Subsidiary condition for Yang-Mills theory

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A subsidiary condition for Yang-Mills theory is given. A prescription is proposed for using such a subsidiary condition to eliminate unphysical degrees of freedom from gauge theories in covariant gauges. It is pointed out that elimination of such unphysical modes can generate explicit nonlocal interactions among particles in the physical subspace. The Coulomb interactions among charged particles in QED is one such nonlocal interaction that can be generated in this way. It is argued that confining forces among color-bearing combinations of quarks and transverse gluons in QCD might be another.

I. INTRODUCTION

Even though it has not yet been possible to find a formulation of non-Abelian gauge theory free of all unphysical degrees of freedom, a procedure is known that, in principle, is available for eliminating such unphysical modes from gauge theories. This procedure has been applied to QED,[1] and could also be effective in a non-Abelian gauge theory provided its Feynman rules are unitary in its physical subspace, as, for example, in Yang-Mills theory[2] and in QCD. The method is based upon the use of a subsidiary condition, which selects states to constitute a physical subspace of the indefinite-metric space in which the manifestly covariant form of the theory must be embedded. A subsidiary condition is a time-independent constraint, and in order to impose such a time-independent constraint it is necessary to use a spectrally pure operator. For example, in the momentum representation, a spectrally pure operator $\Omega(\vec{k})$ satisfies

$$[H,\Omega(\vec{k})]+k\Omega(\vec{k})=0\ , \tag{1.1}$$

so that $\Omega(x)$, given by $\Omega(x)=\sum_{\vec{k}}\Omega(\vec{k})e^{ik_\mu x_\mu}$ with $k_0=|\vec{k}|$, is the correct Heisenberg operator. It is possible to use spectrally pure operators, that impose subsidiary conditions, to eliminate unphysical degrees of freedom from gauge theories.

Spectrally pure operators can generally be represented as

$$\Omega(\vec{k})=\Omega_0(\vec{k})+\Omega_1(\vec{k})\ , \tag{1.2}$$

where $\Omega_0(\vec{k})$ is independent of the charge parameter (e) that defines the strength of the interaction. $\Omega_1(\vec{k})$, on the other hand, vanishes as $e\to 0$. Since $\Omega_0(\vec{k})$ has to obey $[H_0,\Omega_0(\vec{k})]+k\Omega_0(\vec{k})=0$, Eq. (1.1) implies that $[H_1,\Omega_0(\vec{k})]$ must be canceled by $[H,\Omega_1(\vec{k})]+k\Omega_1(\vec{k})$. $\Omega(\vec{k})$ imposes a subsidiary condition

$$\Omega(\vec{k})\,|\,\nu\rangle=0\ , \tag{1.3}$$

which selects the set of states $|\,\nu\rangle$ to be physical states. In order for $\Omega(\vec{k})$ to be useful in imposing this subsidiary condition, not only must it be spectrally pure, and obey Eq. (1.1), but $\Omega_0(\vec{k})$ must serve as a projection operator that selects pure gauge states. This it does by identifying them as the set $|\,n)$ that obey

$$\Omega_0(\vec{k})\,|\,n)=0\ . \tag{1.4}$$

The relation between Eqs. (1.3) and (1.4), and isomorphisms between the sets $|\,\nu\rangle$ and $|\,n)$, are important elements in the strategy for eliminating unphysical degrees of freedom from gauge theories in covariant gauges.

There are various ways of characterizing pure gauge states, but for our purposes the following is very useful: In any manifestly covariant form of a gauge theory there are ghost-particle states, all of which can appear as final states of S-matrix elements. Some of these states contain a single ghost particle only, others consist of combinations of ghost particles. Pure gauge ghost-particle states produce vanishing S-matrix elements when they appear in the role of final states, provided the initial states are ghost-free. In QED the pure gauge ghosts are the excitations of the free gauge-fixing field. In non-Abelian theories the single-ghost-particle states, that are pure gauge ghosts in QED, are pure gauge ghosts still. In addition, in non-Abelian theories combinations of different varieties of ghost particles that express the basic nonlinearity of the theory are also pure gauge ghost states. As we will see in specific examples, not all ghost states that produce vanishing S-matrix elements are pure gauge ghosts.

In QED it is very easy to find an operator $\Omega(\vec{k})$ that is suitable for imposing a subsidiary condition. $\Omega(\vec{k})$ is simply the positive-frequency part of the free gauge-fixing field.[3] The fact that $\Omega(x)$, where $\Omega(x)=\sum_{\vec{k}}\Omega(\vec{k})e^{ik_\mu x_\mu}$, is the positive-frequency part of a local free field, guarantees that $\Omega(\vec{k})$ obeys Eq. (1.1) and is spectrally pure. In addition, since the pure gauge ghosts in QED are excitations of the free gauge-fixing field, $\Omega_0(\vec{k})$ projects pure gauge ghosts as required. Since the gauge-fixing field is not free in non-Abelian theory, it has generally been assumed that it is not possible to construct a spectrally pure operator $\Omega(\vec{k})$, suitable for imposing a subsidiary condition, in non-Abelian gauge theory.

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The main point of this paper is that it is possible to construct an operator $\Omega(\vec{k})$, appropriate for imposing a subsidiary condition, even in non-Abelian gauge theory. A free gauge-fixing field is not essential in the process, though in Abelian gauge theories, in which a free gauge-fixing field exists, it greatly reduces the technical problems of constructing and representing $\Omega(\vec{k})$. In later sections of this paper we will show how to construct $\Omega(\vec{k})$ for Yang-Mills theory, and exhibit its form. We will also discuss how $\Omega(\vec{k})$ for Yang-Mills theory (and QCD) might be an important tool in clarifying the confinement properties of the theory.

II. QED—AN ILLUSTRATIVE EXAMPLE

In QED, in the manifestly covariant Lorentz gauge, there are two photon ghosts. In the Fock representation, in which particle states are eigenstates of the interaction-free Hamiltonian H_0, the one-particle ghost states are given by $a_Q^*(\vec{k})\,|\,0)$ and $a_R^*(\vec{k})\,|\,0)$, where the asterisk indicates the adjoint in the indefinite metric space and where $|\,0)$ is the "bare" vacuum given by $H_0\,|\,0)=0$. $a_Q^*(\vec{k})$ and $a_R^*(\vec{k})$ and their respective adjoints in the indefinite-metric space are given by

$$a_Q^*(\vec{k})=\frac{k_\mu a_\mu^\dagger(\vec{k})}{(2)^{1/2}k}\ , \tag{2.1a}$$

$$a_R^*(\vec{k})=\frac{\bar{k}_\mu a_\mu^\dagger(\vec{k})}{(2)^{1/2}k}\ , \tag{2.1b}$$

$$a_Q(\vec{k})=\frac{k_\mu a_\mu(\vec{k})}{(2)^{1/2}k}\ , \tag{2.1c}$$

and

$$a_R(\vec{k})=\frac{\bar{k}_\mu a_\mu(\vec{k})}{(2)^{1/2}k}\ . \tag{2.1d}$$

$\bar{k}_\mu$ differs from k_μ only in having $\bar{k}_4=-k_4$. The commutation rules for these ghost excitations,

$$[a_Q(\vec{k}),a_Q^*(\vec{k}')]=[a_R(\vec{k}),a_R^*(\vec{k}')]=0$$

and

$$[a_Q(\vec{k}),a_R^*(\vec{k}')]=[a_R(\vec{k}),a_Q^*(\vec{k}')]=\delta_{\vec{k},\vec{k}'}\ ,$$

are characteristic of ghost degrees of freedom. If $a_Q^*(\vec{k})\,|\,0)$ and $a_R^*(\vec{k})\,|\,0)$ were positive-norm states instead of zero-norm ghosts, these commutation rules could not be maintained consistently. In previous work it has been pointed out that $a_Q^*(\vec{k})\,|\,0)$ is the pure gauge ghost in the one-particle sector, and that any state $|\,f)$, given by $|\,f)=a_Q^*(\vec{k})\,|\,f')$, leads to the vanishing of the S-matrix element $(f\,|\,S\,|\,i)$ for ghost-free initial states $|\,i)$.[4] [We will use the symbols $|\,i)$ and $|\,f')$ always to designate ghost-free eigenstates of the interaction-free Hamiltonian H_0.] The fact that S-matrix elements vanish, in QED, when the polarization factor $\epsilon_\mu(k)$ for an external photon line is replaced by k_μ, is a signal that $a_Q^*(\vec{k})\,|\,0)$ is a pure gauge ghost. When $\epsilon_\mu(k)$ is replaced by $\bar{k}_\mu$ the corresponding S-matrix element does not vanish, verifying that the ghost state $a_R^*(\vec{k})\,|\,0)$ is not a pure gauge ghost.

