow-motion, expansion is thus required. The wave-zone coordinates are thus defined as

$$x^i \equiv \epsilon X^i, \quad x^\mu \equiv (t, x^i) \tag{3.1}$$

and all potentials will depend functionally on x^μ. From dimensional considerations and from the form of the near-zone potentials $\vec{\alpha}$ and $\vec{C}^i$ in Eqs. (2.25) and (2.26), the part of the wave-zone expansion proportional to κ is taken as

$$\vec{A}^\mu \sim \kappa[\vec{M}^\mu(x^\nu) + \epsilon \vec{P}^\mu(x^\nu) + \cdots] + O(\kappa^2) \ . \tag{3.2}$$

The Abelian potentials $\vec{M}^\mu$, $\vec{P}^\mu$ satisfy linear, vacuum wave equations and gauge conditions

$$\Box \vec{M}^\mu = 0 \ , \tag{3.3a}$$

$$\vec{M}^\mu{}_{,\mu} = 0 \ , \tag{3.3b}$$

$$\Box \vec{P}^\mu = 0 \ , \tag{3.4a}$$

$$\vec{P}^\mu{}_{,\mu} = 0 \ . \tag{3.4b}$$

In the present problem, a simple matching procedure gives the same result as the more sophisticated method of comparing "intermediate limits"[3]: The leading term $\kappa\epsilon^{-1}\vec{\alpha}$ in the near-zone expansion of $\vec{A}^t$, rewritten in wave-zone coordinates x^μ, has the expansion

$$\kappa\epsilon^{-1}\vec{\alpha} \sim \kappa \left[\frac{\vec{q}_1 + \vec{q}_2}{r} + \epsilon \frac{\vec{d}_M(t) Y_{1M}(\theta,\varphi)}{r^2} + \cdots \right] , \tag{3.5}$$

where $r \equiv |x^i|$, and where

$$\vec{d}_M(t) Y_{1M}(\theta,\varphi) = [\vec{q}_1 R^j_1(t) + \vec{q}_2 R^j_2(t)] \frac{x^j}{r} \ . \tag{3.6}$$

Similarly, the leading term $\kappa\vec{C}^i$ in the near-zone expansion of $\vec{A}^i$, rewritten in wave-zone coordinates, has the expansion

$$\kappa\vec{C}^i \sim \kappa\epsilon \frac{\vec{q}_1 \dot{R}^i_1 + \vec{q}_2 \dot{R}^i_2}{r} + \cdots \ . \tag{3.7}$$

The $O(\kappa)$ potential $\vec{M}^\mu$ that matches to $(\vec{\alpha}, \vec{C}^i)$ is the static monopole

$$\vec{M}^t = \frac{\vec{q}_1 + \vec{q}_2}{r}, \quad \vec{M}^i = 0 \ . \tag{3.8}$$

The most general Abelian electric dipole solutions of Eqs. (3.4) are linear combinations of the form[4,15]

$$\begin{aligned} \vec{P}^t &= \left[\frac{\vec{F}'_M(t \mp r)}{r} \pm \frac{\vec{F}_M(t \mp r)}{r^2} \right] Y_{1M}(\theta,\varphi) \ , \\ \vec{P}^i &= \left[\pm\sqrt{3} \frac{\vec{F}'_M(t \mp r)}{r} \right] (\mathcal{Y}_{10M})_i \ , \end{aligned} \tag{3.9}$$

and

$$\vec{P}^t = \left[\frac{\vec{G}'_M(t \mp r)}{r} \pm \frac{\vec{G}_M(t \mp r)}{r^2} \right] Y_{1M}(\theta,\varphi) \ ,$$

$$\vec{P}^i = \mp \left[\frac{3}{2} \right]^{1/2} \left[\frac{\vec{G}'_M(t \mp r)}{r} \pm \frac{3\vec{G}_M(t \mp r)}{r^2} + \frac{3\vec{G}^{(-1)}_M(t \mp r)}{r^3} \right] (\mathcal{Y}_{12M})_i \ ,$$