In QED the positive-frequency part of the gauge-fixing field is given by

$$\chi^{(+)}(x)=i\sum_{\vec{k}}|\,\vec{k}\,|^{1/2}[a_Q(\vec{k})+J_0(\vec{k})]e^{ik_\mu x_\mu}\ , \tag{2.2}$$

where $J_0(\vec{k})=(2k^{3/2})^{-1}\int j_0(\vec{x})e^{-i\vec{k}\cdot\vec{x}}d\vec{x}$. Except perhaps for trivial c-number factors we can identify $\Omega_0(\vec{k})$ and $\Omega_1(\vec{k})$ for QED as

$$\Omega_0(\vec{k})=a_Q(\vec{k}) \tag{2.3a}$$

and

$$\Omega_1(\vec{k})=J_0(\vec{k})\ . \tag{2.3b}$$

The spectrally pure operator $\Omega(\vec{k})$ imposes the subsidiary condition

$$[a_Q(\vec{k})+J_0(\vec{k})]\,|\,\nu\rangle=0\ , \tag{2.4}$$

and Eq. (2.3a) identifies $a_Q^*(\vec{k})\,|\,0)$ as the one-particle pure gauge ghost state. It is worth noting that it is possible for $(f\,|\,S\,|\,i)$ to vanish because of the pure gauge content of $|\,f)$ even though the state vector $|\,f)$ is not a pure gauge ghost state. For example, the state $|\,f)=a_Q^*(\vec{k}_1)a_R^*(\vec{k}_2)\,|\,f')$ does not obey $a_Q(\vec{k})\,|\,f)=0$, but $(f\,|\,S\,|\,i)=0$ nevertheless. The existence of multiparticle ghost states $|\,f)$, for which $(f\,|\,S\,|\,i)=0$ but $\Omega_0(\vec{k})\,|\,f)\neq0$ is characteristic of non-Abelian gauge theories as well as of QED.

In QED it is possible to construct a unitary operator U (so that $U^*=U^{-1}$), which transforms $\Omega(\vec{k})$ as shown by

$$U^{-1}\Omega(\vec{k})U=\Omega_0(\vec{k})\ . \tag{2.5}$$

If we apply this transformation to H, so that

$$U^{-1}HU=\hat{H}\ , \tag{2.6}$$

then we transform to a new representation of QED in the manifestly covariant Lorentz gauge. In this new representation $a_Q(\vec{k})$ by itself is the form of the spectrally pure $\Omega(\vec{k})$ that imposes the subsidiary condition, since

$$[\hat{H},a_Q(\vec{k})]+ka_Q(\vec{k})=0\ . \tag{2.7}$$

$\hat{H}$ has the form

$$\hat{H}=H_0+H_{1,T}+H_C+H_Q\ , \tag{2.8}$$

where H_0 is the interaction-free Hamiltonian, $H_{1,T}$ describes the interaction of the charged particles and the transverse part of the electromagnetic vector potential, and H_C is the nonlocal Coulomb interaction

$$(8\pi)^{-1}\int d\vec{x}\,d\vec{y}\,j_0(\vec{x})j_0(\vec{y})/\,|\,\vec{x}-\vec{y}\,|\ .$$

H_Q is a part of H that contains a_Q and a_Q^* operators, but never a_R or a_R^* operators. Indeed, a_R or a_R^* operators must never appear in the interaction Hamiltonian in this representation, since if they were to appear there, they would make it impossible for Eq. (2.7) to hold. Since a_Q and a_Q^* commute, H_Q can never play any role in dynamical processes, or in any way affect S-matrix elements to observable states, though it is of course important for maintaining the equation of motion characteristic of the Lorentz gauge. The unitary transformation described by Eqs. (2.5) and (2.6) eliminates all ghost degrees of freedom from the dynamical processes among transverse photons and charged particles, and a surrogate nonlocal interaction between charges is developed in $\hat{H}$.

We have reviewed this material here because we hope to generalize it, and apply it to non-Abelian gauge theory. If one can unitarily transform $\Omega(\vec{k})$ to a representation, in which it has the form of the original $\Omega_0(\vec{k})$, so that in the new representation $\Omega_0(\vec{k})$ becomes the spectrally pure operator that imposes a time-independent constraint, then in that new representation ghost degrees of freedom cease to transmit interactions among the physical degrees of freedom. Instead of ghost-transmitted forces, explicit nonlocal interactions should appear in the transformed Hamiltonian in non-Abelian theories just as in QED. In QED we are very accustomed to a ghost-free version of the theory, since we can achieve it directly in the Coulomb gauge. But in non-Abelian theory the Coulomb gauge still has Faddeev-Popov ghosts, and axial gauges have gluon ghosts. Extending the procedure reviewed in this section is quite likely our best hope for reducing a non-Abelian gauge theory to a form that is both ghost-free, and unitarily equivalent to a manifestly covariant form that is known to imply the familiar Feynman rules in a mathematically reliable fashion. It is very attractive to speculate that long-range nonlocal interactions, that serve as surrogates for ghost-transmitted forces, are a common feature of all gauge theories. That QED is a limiting case, in which the Coulomb interaction, though long range, fails to confine; but that Yang-Mills theory and QCD represent a further step in which the long-range interactions are able to confine.

III. THE GHOST SPECTRUM OF YANG-MILLS THEORY

To fully exploit the resemblance between Abelian and non-Abelian gauge theories, it is helpful to formulate Yang-Mills theory as a canonical field theory, so that Faddeev-Popov ghosts and gluon ghosts are treated in equivalent fashion, and so that all ghost excitations are exhibited very clearly. We have previously discussed such a formulation of Yang-Mills theory, and demonstrated that the Dyson-Ward expansion of the scattering matrix, applied to that formulation, leads to the familiar Feynman rules.[5] It is therefore possible to now investigate nonperturbative implications of this formalism with the confidence that we are dealing with a theory that has the intended perturbative properties.

The Lagrangian for this theory is given by

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{4}\vec{f}_{\mu\nu}\cdot\vec{f}_{\mu\nu} - \bar{\Psi}[m+\gamma_\mu(\partial_\mu - ie\vec{b}_\mu\cdot\vec{\tau})]\Psi \\ &+\partial_\mu\vec{b}_\mu\cdot\vec{\chi} + \tfrac{1}{2}\vec{\chi}\cdot\vec{\chi} - i\partial_\mu\vec{\sigma}_a\cdot\partial_\mu\sigma_b \\ &-2ie\vec{b}_\mu\cdot(\vec{\sigma}_a\times\partial_\mu\vec{\sigma}_b)\ . \end{aligned} \tag{3.1}$$

The Faddeev-Popov fields are treated as anticommuting scalar operator-valued fields with fermion ghost excitations; the Faddeev-Popov fields obey canonical anticommutation rules. The theory is embedded in an indefinite metric Hilbert space in which the spin-statistics theorem does not apply.[6] The formulation leads to a Hamiltonian H, whose interaction-free part, H_0, has a normal, positive-semidefinite energy spectrum. The Lagrangian yields a canonically conjugate momentum for every field, and has a conserved current, $\vec{J}_\mu$, given by

$$\begin{aligned}\vec{J}_\mu = &\ \vec{j}_\mu + 2e\vec{b}_\nu\times\vec{f}_{\mu\nu} + 2e\vec{b}_\mu\times\vec{\chi} \\ &-2ie[\vec{\sigma}_a\times\partial_\mu\vec{\sigma}_b - \partial_\mu\vec{\sigma}_a\times\vec{\sigma}_b] \\ &+4ie^2[(\vec{b}_\mu\times\vec{\sigma}_a)\times\vec{\sigma}_b]\ ,\end{aligned} \tag{3.2}$$

where the arrow designates isovectors, and $\vec{j}_\mu$ is the quark current $\vec{j}_\mu = ie\bar{\Psi}\gamma_\mu\vec{\tau}\Psi$. $\vec{f}_{\mu\nu}$ is given by $\vec{f}_{\mu\nu} = \partial_\nu\vec{b}_\mu - \partial_\mu\vec{b}_\nu - 2e\vec{b}_\mu\times\vec{b}_\nu$, and the gauge field $\vec{b}_\mu$ is expressed in terms of gluon excitations in the exact same way as A_μ is expressed in terms of photon excitations in manifestly covariant formulations of QED, with the sole exception that all fields and excitations in the Yang-Mills case carry an isospin index. The Faddeev-Popov fields are expressed in terms of fermion ghost excitations by

$$\sigma_a^i(\vec{x}) = \sum_{\vec{k}} \frac{1}{\sqrt{2k}}[g_b^i(\vec{k})e^{i\vec{k}\cdot\vec{x}} + g_b^{i*}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}] \tag{3.3a}$$

and

$$\sigma_b^i(\vec{x}) = -i\sum_{\vec{k}} \frac{1}{\sqrt{2k}}[g_a^i(\vec{k})e^{i\vec{k}\cdot\vec{x}} - g_a^{i*}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}]\ , \tag{3.3b}$$

where

$$\{g_a^i(\vec{k}), g_a^{j*}(\vec{k}\,')\} = \{g_b^i(\vec{k}), g_b^{j*}(\vec{k}\,')\} = 0$$

and

$$\{g_a^i(\vec{k}), g_b^{j*}(\vec{k}\,')\} = \{g_b^i(\vec{k}), g_a^{j*}(\vec{k}\,')\} = \delta_{i,j}\delta_{\vec{k},\vec{k}'}\ .$$

Earlier investigations have shown that $(f\,|\,S\,|\,i) = 0$ for states $|f)$ given by $a_Q^{i*}(\vec{k})\,|f')$, $g_a^{i*}(\vec{k})\,|f')$, or $g_b^{i*}(\vec{k})\,|f')$ where $|f')$ is devoid of ghost particles.[7] We therefore classify the single-ghost-particle states $a_Q^{i*}(\vec{k})\,|0)$, $g_a^{i*}(\vec{k})\,|0)$, and $g_b^{i*}(\vec{k})\,|0)$ as pure gauge ghost states. Moreover, just as in QED, the corresponding multiparticle states that consist of a single variety of ghost particles, i.e.,

$$a_Q^{i(1)*}(\vec{k}_1)\cdots a_Q^{i(n)*}(\vec{k}_n)\,|0)\ ,$$
$$g_a^{i(1)*}(\vec{k}_1)\cdots g_a^{i(n)*}(\vec{k}_n)\,|0)\ ,$$

and

$$g_b^{i(1)*}(\vec{k}_1)\cdots g_b^{i(n)*}(\vec{k}_n)\,|0)$$

similarly produce vanishing S-matrix elements. We should therefore expect the same pure gauge structure in this sector of Hilbert space as in QED, and construct $\Omega_0^i(\vec{k})$ for Yang-Mills theory so that $\Omega_0^i(\vec{k})\,|n) = 0$ for the one- and n-ghost-particle states listed above. Moreover, in Yang-Mills theory there is an additional two-particle ghost state, that is a pure gauge ghost specifically because of the nonlinearities implicit in non-Abelian gauge theories. The existence of this state stems from the fact that, in Yang-Mills theory, the S-matrix element $(f'\,|\,a_Q^{i(1)}(\vec{k}_1)a_R^{i(2)}(\vec{k}_2)S\,|\,i)$ does not vanish; in QED, the appearance of $a_Q(\vec{k})$ in $(f\,|$ suffices to guarantee that the corresponding S-matrix element does vanish. The nonlinearity in Yang-Mills theory gives rise to vertices, that couple a Q and R pair of gluon ghosts to transverse gluons, so that probability measure can be transferred from the physical to the unphysical subspace. This fact was realized long ago by Feynman, who pointed out that in Yang-Mills theory, without any Faddeev-Popov ghosts, unitarity in the physical subspace is not preserved.[8] It was Feynman's prescription for a remedy that originally motivated the inclusion of scalar fermion ghost loops in the Feynman rules for Yang-Mills theory.

In previous work we have described the two-particle pure gauge ghost state[9]

$$|\,\Psi(1,2))=\Psi^{i(1),i(2)*}(\vec{k}_1,\vec{k}_2)\,|\,0) \tag{3.4}$$

with

$$\Psi^{i(1),i(2)*}(\vec{k}_1,\vec{k}_2)=(2k_1k_2)^{-1/2}[\,k_1a_Q^{i(1)*}(\vec{k}_1)a_R^{i(2)*}(\vec{k}_2)+k_2g_b^{i(1)*}(\vec{k}_1)g_a^{i(2)*}(\vec{k}_2)]\,. \tag{3.5}$$

We demonstrated that,[10] for ghost-free states $|f'),(f'\,|\,\Psi^{i(1),i(2)}(\vec{k}_1,\vec{k}_2)S\,|\,i)=0$ where $\Psi^{i(1),i(2)}(\vec{k}_1,\vec{k}_2)$ is given by

$$\Psi^{i(1),i(2)}(\vec{k}_1,\vec{k}_2)=(2k_1k_2)^{-1/2}[\,k_1a_R^{i(2)}(\vec{k}_2)a_Q^{i(1)}(\vec{k}_1)+k_2g_a^{i(2)}(\vec{k}_2)g_b^{i(1)}(\vec{k}_1)]\,. \tag{3.6}$$

There is another two-particle ghost state in Yang-Mills theory that we will use in our analysis. It is given by

$$|\,\phi(1,2))=\phi^{i(1),i(2)*}(\vec{k}_1,\vec{k}_2)\,|\,0) \tag{3.7}$$

with

$$\phi^{i(1),i(2)*}(\vec{k}_1,\vec{k}_2)=(2k_1k_2)^{-1/2}[\,k_1a_Q^{i(1)*}(\vec{k}_1)a_R^{i(2)*}(\vec{k}_2)-k_2g_b^{i(1)*}(\vec{k}_1)g_a^{i(2)*}(\vec{k}_2)]\,. \tag{3.8}$$

$|\,\phi(1,2))$ is not a pure gauge state and $(f'\,|\,\phi^{i(1),i(2)}(\vec{k}_1,\vec{k}_2)S\,|\,i)$ does not vanish, but since the norm $(f'\,|\,\phi^{i'(1),i'(2)}(\vec{k}\,'_1,\vec{k}\,'_2)\phi^{i(1),i(2)*}(\vec{k}_1,\vec{k}_2)\,|\,f')=0$ no probability is lost through $(f'\,|\,\phi^{i(1),i(2)}(\vec{k}_1,\vec{k}_2)S\,|\,i)$ to a ghost state.

It is instructive to analyze how the participation of pure gauge states safeguards the unitarity of the S matrix within the quotient space of positive-norm states that describe transverse gluons, and quark excitations of the spinor field. For example, in the one-ghost-particle sector the unit operator has the form

$$\mathbb{1}_{[1-G]}=a_Q^*(1)\,|\,0)(0\,|\,a_R(1)+a_R^*(1)\,|\,0)(0\,|\,a_Q(1)+g_a^*(1)\,|\,0)(0\,|\,g_b(1)+g_b^*(1)\,|\,0)(0\,|\,g_a(1)\,, \tag{3.9}$$

where the numerical index (1) designates both the momentum and isospin, and is to be summed over its full range of values. The component of the scattering wave function in this sector, in the limit $t\to\infty$ (ignoring renormalization effects), has the form

$$[\Psi(t\to\infty)]_{[1-G]}=a_Q^*(1)\,|\,f')(f'\,|\,a_R(1)S\,|\,i)+a_R^*(1)\,|\,f')(f'\,|\,a_Q(1)S\,|\,i)+g_a^*(1)\,|\,f')(f'\,|\,g_b(1)S\,|\,i)+g_b^*(1)\,|\,f')(f'\,|\,g_a(1)S\,|\,i)\,. \tag{3.10}$$

Since $(f'\,|\,a_R(1)S\,|\,i)$ is the only matrix element in Eq. (3.10) that does not vanish, Eq. (3.10) reduces to

$$[\Psi(t\to\infty)]_{[1-G]}=a_Q^*(1)\,|\,f')(f'\,|\,a_RS\,|\,i)\,. \tag{3.10a}$$

As $a_Q^*(1)\,|\,f')$ is a zero-norm state, the norm of $[\Psi(t\to\infty)]_{[1-G]}$ vanishes.

In the two-ghost-particle sector the unit operator has the form

$$\mathbb{1}_{[2-G]}=\tfrac{1}{2}[a_Q^*(1)a_Q^*(2)\,|\,0)(0\,|\,a_R(2)a_R(1)+a_R^*(1)a_R^*(2)\,|\,0)(0\,|\,a_Q(2)a_Q(1)+g_b^*(1)g_b^*(2)\,|\,0)(0\,|\,g_a(2)g_a(1)+g_a^*(1)g_a^*(2)\,|\,0)(0\,|\,g_b(2)g_b(1)]+\Psi^*(1,2)\,|\,0)(0\,|\,\phi(2,1)+\phi^*(1,2)\,|\,0)(0\,|\,\Psi(2,1)\,. \tag{3.11}$$

Since $(f'\,|\,a_Q(1)a_Q(2)S\,|\,i)$, $(f'\,|\,g_b(1)g_b(2)S\,|\,i)$, $(f'\,|\,g_a(1)g_a(2)S\,|\,i)$, and $(f'\,|\,\Psi(1,2)S\,|\,i)$ all vanish for ghost-free $(f'\,|$, the component of $\Psi(t\to\infty)$ in this sector reduces to

$$[\Psi(t\to\infty)]_{[2-G]}=\tfrac{1}{2}a_Q^*(1)a_Q^*(2)\,|\,f')(f'\,|\,a_R(2)a_R(1)S\,|\,i)+\Psi^*(1,2)\,|\,f')(f'\,|\,\phi(2,1)S\,|\,i)\,. \tag{3.12}$$

$a_Q^*(1)a_Q^*(2)\,|\,f')$ and $\Psi^*(1,2)\,|\,f')$ both have zero norm, and the inner product of the two states, $(0\,|\,a_Q(2')a_Q(1')\Psi^*(1,2)\,|\,0)$, vanishes for all possible values of the arguments of Ψ and a_Q. The probability measure contained in $[\Psi(t\to\infty)]_{[2-G]}$ therefore also vanishes. In the three-ghost-particle sector the situation is similar except for one additional feature that arises in the part of the three-ghost sector spanned by the unit operator, $\mathbb{1}_{[\gamma]}$ given by

$$\mathbb{1}_{[\gamma]}=\tfrac{1}{2}[a_Q^*(1)a_Q^*(2)a_R^*(3)\,|\,0)(0\,|\,a_Q(3)a_R(2)a_R(1)+a_R^*(1)a_Q^*(2)a_R^*(3)\,|\,0)(0\,|\,a_Q(3)a_R(2)a_Q(1)]$$
$$+a_Q^*(1)g_b^*(2)g_a^*(3)\,|\,0)(0\,|\,g_b(3)g_a(2)a_R(1)+a_R^*(1)g_b^*(2)g_a^*(3)\,|\,0)(0\,|\,g_b(3)g_a(2)a_Q(1)\;. \tag{3.13}$$