$$\mathcal{Y}_{10M} \equiv \text{vector spherical harmonic} \ . \tag{3.10}$$

For these Abelian potentials, we impose the outgoing-wave condition that only functions of the form $\vec{F}_M(t-r)$ and $\vec{G}_M(t-r)$ can occur. The condition that the outer expansion

$$\vec{A}^\mu \sim \kappa(\vec{M}^\mu + \epsilon\vec{P}^\mu + \cdots)$$

match to the inner expansion

$$\vec{A}^i \sim \frac{\kappa}{\epsilon}\vec{\alpha} + \cdots, \quad \vec{A}^i \sim \kappa\vec{C}^i + \cdots$$

determines $\vec{F}_M(t)$ and $\vec{G}_M(t)$:

$$\vec{F}_M(t) = \vec{d}_M(t) \ , \tag{3.11}$$

$$\vec{G}_M(t) = 0 \ . \tag{3.12}$$

Thus, the dipole $O(\kappa\epsilon)$ part of the wave-zone expansion is

$$\vec{P}^t = \left[\frac{\vec{d}^j(u)}{r^2} + \frac{\vec{d}^{j\prime}(u)}{r} \right] \frac{x^j}{r} \ , \tag{3.13a}$$

$$\vec{P}^j = \frac{\vec{d}^{j\prime}(u)}{r} \ , \tag{3.13b}$$

$$\vec{d}^{\,j}(t) \equiv \vec{q}_1 R_1^j(t) + \vec{q}_2 R_2^j(t) \,, \tag{3.13c}$$

$$u \equiv t - r \,. \tag{3.13d}$$

What has been obtained so far is essentially the multipole expansion of a Lienard-Wiechert potential due to two slow-motion charges. Inserting (3.2) into Eqs. (2.5), one sees immediately that nonlinear terms require the addition of terms proportional to κ^2, etc., to the expansion (3.2):

$$\vec{A}^{\mu} \sim \kappa(\vec{M}^{\mu} + \epsilon \vec{P}^{\mu} + \cdots) + \kappa^2(\epsilon \vec{\pi}^{\mu} + \cdots) + O(\kappa^3) \,. \tag{3.14}$$

The equations satisfied by $\vec{\pi}^{\mu}$, are from (2.4) and (2.5),

$$\Box \vec{\pi}^{\nu} = -2(\vec{M}^{\mu} \times \vec{P}^{\nu}{}_{,\mu} + \vec{P}^{\mu} \times \vec{M}^{\nu}{}_{,\mu}) + \vec{M}^{\mu} \times \vec{P}_{\mu,}{}^{\nu} + \vec{P}^{\mu} \times \vec{M}_{\mu,}{}^{\nu} \,, \tag{3.15a}$$

$$\vec{\pi}^{\nu}{}_{,\nu} = 0 \,. \tag{3.15b}$$

In fact, the solution for $\vec{\pi}^{\mu}$ will not be needed in what follows.

IV. RESISTIVE POTENTIALS

In order to carry out matching to an order sufficient to see radiation reaction, one expresses

$$\vec{A}^{\mu} \sim \kappa(\vec{M}^{\mu} + \epsilon \vec{P}^{\mu} + \cdots)$$

in near-zero coordinates and carries out the ϵ expansion; i.e., functions $f(t - \epsilon R)$ are expanded in a Taylor series about $f(t)$. The leading terms thus generated will come from the Abelian dipole $\vec{P}^{\mu}$:

$$\vec{P}^{t} = [\epsilon^{-2} \vec{d}^{\,j}(t) R^{-2} - \tfrac{1}{2} \vec{d}^{\,j(2)}(t) + \tfrac{1}{3} \epsilon \vec{d}^{\,j(3)}(t) R + \cdots] \frac{X^j}{R} \,, \tag{4.1}$$

$$\vec{P}^{j} = \epsilon^{-1} \vec{d}^{\,j(1)}(t) R^{-1} - \vec{d}^{\,j(2)}(t) + \cdots \,, \tag{4.2}$$

where

$$R \equiv |X^i| \,, \tag{4.3}$$

$$\vec{d}^{\,j(3)}(t) = \frac{d^3}{dt^3} \vec{d}^{\,j}(t), \text{ etc.} \tag{4.4}$$

Note that the first influence of the outgoing-radiation condition will be generated by the term proportional to ϵ^1 in $\vec{P}^t$ and the term proportional to ϵ^0 in $\vec{P}^j$. (These terms would change sign under the replacement $t - r \to t + r$).