In this part of the three-ghost sector there is a state $|\,\chi(1,2,3))$ given by

$$|\,\chi(1,2,3))=\chi^{i(1),i(2),i(3)*}(\vec{k}_1,\vec{k}_2,\vec{k}_3)\,|\,0) \tag{3.14}$$

with

$$\chi^{i(1),i(2),i(3)*}(\vec{k}_1,\vec{k}_2,\vec{k}_3)=a_Q^{i(1)*}(\vec{k}_1)a_R^{i(2)*}(\vec{k}_2)a_R^{i(3)*}(\vec{k}_3)+(k_2/k_1)g_b^{i(1)*}(\vec{k}_1)g_a^{i(2)*}(\vec{k}_2)a_R^{i(3)*}(k_3)$$
$$+(k_3/k_1)g_b^{i(1)*}(\vec{k}_1)a_R^{i(2)*}(\vec{k}_2)g_a^{i(3)*}(\vec{k}_3) \tag{3.15}$$

for which $(f'\,|\,\chi(1,2,3)S\,|\,i)=0$ even though $|\,\chi(1,2,3))$ is not a pure gauge state. There is a close parallel between the roles of $(f'\,|\,\chi(1,2,3)S\,|\,i)$ in Yang-Mills theory and that of $(f'\,|\,a_Q(1)a_R(2)S\,|\,i)$ in QED. In both cases the S-matrix elements vanish because of the pure gauge content of the final states, even though the final states themselves do not satisfy $\Omega_0(\vec{k})\,|\,n)=0$ and are not pure gauge ghost states. By using $(f'\,|\,\chi(1,2,3)S\,|\,i)=0$ it is possible to express the component of $\Psi(t\to\infty)$, in the ghost sector spanned by $\mathbb{1}_{[\gamma]}$, in the form

$$[\Psi(t\to\infty)]_{[\gamma]}=\tfrac{1}{2}\chi^*(1,2,3)\,|\,f')(f'\,|\,a_Q(3)a_Q(2)a_R(1)S\,|\,i)$$
$$+\tfrac{1}{2}a_Q^*(1)a_Q^*(2)a_R^*(3)\,|\,f')(f'\,|\,a_Q(3)a_R(2)a_R(1)S\,|\,i)+a_Q^*(1)g_b^*(2)g_a^*(3)\,|\,f')(f'\,|\,g_b(3)g_a(2)a_R(1)S\,|\,i)\;. \tag{3.16}$$

The inner products among the state vectors $a_Q^*(1)a_Q^*(2)a_R^*(3)\,|\,f')$, $\chi^*(1,2,3)\,|\,f')$, and $a_Q^*(1)g_b^*(2)g_a^*(3)\,|\,f')$ do not all vanish, and it is not manifest, from Eq. (3.16), that the probability measure in the ghost sector spanned by $\mathbb{1}_{[\gamma]}$ vanishes. When that probability measure is evaluated it turns out to be

$$|\,[\Psi(t\to\infty)]_{[\gamma]}\,|^2=\tfrac{1}{2}(i\,|\,S^*\chi^*(3,2,1)\,|\,f')(f'\,|\,a_Q(3)a_Q(2)a_R(1)S\,|\,i)+\tfrac{1}{2}(i\,|\,S^*a_R^*(1)a_Q^*(2)a_R^*(3)\,|\,f')(f'\,|\,\chi(1,2,3)S\,|\,i)\;, \tag{3.17}$$

and, in that form, inspection suffices to demonstrate that $[\Psi(t\to\infty)]_{[\gamma]}\,|^2$ vanishes.

The fact that the probability measure in the ghost sector vanishes is of crucial importance. Were that not the case, then the probability measure in the ghost-free sectors might well exceed unity, while the probability measure in the ghost sectors would then be negative. While the self-adjointness of the Hamiltonian would still mandate a total unit probability summed over the entire indefinite-metric Hilbert space, such a formalism would be inherently uninterpretable as a physical theory.

We have constructed an operator $\Gamma^i(\vec{k})$ to perform the role of $\Omega_0(\vec{k})$ and to project the pure gauge states, in Yang-Mills theory.[11] $\Gamma^i(\vec{k})$ is given by

$$\Gamma^i(\vec{k})=a_Q^i(\vec{k})+\sum_{\vec{q},n}(q/k)a_Q^{n*}(\vec{q})g_a^n(\vec{q})g_b^i(\vec{k})\;. \tag{3.18}$$

$\Gamma^i(\vec{k})\,|\,n)=0$ selects a set of states which includes all the one-ghost-particle states that produce vanishing S-matrix elements, and which we previously classified as pure gauge states. These include $a_Q^{i*}(\vec{k})\,|\,0)$, $g_a^{i*}(\vec{k})\,|\,0)$, $g_b^{i*}(\vec{k})\,|\,0)$ but properly exclude $a_R^{i*}(\vec{k})\,|\,0)$. The n-particle states $a_Q^{i(1)*}(\vec{k}_1)\cdots a_Q^{i(n)*}(\vec{k}_n)\,|\,0)$, $g_a^{i(1)*}(\vec{k}_1)\cdots g_a^{i(n)*}(\vec{k}_n)\,|\,0)$, and $g_b^{i(1)*}(\vec{k}_1)\cdots g_b^{i(n)*}(\vec{k}_n)\,|\,0)$ are included among the pure gauge states, as is $|\,\Psi(1,2))$, while the states $|\,\phi(1,2))$ and $|\,\chi(1,2,3))$ are properly excluded. $\Gamma^i(\vec{k})$ therefore satisfies the requirement that it project the desired states. The remaining crucial test for $\Gamma^i(\vec{k})$, as an operator suitable to serve as $\Omega_0^i(\vec{k})$ in non-Abelian theory, is whether it is possible to complete it by constructing $\Omega_1^i(\vec{k})$ so that $\Gamma^i(\vec{k})+\Omega_1^i(\vec{k})$ is a spectrally pure operator. In the next section we will show that $\Gamma^i(\vec{k})$ satisfies that requirement too, and we will give an explicit representation of $\Omega^i(\vec{k})$.

IV. REPRESENTATION OF SPECTRALLY PURE $\Omega^i(\vec{k})$

We first define the operator $\Lambda^i(\vec{k})$ by

$$\Lambda^i(\vec{k})=[a_Q^i(\vec{k})+J_0^i(\vec{k})]+\sum_{\vec{q},n}(q/k)[a_Q^{n*}(\vec{q})+J_0^n(-\vec{q})]g_a^n(\vec{q})g_b^i(\vec{k})\;, \tag{4.1}$$

where $J_0^i(\vec{k})=(2k^{3/2})^{-1}\int d\vec{x}\,e^{-i\vec{k}\cdot\vec{x}}J_0^i(\vec{x})$, and $J_0^i(\vec{x})$ is defined by Eq. (3.2). $\Lambda^i(\vec{k})$ is obtained by substituting $[a_Q^i(\vec{k})+J_0^i(\vec{k})]$ for $a_Q^i(\vec{k})$, and $[a_Q^{n*}(\vec{q})+J_0^n(-\vec{q})]$ for $a_Q^{n*}(\vec{q})$ in $\Gamma^i(\vec{k})$. In order to construct a spectrally pure operator $\Omega^i(\vec{k})$, we also define a residue operator $\mathscr{R}^i(\vec{k})$, by

$$[H,\Lambda^i(\vec{k})]+k\Lambda^i(\vec{k})=\mathscr{R}^i(\vec{k})\;. \tag{4.2}$$

$\mathscr{R}^i(\vec{k})$ can easily be obtained from the explicit expressions for H and Λ, and from the commutation (and anticommutation) rules for the gluon, quark, and Faddeev-

Popov fields, each with its corresponding canonical momentum. The result of that calculation is reported in Sec. 5 of Ref. 7. For our purposes the most useful form of $\mathscr{R}^i(\vec{k})$ is one in which we have already made use of the identities

$$g_b^i(\vec{k})|\alpha\rangle = -(E_\alpha - H - k)^{-1} Y^i(\vec{k})|\alpha\rangle \tag{4.3a}$$

and

$$g_a^i(\vec{k})|\alpha\rangle = -(E_\alpha - H - k)^{-1} Z^i(\vec{k})|\alpha\rangle\ , \tag{4.3b}$$

where $|\alpha\rangle$ designates an eigenstate of H with $(H - E_\alpha)|\alpha\rangle = 0$. The $Y^i(\vec{k})$ and $Z^i(\vec{k})$ that appear in these identities are given by

$$Y^i(\vec{k}) = -\sum_{\vec{k},j} (k/\kappa)[Q^{ij}(\vec{k},\vec{\kappa}) g_b^j(\vec{\kappa}) + g_b^{j*}(\vec{\kappa}) Q^{ij}(\vec{k},-\vec{\kappa})] \tag{4.4}$$

and

$$Z^i(\vec{k}) = \sum_{\vec{k},j} (\kappa/k)[Q^{ji}(-\vec{\kappa}_j - \vec{k}) g_a^j(\vec{\kappa}) + g_a^{j*}(\vec{\kappa}) Q^{ji}(\vec{\kappa},-\vec{k})]\ . \tag{4.5}$$

$Q^{ij}(\vec{k},\vec{\kappa})$ is an important auxiliary quantity that appears in $Y^i(\vec{k})$ and $Z^i(\vec{k})$ as well as in other expressions in this work. It is given by

$$Q^{ij}(\vec{k},\vec{\kappa}) = iek^{-3/2}\kappa^{1/2}\epsilon_{ijn} \int d\vec{x}\, e^{-i(\vec{k}-\vec{\kappa})\cdot\vec{x}} k_\mu b_\mu^n(\vec{x})\ . \tag{4.6}$$

We follow the convention that the signs of the momentum arguments in Q (and later, in F, D, and S) only refer to the signs of $\vec{\kappa}$ and $\vec{k}$ in the exponential, but not in other parts of the expression.[12] We will write $\mathscr{R}^i(\vec{k})$ in the form

$$\mathscr{R}^i(\vec{k}) = \sum_\alpha \mathscr{R}^i(\vec{k},\alpha)|\alpha\rangle\langle\bar{\alpha}|\ ,$$

where the unit operator $\sum_\alpha |\alpha\rangle\langle\bar{\alpha}|$, is explicitly shown as a series that represents closure over a complete set of states. The overbar on $\langle\bar{\alpha}|$ indicates that in the indefinite-metric space, in which this theory is formulated, closure does not always involve direct products of states with their own adjoints. We will write $\mathscr{R}^i(\vec{k})$ in this form but suppress the $\sum_\alpha |\alpha\rangle\langle\bar{\alpha}|$ in all subsequent expressions. $\mathscr{R}^i(\vec{k})$ then is given by

$$\begin{aligned}\mathscr{R}^i(\vec{k}) = {} & [Q^{ij}(\vec{k},\vec{\kappa}), J_0^j(\vec{\kappa})] - X_g^i(\vec{k}) - X_s^i(\vec{k}) + X^j(-\vec{\kappa})(E_\alpha - H - k - \kappa)^{-1} Q^{ij}(\vec{k},-\vec{\kappa}) \\ & - Q^{ij}(\vec{k},\vec{\kappa})(E_\alpha - H - \kappa)^{-1} X^j(\vec{\kappa}) + (\kappa/k) X^j(-\vec{\kappa})(E_\alpha - H - k - \kappa)^{-1} Z^j(\kappa)(E_\alpha - H - k)^{-1} Y^i(\vec{k}) \\ & - (\kappa/k) X^j(-\vec{\kappa})(E_\alpha - H - k - \kappa)^{-1} Y^i(\vec{k})(E_\alpha - H - \kappa)^{-1} Z^j(\vec{\kappa}) \\ & - (\kappa/k)[a_Q^{j*}(\kappa) + J_0^j(-\kappa)][Z^j(\vec{\kappa})(E_\alpha - H - k)^{-1} Y^i(\vec{k}) - Y^i(\vec{k})(E_\alpha - H - \kappa)^{-1} Z^j(\vec{\kappa})]\ .\end{aligned} \tag{4.7}$$

Repeated momenta and isospin indices in Eq. (4.7) are to be summed over their full ranges. $X^i(\vec{k})$ originates from an expression that is similar to Eqs. (4.3a) and (4.3b), i.e.,

$$[a_Q^i(\vec{k}) + J_0^i(\vec{k})]|\alpha\rangle = (E_\alpha - H - k)^{-1} X^i(\vec{k})|\alpha\rangle\ . \tag{4.3c}$$

$X^i(\vec{k})$ has the form

$$X^i(\vec{k}) = X_b^i(\vec{k}) + X_g^i(\vec{k}) + X_s^i(\vec{k})\ , \tag{4.8a}$$

the components of $X^i(\vec{k})$, given in Eq. (4.8a) are given by the following:

$$X_b^i(\vec{k}) = \sum_{\vec{\kappa},j} [Q^{ij}(\vec{k},\vec{\kappa}) a_Q^j(\vec{\kappa}) - a_Q^{j*}(\vec{\kappa}) Q^{ij}(\vec{k},-\vec{\kappa})]\ , \tag{4.8b}$$

$$X_g^i(\vec{k}) = \sum_{\vec{\kappa},j} [F^{ij}(\vec{k},\vec{\kappa}) g_b^j(\vec{\kappa}) + g_b^{j*}(\vec{\kappa}) F^{ij}(\vec{k},-\vec{\kappa})]\ , \tag{4.8c}$$

with

$$F^{ij}(\vec{k},\vec{\kappa}) = e(2\kappa)^{-1/2} k^{-3/2} \epsilon_{ijn} \int d\vec{x}\, e^{-i(\vec{k}-\vec{\kappa})\cdot\vec{x}} k_\mu \kappa_\mu \sigma_b^n(\vec{x})\ ; \tag{4.8d}$$

$$X_s^i(\vec{k}) = D^i(\vec{k}) + \sum_{\vec{\kappa},j} [S^{ij}(\vec{k},\vec{\kappa}) g_b^j(\vec{\kappa}) - g_b^{j*}(\kappa) S^{ij}(\vec{k},-\vec{\kappa})]\ , \tag{4.8e}$$

with

$$D^i(\vec{k}) = \frac{2e^2}{k^{3/2}} \left[\frac{1}{(2\pi)^3} \int \frac{d\vec{p}}{|\vec{p}|} \right] \int d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}} k_n b_n^i(\vec{x}) \tag{4.8f}$$

and

$$S^{ij}(\vec{k},\vec{\kappa}) = 2ie^2(2\kappa)^{-1/2} k^{-3/2} \epsilon_{isk}\epsilon_{rjk} \int d\vec{x}\, e^{-i(\vec{k}-\vec{\kappa})\cdot\vec{x}} k_n b_n^r(\vec{x}) \sigma_b^s(\vec{x})\ . \tag{4.8g}$$

The presence of $\mathscr{R}^i(\vec{k})$ on the right-hand side of Eq. (4.2) is a sign that $\Lambda^i(\vec{k})$ is not a spectrally pure operator, and that it can therefore not be used to impose a constraint. It is possible to invert Eq. (4.2) and to define an operator $\omega^i(\vec{k})$ by

$$\omega^i(\vec{k})=\Lambda^i(\vec{k})+(E_\alpha-H-k)^{-1}\mathscr{R}^i(\vec{k})\,|\,\alpha\rangle\langle\bar{\alpha}\,| \ . \qquad (4.9a)$$

Formally $\omega^i(\vec{k})$ obeys $[H,\omega^i(\vec{k})]+k\omega^i(\vec{k})=0$ but that, by itself, still does not entitle us to use $\omega^i(\vec{k})$ to impose a constraint. The energy denominator $(E_\alpha-H-k)^{-1}$ in Eq. (4.9) makes $\omega^i(\vec{k})$ potentially too singular to be part of a useful subsidiary condition. $\mathscr{R}^i(\vec{k})$ might well have nonvanishing matrix elements $\langle\alpha'\,|\,\mathscr{R}^i(\vec{k})\,|\,\alpha\rangle$ where $|\,\alpha'\rangle$ obeys $(H-E_{\alpha'})\,|\,\alpha'\rangle=0$ with $E_{\alpha'}+k=E_\alpha$. This would leave us with a vanishing denominator in Eq. (4.9). It is necessary to verify that this catastrophe cannot occur, before we can proceed to determine the form of $\Omega^i(\vec{k})$.

It has been possible to formulate a series representation of an operator $\Omega_1^i(\vec{k})$ such that $\mathscr{R}^i(\vec{k})$ can be expressed in the form

$$\mathscr{R}^i(\vec{k})=-\{[H,\Omega_1^i(\vec{k})]+k\Omega_1^i(\vec{k})\} \ . \qquad (4.10)$$

This fact implies that $\langle\bar{\alpha}\,|\,\mathscr{R}^i(\vec{k})\,|\,\alpha\rangle=0$ when $E_\alpha=E_{\alpha'}+k$, and that $(E_\alpha-H-k)^{-1}\mathscr{R}^i(\vec{k})\,|\,\alpha\rangle\langle\bar{\alpha}\,|$ retains meaning even when H has the eigenvalue $E_{\alpha'}=E_\alpha-k$. We can therefore rewrite Eq. (4.9a) in the form

$$\omega^i(\vec{k})=\Lambda^i(\vec{k})+\Omega_1^i(\vec{k}) \qquad (4.9b)$$

and identify $[\Lambda^i(\vec{k})+\Omega_1^i(\vec{k})]$ as the spectrally pure operator $\Omega^i(\vec{k})$.

The technical problems associated with implementing Eq. (4.10) require us to express $\mathscr{R}^i(\vec{\kappa})$ in the form of Eq. (4.7) and to represent $\Omega_1^i(\vec{k})$ as a formal operator series. The individual terms in this series contain integrals over singular denominators typified by

$$\sum_{\vec{\kappa},j} g_b^{j*}(\vec{\kappa})(E_\alpha-H-k-\kappa)^{-1}F^{ij}(\vec{k},\vec{\kappa}) \ .$$

All the integrals of this type, appearing in $\Omega_1^i(\vec{k})$, involve integration over at least one momentum variable in addition to the summation over the complete set of states $|\,\alpha\rangle\langle\bar{\alpha}\,|$. Such improper integrals, when they are convergent, define multivalued functions. The contours of the integrations need to be specified to remove all ambiguities, but the exact nature of the contours is not relevant for the discussion here.