The leading effects of radiation reaction will reside in corrections to the near-zone expansion (2.12) that match the time-odd terms of Eqs. (4.1) and (4.2). Since, in expansion (3.2), $\vec{P}^{\mu}$ is multiplied by $\kappa\epsilon$, one is tempted to assume that radiation arises in the near-zone expansion at $O(\kappa\epsilon^2)$ in $\vec{\phi}$ and at $O(\kappa\epsilon)$ in $\vec{A}^j$. However, by the "Coulomb" force law (2.27) and the definition (4.3) of $\vec{d}^{\,j}$, both $\vec{d}^{\,j(3)}(t)$ and $\vec{d}^{\,j(2)}(t)$ are of $O(\kappa/\epsilon)^2$.

Therefore, the near-zone expansions (2.12) are augmented to include time-odd term $\vec{b}$ at $O(\kappa^2)$ in $\vec{\phi}$ and a time-odd term $\vec{I}^{\,i}$ at $O(\kappa^2/\epsilon)$ in $\vec{A}^i$:

$$\vec{\phi} \sim \frac{\kappa}{\epsilon}(\vec{\alpha} + \epsilon^2 \vec{\beta} + \cdots) + \kappa^2(\vec{a} + \vec{b} + \cdots) + O(\kappa^3) \,, \tag{4.5}$$

$$\vec{A}^{i} \sim \frac{\kappa}{\epsilon}(\epsilon \vec{C}^{i} + \epsilon^3 \vec{D}^{i} + \cdots) + \kappa^2(\epsilon^{-1} \vec{H}^{i} + \epsilon^{-1} \vec{I}^{\,i} + \cdots) + O(\kappa^3) \,. \tag{4.6}$$

The terms $\vec{b}$ and $\vec{I}^{\,i}$ will be referred to below as "resistive potentials," for reasons that will become clear momentarily. The equations satisfied by $\vec{b}$ and $\vec{I}^{\,j}$ are homogeneous:

$$\nabla_X{}^2 \vec{b} = 0 \,, \tag{4.7}$$

$$\nabla_X{}^2 \vec{I}^{\,i} = 0 \,. \tag{4.8}$$

The potentials $\vec{b}$ and $\vec{I}^{\,j}$ are required to be regular at the origin and to match the leading time-odd terms in expansions (4.1) and (4.2).

The solutions of (4.7) and (4.8) fulfilling these boundary and matching conditions are

$$\vec{b} = \tfrac{1}{3} \vec{d}^{\,j(3)}(t) X^j \,, \tag{4.9}$$

$$\vec{I}^{\,j} = -\vec{d}^{\,j(2)}(t) \,. \tag{4.10}$$

One might ask why time-odd terms were not introduced in the near-zone expansions (4.5) and (4.6) at $O(\kappa)$. If such terms had been included, matching would have led to a zero solution. As for the "time-even" terms $\vec{\beta}$, $\vec{D}^{i}$, etc., no further information is gained by matching their solutions to the wave-zone expansion. Moreover, they generate terms in the equations of motion unrelated to radiation reaction (although in order of magnitude much larger).

We now use the method of multiple time scales[16] to examine the effects of the resistive potentials $\vec{b}$ and $\vec{I}^{\,i}$ in the covariant charge conservation identity (2.3). Because $\vec{b}$ is of $O(\kappa^2)$ and because $\vec{j}^{\,\mu}$ itself is of $O(\kappa)$, changes in $\vec{q}_A$ (if they occur at all) should be proportional to κ^2 over a time of order unity, or conversely, changes of $O(1)$ in $\vec{q}_A$ should be expected in a time of $O(1/\kappa^2)$. Now, the charge vectors $\vec{q}_1$ and $\vec{q}_2$ have been tacitly assumed throughout to be time independent. However, all questions encountered until now remain unaffected if in fact $\vec{q}_1$ and $\vec{q}_2$ are permitted to depend on the damping time parameter $\kappa^2 t$.