Later we will discuss the mathematical existence of these integrals, but here we emphasize that their singular integrands do not present the problem inherent in the singular $(E_\alpha-H-k)^{-1}$ in Eq. (4.9a), which is summed over the states $|\,\alpha\rangle\langle\bar{\alpha}\,|$ but not integrated over any momentum variables. To clarify what that hazard is, we offer the following illustrative example: Suppose we had attempted to use $a_Q^i(\vec{k})$ alone, instead of $\Gamma^i(\vec{k})$, as the $\Omega_0^i(\vec{k})$ component of a spectrally pure operator (as would be appropriate for QED). We would then have found $[a_Q^i(\vec{k})+J_0^i(\vec{k})]$ replacing $\Lambda^i(\vec{k})$, and $-X^i(\vec{k})$ replacing $\mathscr{R}^i(\vec{k})$ given in Eq. (4.7). One of the constituent elements of $X^i(\vec{k})$ is the $\sum_{\vec{\kappa},j}Q^{ij}(\vec{k},\vec{\kappa})a_Q^j(\vec{\kappa})$ that is part of $X_b^i(\vec{k})$, as shown in Eqs. (4.8a) and (4.8b). From Eq. (4.3c) we find that one of the contributions, that $\sum_{\vec{\kappa},j}Q^{ij}(\vec{k},\vec{\kappa})a_Q^1(\vec{\kappa})$ makes to $\langle\bar{\alpha}'\,|\,X^i(\vec{k})\,|\,\alpha\rangle$ is

$$-\left\langle\bar{\alpha}'\left|\sum_{\vec{\kappa},j}Q^{ij}(\vec{k},\vec{\kappa})J_0^j(\vec{\kappa})\right|\alpha\right\rangle .$$

We express $Q^{ij}(\vec{k},\vec{\kappa})J_0^j(\vec{\kappa})$ as half of the sum of the commutator and the anticommutator of $Q^{ij}(\vec{k},\vec{\kappa})$ and $J_0^j(\vec{\kappa})$. The anticommutator $\{Q^{ij}(\vec{k},\vec{\kappa}),J_0^j(\vec{\kappa})\}$ has terms trilinear and quartic in operator valued fields; the commutator, for which we define $\xi=\frac{1}{2}[Q^{ij}(\vec{k},\vec{\kappa}),J_0^j(\vec{\kappa})]$, is linear in the gluon field b_u. The formally "spectrally pure" operator $\mu^i(\vec{k})$ is given by

$$\mu^i(\vec{k})=(a_Q^i(\vec{k})+J_0^i(\vec{k}))-\sum_\alpha(E_\alpha-H-k)^{-1}X^i(\vec{k})\,|\,\alpha\rangle\langle\bar{\alpha}\,| \ . \qquad (4.11)$$

One of the components of

$$-\sum_\alpha(E_\alpha-H-k)^{-1}X^i(\vec{k})\,|\,\alpha\rangle\langle\bar{\alpha}\,|$$

is $\sum_\alpha(E_\alpha-H-k)^{-1}\xi\,|\,\alpha\rangle\langle\bar{\alpha}\,|$. We express this component as

$$\sum_{\alpha,\alpha'}(E_\alpha-E_{\alpha'}-k)^{-1}\,|\,\alpha'\rangle\langle\bar{\alpha}'\,|\,\xi\,|\,\alpha\rangle\langle\bar{\alpha}\,| \ ,$$

and study the case of

$$|\,\alpha\rangle=\{a_R^{n*}(\vec{p})+(E_\alpha-H-p)^{-1}[H_1,a_R^{n*}(\vec{p})]\}\,|\,\alpha'\rangle \ .$$

We note that if $(H-E_{\alpha'})\,|\,\alpha'\rangle=0$ then $(H-E_\alpha)\,|\,\alpha\rangle=0$ with $E_\alpha=E_{\alpha'}+p$. We next extract, as one contribution to

$$\sum_{\alpha,\alpha'}(E_\alpha-E_{\alpha'}-k)^{-1}\,|\,\alpha'\rangle\langle\bar{\alpha}'\,|\,\xi\,|\,\alpha\rangle\langle\alpha\,| \ ,$$

the quantity σ, given by

$$\sigma=(p-k)^{-1}\sum_{\alpha,\alpha'}|\,\alpha'\rangle\langle\bar{\alpha}'\,|\,[\xi,a_R^{n*}(\vec{p})]\,|\,\alpha'\rangle\langle\bar{\alpha}'\,|\,a_Q^n(\vec{p}) \ . \qquad (4.12a)$$

Explicit calculation leads to

$$\sigma=\frac{1}{(p-k)}\frac{e^2}{k}\sum_{\vec{\kappa}}\frac{1}{|\,\vec{\kappa}\,|}\delta_{\vec{k},\vec{p}}a_Q^i(\vec{p}) \ . \qquad (4.12b)$$

Besides the divergent integral, which can be regularized, σ contains the catastrophic infinity $(p-k)^{-1}\delta_{\vec{k},\vec{p}}$. When an expression as singular as σ is not canceled by other contributions, and when it remains part of a spectrally pure operator, it is impossible to use the latter to impose a subsidiary condition.

Whether a residue develops the kind of catastrophic singularity discussed in the preceding illustrative example is sensitively dependent on the form of $\Omega_0^i(\vec{k})$. When $\Gamma^i(\vec{k})$ is chosen to be $\Omega_0^i(\vec{k})$ for Yang-Mills theory, this catastrophe is avoided. The residue $\mathscr{R}^i(\vec{k})$ combines parts from $[a_Q^i(\vec{k})+J_0^i(\vec{k})]$ and from the trilinear combination

$$Q_{AB}=\sum_{\vec{p},j}(p/k)[a_Q^{j*}(\vec{p})+J_0^j(-\vec{p})]g_a^j(\vec{p})g_b^i(\vec{k})\ .$$

The $a_Q^{j*}(\vec{\kappa})Q^{ij}(\vec{k},-\vec{\kappa})$ that stems from $[a_Q^i(\vec{k})+J_0^i(\vec{k})]$ is canceled by a contribution from Q_{AB}; the $Q^{ij}(\vec{k},\vec{\kappa})J_0^j(\vec{\kappa})$ from $[a_Q^i(\vec{k})+J_0^i(\vec{k})]$ combines with $-J_0^j(\vec{\kappa})Q^{ij}(\vec{k},\vec{\kappa})$ from Q_{AB} to form the commutator $[Q^{ij}(\vec{k},\vec{\kappa}),J_0^j(\vec{\kappa})]$ and the latter is canceled by a combination of contributions from other parts of $\mathscr{R}^i(\vec{\kappa})$. Extensive and remarkable cancellations between contributions to the residue from $[a_Q^i(\vec{k})+J_0^i(\vec{k})]$ and Q_{AB} exclude from $\mathscr{R}^i(\vec{k})$ all those components for which $\langle\bar{\alpha}'\,|\,\mathscr{R}^i(\vec{k})\,|\,\alpha\rangle$ might not vanish for $E_\alpha=E_{\alpha'}+k$. The fact that these cancellations take place lends support to $\Gamma^i(\vec{k})$ as an appropriate choice for $\Omega_0^i(\vec{k})$ for Yang-Mills theory, and to the use of $[\Lambda^i(\vec{k})+\Omega_1^i(\vec{k})]\,|\,\nu\rangle=0$ as a time-independent constraint.

We will give $\Omega^i(\vec{k})$ for Yang-Mills theory in the form

$$\Omega^i(\vec{k})=\Lambda^i(\vec{k})+e^{-ikt}\lambda^i(\vec{k};t) \tag{4.13a}$$

and $\lambda^i(\vec{k};t)$ will be represented as

$$\lambda^i(\vec{k};t)=\alpha^i(\vec{k};t)+\beta^i(\vec{k};t)+\gamma^i(\vec{k};t)+\delta^i(\vec{k};t)\ , \tag{4.13b}$$

where $\alpha,\beta,\gamma,\delta$ each represent an operator series. Each series involves time-ordered products of Heisenberg fields. Certain spatial integrals containing these operator-valued fields, that are basic to the dynamics of the Yang-Mills theory, appear in α, β, γ, and δ in different combinations. The recurrence of these same operator-valued quantities in definite patterns in different expressions is instrumental in the cancellation mechanism that permits construction of $\Omega_1^i(\vec{k})$. Prominent among these quantities are $B^{ij}(\vec{k},\vec{q};x_0)$ and $f^{ij}(\vec{k},\vec{q};x_0)$, given by

$$B^{ij}(\vec{k},\vec{q};x_0)=k^{3/2}q^{-1/2}\exp[iHx_0]Q^{ij}(\vec{k},\vec{q})\exp[-iHx_0] \tag{4.14a}$$

and by

$$f^{ij}(\vec{k},\vec{q};x_0)=k^{3/2}q^{1/2}\exp[iHx_0]F^{ij}(\vec{k},\vec{q})\exp[-iHx_0]\ , \tag{4.14b}$$

so that $B^{ij}(\vec{k},\vec{q};x_0)$ and $f^{ij}(\vec{k},\vec{q};x_0)$ have the form

$$B^{ij}(\vec{k},\vec{q};x_0)=ie\epsilon_{ijn}\int d\vec{x}\,e^{-i(\vec{k}-\vec{q})\cdot\vec{x}}k_\mu b_\mu^n(x) \tag{4.14c}$$

and

$$f^{ij}(\vec{k},\vec{q};x_0)=2^{-1/2}e\epsilon_{ijn}\int d\vec{x}\,e^{-i(\vec{k}-\vec{q})\cdot\vec{x}}k_\mu q_\mu\sigma_b^n(x)\ , \tag{4.14d}$$

respectively.

$\alpha^i(\vec{k},t)$ is given by

$$\alpha^i(\vec{k};t)=k^{-3/2}\sum_{\vec{p},j}p^{-1/2}g_b^{j*}(\vec{p};t)e^{-ipt}\varphi^{ij}(\vec{k},-\vec{p};t)\ , \tag{4.15}$$

where $\varphi^{ij}(\vec{k},-\vec{p};t)$ represents the series $\varphi=\varphi^{(1)}+\varphi^{(2)}+\cdots+\varphi^{(N)}$. $\varphi^{(1)}$ is given by

$$\varphi^{(1)}=i\int_{-\infty}^{t}dx_0 f^{ij}(\vec{k},-\vec{p};x_0)e^{i(k+p)x_0}\ . \tag{4.16}$$

Time dependence always designates Heisenberg picture operators in this work, and convergence factors in integrals at $t\rightarrow-\infty$ are understood. Temporal variables are dummy indices in these expressions. The time variable drops out of the final expression for $e^{-ikt}\lambda^i(\vec{k},t)$, making $\Omega^i(\vec{k})$ a time-independent operator. The Nth-order term $\varphi^{(N)}$ is given by

$$\begin{aligned}\varphi^{(N)}=(i)^N\sum\,|\vec{q}_1|^{-1}\cdots|\vec{q}_{(N-1)}|^{-1}\mathrm{sgn}(\alpha)\\ \times\int_{-\infty}^{t}dx_0(1)\cdots\int_{-\infty}^{x_0(N-1)}dx_0(N)\\ \times f^{in(1)}(\vec{k},\xi\vec{q}_1;x_0(1))\exp[i(k+\epsilon(q_1))x_0(1)]B^{n(1),n(2)}(\xi\vec{q}_1,\xi\vec{q}_2;x_0(2))\\ \times\exp[i(\epsilon(q_1)+\epsilon(q_2))x_0(2)]\cdots B^{n(N-1),j}(\xi\vec{q}_{(N-1)},-\vec{p};x_0(N))\\ \times\exp[i(\epsilon(q_{N-1})+p)x_0(N)]\ .\end{aligned} \tag{4.17}$$

The summation $\sum$ in Eq. (4.17) includes integration over all momenta $\vec{q}_i$, summation over all isospin indices $n(i)$, and summation over the following permutations: (1) Exchanges of f and B, in which the arguments of f and B (momenta and isospin indices) remain fixed in position, but the functions f and B change places, so that in the new position B has the arguments that f had before the exchange, and f, in the new position, has the arguments that B had before the exchange. (2) Exchanges in which f and B exchange positions, but each of them is accompanied by all of its momentum and isospin indices. (3) Exchanges in which two B terms change places, each of them accompanied by all momentum and isospin indices, except that we exclude those exchanges in which both B terms have dummy momentum and isospin indices so that the original and permuted expressions would be identical.

The dummy time argument is always defined by position in the time-ordered array of operators. $\epsilon(q)$ is a position-dependent quantity, and represents either $|\vec{q}|$ or $-|\vec{q}|$ depending upon its position in the integrand. For each q_i, $\epsilon(q_i)$ appears exactly twice in each expression. It represents $|\vec{q}|$ in the *rightmost* position, and $-|\vec{q}|$ in the *leftmost* position. $\xi\vec{q}$ is also position dependent, and represents either $\vec{q}$ or $-\vec{q}$, depending upon its position. For each $\vec{q}_i$, $\xi\vec{q}_i$ appears exactly twice in the argument of an f or a B term, once in the first argument position, and once in the second [e.g., $B(\xi\vec{q}_i,\vec{p};x_0)$ and $B(\vec{k},\xi\vec{q}_i;x_0)$, respectively]. If the function (f or B) in which $\xi\vec{q}$ is in the first argument position is to the *right* of the function in which $\xi\vec{q}_i$ is in the second argument position, $\xi\vec{q}_i$ designates $\vec{q}_i$. If the order is reversed, $\xi\vec{q}_i$ designates $-\vec{q}_i$. Sgn(α) indicates the overall sign of each permutation. It is a multiplicative combination that includes (-1) for each integrated dummy momentum for which $\xi\vec{q}_i=-\vec{q}_i$, except that if the dummy momentum is in the second argument position of f, it is not counted. Sgn(α) also includes (-1) if $-\vec{p}$ appears as an argument in the second argument position of f. To illustrate, we cite $\varphi^{(2)}$, which is given by

$$\begin{aligned}\varphi^{(2)}=-\sum_{\vec{q},n}|\vec{q}|^{-1}\int_{-\infty}^{t}dx_0\int_{-\infty}^{x_0}dy_0\{&f^{in}(\vec{k},\vec{q};x_0)\exp[i(k-q)x_0]B^{nj}(\vec{q},-\vec{p};y_0)\exp[i(p+q)y_0]\\&+B^{nj}(-\vec{q},-\vec{p};x_0)\exp[i(p-q)x_0]f^{in}(\vec{k},-\vec{q};y_0)\exp[i(k+q)y_0]\\&+f^{nj}(-\vec{q},-\vec{p};x_0)\exp[i(p-q)x_0]B^{in}(\vec{k},-\vec{q};y_0)\exp[i(k+q)y_0]\\&-B^{in}(\vec{k},\vec{q};x_0)\exp[i(k-q)x_0]f^{nj}(\vec{q},-\vec{p};y_0)\exp[i(p+q)y_0]\}\ .\end{aligned}\tag{4.18}$$

β is given by

$$\beta^{i}(\vec{k},t)=\sum_{\vec{p},j}k^{-3/2}p\,[a_Q^{j*}(\vec{p};t)+J_0^j(-\vec{p};t)]e^{-ipt}\int_{-\infty}^{t}dx_0\Psi^{ij}(\vec{k},-\vec{p};x_0)\tag{4.19}$$

with

$$\Psi^{ij}(\vec{k},-\vec{p};x_0)=\sum_{\vec{q},n}q^{-1/2}\left[g_b^{n*}(\vec{q},x_0)e^{-iqx_0}T^{ijn}(\vec{k},\vec{p},\vec{q};x_0)+iY^n(-\vec{q};x_0)e^{-iqx_0}\int_{-\infty}^{x_0}dy_0T^{ijn}(\vec{k},\vec{p},\vec{q};y_0)\right],\tag{4.20}$$

where $T^{ijn}(\vec{k},\vec{p},\vec{q};t)$ is a series given by $T=T^{(2)}+\cdots+T^{(N)}$. $T^{(2)}$ is given by

$$T^{(2)}=\int_{-\infty}^{t}dx_0[B^{in}(\vec{k},-\vec{q};t)e^{i(k+q)t}Z^j(\vec{p};x_0)e^{ipx_0}+Z^j(\vec{p};t)e^{ipt}B^{in}(\vec{k},-\vec{q};x_0)e^{i(k+q)x_0}]\tag{4.21}$$

and the Nth-order term is given by

$$\begin{aligned}T^{(N)}=-\sum(-i)^N|\vec{\kappa}_1|^{-1}\cdots|\vec{\kappa}_{(N-2)}|^{-1}\mathrm{sgn}(\beta)\\\times\int_{-\infty}^{t}dx_0(1)\cdots\int_{-\infty}^{x_0(N-2)}dx_0(N-1)&B^{is(1)}(\vec{k},\xi\vec{\kappa}_1;t)\exp[i(k+\epsilon(\kappa_1))t]B^{s(1)s(2)}(\xi\vec{\kappa}_1,\xi\vec{\kappa}_2;x_0(1))\\&\times\exp[i(\epsilon(\kappa_1)+\epsilon(\kappa_2))x_0(1)]\cdots B^{s(N-2)n}(\xi\vec{\kappa}_{(N-2)},-\vec{q};x_0(N-2))\\&\times\exp[i(\epsilon(\kappa_{N-2})+q)x_0(N-2)]Z^j(\vec{p};x_0(N-1))\\&\times\exp[ipx_0(N-1)]\ .\end{aligned}\tag{4.22}$$

In the case of $T^{(N)}$ the summation $\sum$ includes all dummy momenta $\vec{\kappa}_i$ and isospin indices $s(i)$, and the following permutations: (1) Exchanges in which Z and a B exchange positions, each accompanied by all its momentum and isospin indices (Z always has the momentum argument p and isospin j). (2) Exchanges in which two B terms change places, each of them accompanied by all momentum and isospin indices, except that we exclude those exchanges in which both B terms have dummy momentum and isospin indices so that the original and permutated expressions would be identical. Sgn(β) indicates the sign of each permutation. It is a multiplicative combination that counts (-1) for each dummy momentum for which $\xi\vec{\kappa}=-\vec{\kappa}$. To illustrate we cite $T^{(3)}$ which is given by

$$T^{(3)}=-i\sum_{\vec{\kappa}}|\vec{\kappa}|^{-1}\int_{-\infty}^{t}dx_0\int_{-\infty}^{x_0}dy_0\{B^{is}(\vec{k},\vec{\kappa};t)\exp[i(k-\kappa)t]Z^{j}(\vec{p};x_0)\exp[ipx_0]B^{sn}(\vec{\kappa},-\vec{q};y_0)\exp[i(q+\kappa)y_0]$$
$$+B^{is}(\vec{k},\vec{\kappa};t)\exp[i(k-\kappa)t]B^{sn}(\vec{\kappa},-\vec{q};x_0)\exp[i(q+\kappa)x_0]Z^{j}(\vec{p};y_0)\exp[ipy_0]$$
$$-B^{sn}(-\vec{\kappa},\vec{q};t)\exp[i(q-\kappa)t]Z^{j}(\vec{p};x_0)\exp[ipx_0]B^{is}(\vec{k},-\vec{\kappa};y_0)\exp[i(k+\kappa)y_0]$$
$$-B^{sn}(-\vec{\kappa},\vec{q};t)\exp[i(q-\kappa)t]B^{is}(\vec{k},-\vec{\kappa};x_0)\exp[i(k+\kappa)x_0]Z^{j}(\vec{p};y_0)\exp[ipy_0]$$
$$-Z^{j}(\vec{p};t)\exp[ipt]B^{sn}(-\vec{\kappa},-\vec{q};x_0)\exp[i(q-\kappa)x_0]B^{is}(\vec{k},-\vec{q};y_0)\exp[i(q+\kappa)y_0]$$
$$+Z^{j}(\vec{p};t)\exp[ipt]B^{is}(\vec{k},\vec{\kappa};x_0)\exp[i(k-\kappa)x_0]B^{sn}(\vec{\kappa},-\vec{q};y_0)\exp[i(q+\kappa)y_0]\}\ . \tag{4.23}$$

$\delta^{ij}(\vec{k},t)$ is given by

$$\delta^{ij}(\vec{k},t)=k^{-3/2}\sum_{\vec{p},j}g_b^{j*}(\vec{p},t)e^{-ipt}\eta^{ij}(\vec{k},-\vec{p};t)\ . \tag{4.24}$$

$\eta^{ij}(\vec{k},-\vec{p};t)$ differs from $\phi^{ij}(\vec{k},-\vec{p};t)$ in two respects only. One is the substitution of $S^{ij}(\vec{k},\vec{\kappa};t)$ which is $O(e^2)$, in the case of η, for $f^{ij}(\vec{k},\vec{\kappa};t)$ which is $O(e)$ in the case of φ. The other is the substitution of sgn(δ) for sgn(α). Sgn(δ) is a multiplicative combination, that counts (-1) for each integrated momentum, $\vec{\kappa}$, in which $\xi\vec{\kappa}$ signifies $-\vec{\kappa}$, including momenta appearing in both argument positions of S.