Consider the leading time-odd contributions to the covariant charge conservation identity (2.3a), which occur at $O(\kappa^3)$. Integrating this identity at some time t over an arbitrarily small volume surrounding $\vec{q}_1$, one obtains, using expansions (4.5) and (4.6),

$$\begin{aligned} \dot{\vec{q}}_1 &\equiv \frac{d}{dt} \vec{q}_1(\kappa^2 t) \\ &= \kappa^2 \vec{q}_1{}'(\kappa^2 t) \\ &= \int_{R_1} \left[\kappa^2(\vec{b} \times \vec{\rho}) - \frac{\kappa^2}{\epsilon}(\vec{I}^{\,j} \times \vec{j}^{\,j}) \right] d^3X + \cdots \,. \end{aligned} \tag{4.11}$$

Substitution of Eqs. (2.8), (4.9), and (4.10) then yields

$$\begin{aligned}\dot{\vec{q}}_1 &= \sum_{A=1,2} \kappa\epsilon^2 \int_{R_1} \{ \tfrac{1}{3}\vec{d}^{\,j(3)} X^j \times [\vec{q}_A \delta_3(\epsilon X^i - \epsilon R_A^i)] + \vec{d}^{\,j(2)} \times [\vec{q}_A \dot{R}_A^i \delta(\epsilon X^i - \epsilon R_A^i)] + \cdots \} d^3X \\ &= \kappa\epsilon^2 \vec{q}_1 \times \vec{q}_2 \left[\tfrac{1}{3} \dddot{R}_2^j R_1^j + \ddot{R}_2^j \dot{R}_1^j \right] + \cdots \\ &= \kappa\epsilon^2 \vec{q}_1 \times \vec{q}_2 \left[\tfrac{2}{3} \dddot{R}_2^j R_1^j - \frac{d}{dt}(\ddot{R}_2^j R_1^j) \right] + \cdots .\end{aligned} \tag{4.12}$$

The expression for particle 2 is obtained by reversing the indices 1 and 2 :

$$\dot{\vec{q}}_2 = \kappa\epsilon^2 \vec{q}_2 \times \vec{q}_1 \left[\tfrac{2}{3} \dddot{R}_1^j R_2^j - \frac{d}{dt}(\ddot{R}_1^j R_2^j) \right] + \cdots . \tag{4.13}$$

Several properties of Eqs. (4.12) and (4.13) are worth noting.

(1) If the particles have identical charge-to-mass ratios, then the sum $\dot{\vec{q}}_1 + \dot{\vec{q}}_2$ vanishes, just as the energy loss due to ordinary electromagnetic dipole radiation vanishes when the charge-to-mass ratios are equal. (Of course, one expects analogously that higher-order terms beginning with the quadrupole resistive potentials would still contribute).

(2) The Abelian solution with $\vec{q}_1$ parallel (or antiparallel) to $\vec{q}_2$ gives rise to no charge radiation reaction.

(3) The magnitude of each charge vector remains constant, at least up to the order considered here; each charge vector "rotates" with respect to the spatially independent basis $\vec{e}_j$. (Of course, such a choice of basis singles out a particular gauge.) With respect to this basis, $\vec{q}_1$ and $\vec{q}_2$ rotate in an oppositely directed sense.

The Coulomb force law (2.27) give an adequate approximation for $\ddot{R}_A^j(t)$, $A = 1,2$:

$$\ddot{R}_1^j = \frac{1}{m_1} \vec{q}_1 \cdot \vec{q}_2 \frac{\kappa}{\epsilon^2} \frac{R_1^j - R_2^j}{|R_1^i - R_2^i|^3} + \cdots , \tag{4.14}$$

$$\ddot{R}_2^j = \frac{1}{m_2} \vec{q}_1 \cdot \vec{q}_2 \frac{\kappa}{\epsilon^2} \frac{R_2^j - R_1^j}{|R_1^i - R_2^i|^3} + \cdots . \tag{4.15}$$

Thus, the time derivatives of $\vec{q}_1$ and $\vec{q}_2$ due to radiation reaction are indeed of order $\kappa^2 = \kappa\epsilon^2 \cdot \kappa\epsilon^{-2}$, as claimed above. In particular, since the Coulomb force is zero when $\vec{q}_1 \cdot \vec{q}_2 = 0$, no charge rotation occurs in this case.

V. SUMMARY AND DISCUSSION

The above calculation indicates that the leading effects of radiation reaction in the slow-motion system of two isocharges discussed above are due to resistive potentials $\vec{b}$ and $\vec{I}^j$ at orders κ^2 and κ^2/ϵ in $\vec{A}^t$ and $\vec{A}^i$, respectively. Their effects lead to "rotations" of the two isocharge vectors given by Eqs. (4.12) and (4.15).

An important technical question is whether terms other than those considered above could make comparable or larger contributions to the charge radiation reaction. Unless the charge-mass ratios of the two particles are equal, one can easily show that the quadrupole and higher multipole terms proportional to κ in the wave-zone expansion (3.14) produce smaller contributions to the radiation reaction, as in ordinary electromagnetic radiation damping. As for the nonlinear orders (i.e., those proportional to $\kappa^2, \ldots$), Eqs. (3.15), (2.17), and (2.18) indicate that no time-odd contributions arise from nonlinearities before $O(\kappa^2\epsilon)$ in $\vec{\phi}$ and $O(\kappa^2)$ in $\vec{A}^j$. Their effects would thus not arise until $O(\kappa^2\epsilon)$ in $\vec{q}_1$ and $\vec{q}_2$, at the earliest. Nevertheless, one might expect complications if one were to study "fast-motion" charge radiation damping,[8,9] since the nonlinear contributions arise at the same κ order as the linear ones.

Indeed, expressions (4.12)—(4.15) agree with the slow-motion limit (1.1) of the truncated "fast-motion" expansion. However, as emphasized above, nonlinear contributions to Eq. (1.1) not included in this "truncated limit" enter at the same "κ order" as the linear ones (those that appear explicitly in the first term).

A related technical point is that the phenomenon of "switchback"[16] can always arise in nonlinear singular perturbation problems such as this one: a term proportional to κ^n in the wave-zone expansion could generate additional terms in the near-zone via matching. As long as one maintains the assumption $\kappa \ll \epsilon^2$, this possibility appears unlikely, given the known sources for the $O(\kappa^2)$ wave-zone equations (3.15)—(3.16), but if one wanted to study the case $\kappa \propto \epsilon^2$, one could not rule out contributions from high orders in κ multiplied by large negative powers of ϵ (see last paragraph below).

A second and more fundamental question—one that arises both here and in Trautman's[7] approach— is the significance of "charge rotation" in physical terms. The very definition of "global charge" in pure gauge theories is still a matter of some contention.[17–21]. What one would like to have is a locally definable, gauge-invariant quantity analogous to "the charge of a particle at time t." As we shall see below, a gauge-invariant statement *can* be made asymptotically for late times in this problem.

Now, it has been assumed throughout that the field-strength parameter κ and the slow-motion parameter ϵ are independent. However, aside from worries about "switchback" as mentioned above, one can also apply this calculation to the case in which the two-particle system is bound by the Coulomb force. The parameters κ and ϵ are then related by

$$\epsilon^2 = \text{coefficient} \times \frac{\kappa q^2}{mL} .$$

If one waits for a time large compared to λ/κ^2, the system becomes in a practical sense asymptotically Abelian. That is, the nonlinear product $\vec{A}^\mu \times \vec{F}_{\mu\nu}$ becomes arbitrarily small in any gauge compatible with our expansions (4.5) and (4.6) after a sufficiently long time has passed. This asymptotic attainment of an Abelian configuration suggests a classical analog to stability of the "color-singlet" state in the quantized version of the theory.

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