$\gamma^{i}(k,t)$ is given by $\gamma=\gamma^{(1)}+\cdots+\gamma^{(N)}$ with

$$\gamma^{(1)}=-e(2k)^{-1/2}\sum_{\vec{q},j,n}|\vec{q}|^{-1}|\vec{k}-\vec{q}|^{-1}\exp[i(k-q-|\vec{k}-\vec{q}|)t]\epsilon_{ijn}\times\int_{-\infty}^{t}dx_0B^{nj}(\vec{q},\vec{q}-\vec{k};x_0)\exp[i(q+|\vec{k}-\vec{q}|)x_0] \tag{4.25}$$

and

$$\gamma^{(2)}=-ie(2k)^{-1/2}\sum_{\vec{q},j,\vec{\kappa},s,n}|\vec{q}|^{-1}|\vec{\kappa}|^{-1}|\vec{k}-\vec{\kappa}|^{-1}\exp[i(k-q-|\vec{k}-\vec{\kappa}|)t]\epsilon_{isj}$$
$$\times\int_{-\infty}^{t}dx_0\int_{-\infty}^{x_0}dy_0\left\{B^{nj}(-\vec{q},(\vec{\kappa}-\vec{k});x_0)\exp[i(|\vec{\kappa}-\vec{k}|-\kappa)x_0]B^{sn}(\vec{\kappa},-\vec{q};y_0)\exp[i(q+\kappa)y_0]\right.$$
$$\left.-B^{sn}(\vec{\kappa},\vec{q};x_0)\exp[i(q-\kappa)x_0]B^{nj}(\vec{q},(\vec{\kappa}-\vec{k});y_0)\exp[i(|\vec{k}-\vec{\kappa}|+\kappa)y_0]\right\}$$
$$-ie(2k^3)^{-1/2}\sum_{\vec{q},j,\vec{\kappa},s,n}|\vec{q}|^{-1}|\vec{q}+\vec{\kappa}|^{-1}\exp[-i(q+\kappa+|\vec{q}+\vec{\kappa}|)t]\epsilon_{snj}$$
$$\times\int_{-\infty}^{t}dx_0\int_{-\infty}^{x_0}dy_0\{B^{nj}(\vec{q},(\vec{q}+\vec{\kappa});x_0)\exp[i(q+|\vec{q}+\vec{\kappa}|)x_0]B^{is}(\vec{k},-\vec{\kappa};y_0)\exp[i(k+\kappa)y_0]$$
$$+B^{is}(\vec{k},-\vec{\kappa};x_0)\exp[i(k+\kappa)x_0]B^{nj}(\vec{q},(\vec{q}+\vec{\kappa});y_0)\exp[i(q+|\vec{q}+\vec{\kappa}|)y_0]\}\ . \tag{4.26}$$

It is difficult to give an explicit expression for $\gamma^{(N)}$, but arbitrary orders can be generated straightforwardly from recursion rules. The expressions for φ, η, and T are formal operator series, and provide a formulation for computation rather than an explicit expression for $\Omega_1^i(\vec{k})$. $\Omega_1^i(\vec{k})$ contains series, in particular φ, η, and T, whose forms are suggestive of expressions in formal scattering theory such as, for example, the Møller wave matrix.[13] The obvious differences are that the operator constituents of φ, η, and T are Heisenberg fields, so that the Green's functions that stem from the time-ordered integrations in φ, η, and T contain the total Hamiltonian H. In contrast the Green's functions in the series representations of the scattering wave matrix contain the interaction-free Hamiltonian H_0. In the case of φ, η, and T, f, S, or B appear between adjacent Green's functions $(E-H)^{-1}$, in lieu of the vertices that appear between adjacent Green's functions $(E-H_0)^{-1}$ in formal scattering theory.[14] A natural way to exploit this resemblance is to try to sum the series for φ, η, and T formally, and represent it, as far as may be possible, in a more compact form. It is known that operator series often admit of such formal summation. One example from scattering theory is the sum

$$G=G_0+G_0H_1G_0+\cdots+(G_0H_1G_0\cdots G_0H_1G_0)\ ,$$

where $G=(E-H)^{-1}$ and $G_0=(E-H_0)^{-1}$. G contains information about the discrete spectrum of H even though the Born series given above may lack the necessary convergence properties to account for that fact in a mathematically rigorous way.[15]

The question arises whether the improper integrals implicit in $\Omega_1^i(\vec{k})$ are convergent. It is known that QCD and Yang-Mills theory are ultraviolet renormalizable.[16] Demonstrations have also been given that, in perturbation theory, on the leading twist level, they are infrared finite very much like QED; that, in these cases, radiative corrections to elastic cross sections combine with radiative inelastic soft-gluon processes to give infrared-finite results. This has been verified for low orders in perturbation theory[17]; for soft gluons radiated by massive quarks, in cases in which a single quark in the initial state is scattered in a colorless external potential, a proof has been given to all orders.[18] Whether soft-gluon contributions in QCD and Yang-Mills theory are finite when all orders of perturbation theory are summed, or whether a nonperturbative infrared divergence reflects a confinement mechanism, is still an open question.[19] It is not clear, at this point, what implications renormalizability, and order-by-order cancellation of divergences in virtual and real soft-gluon processes have for the integrals that are implicit in $\Omega_1^i(\vec{k})$. Nor is it clear to what extent the implicit presence of divergent integrals in the series, contained in $\Omega_1^i(\vec{k})$, interfere with the program to impose $[\Lambda^i(\vec{k})+\Omega_1^i(\vec{k})]\,|\,\nu\rangle=0$ as a constraint. Reference 7 demonstrates that the algebraic identities responsible for unitarity in the physical subspace operate through the cancellation of identical divergent integrals that require trivial regularization only. The essential characteristic in $\Omega_1^i(\vec{k})$ is the pattern in which f, B, and s appear in the series α, β, γ, and δ; the values of the individual integrals involved in the cancellation seem to be less important.

V. DISCUSSION

$\Omega^i(\vec{k})$ imposes the subsidiary condition

$$\Omega^i(\vec{k})\,|\,\nu\rangle=0 \tag{5.1}$$

which selects a subset of states to constitute a physical subspace. Since $\Omega^i(\vec{k})$ is a spectrally pure operator, time evolution will not violate this constraint. The states $|\,\nu\rangle$ therefore take on a particular importance in Yang-Mills theory. They will in general be coherent superpositions of the Fock states that obey $(H_0-E_n)\,|\,n\rangle=0$ and physical states, because they obey Eq. (5.1), constitute stable admixtures of quarks, gluons, and Faddeev-Popov ghosts that remain undisturbed by time evolution. The vacuum state for Yang-Mills theory will be one of these states. Equation (5.1) allows us to define a nonlocal field operator $W^i(x)$ and its time derivative, for which

$$\langle\nu\,|\,W^i(x)\,|\,\nu\rangle=0 \tag{5.2}$$

and

$$\partial_0\langle\nu\,|\,W^i(x)\,|\,\nu\rangle=0\ , \tag{5.3}$$

where

$$W^i(x)=i\sum_{\vec{k}}|\,\vec{k}\,|^{1/2}[\Omega^i(\vec{k})e^{ik_\mu x_\mu}-\Omega^{i*}(\vec{k})e^{-ik_\mu x_\mu}] \tag{5.4}$$

and

$$\partial_0 W^i(x)=\sum_{\vec{k}}|\,k\,|^{3/2}[\Omega^i(\vec{k})e^{ik_\mu x_\mu}+\Omega^{i*}(\vec{k})e^{-ik_\mu x_\mu}]\ . \tag{5.5}$$

Ampere's law and Gauss's law, when implemented as subsidiary conditions in QED in a covariant gauge, are the Abelian analogs of Eqs. (5.4) and (5.5), respectively. On the basis of that analogy it is reasonable to surmise that these equations, particularly Eq. (5.5), contain information about the long-range properties of Yang-Mills theory.

The Fock space that underlies the perturbative rules for non-Abelian gauge theories is almost certainly inappropriate for the description of the observed particle spectrum, since the basic states of that space, i.e., single quarks and gluons, are never observed to appear as asymptotic states. To some extent, Yang-Mills theory and QCD share this problem with QED. In QED, the covariant Feynman rules represent electrons as spin-$\frac{1}{2}$ fermions that emerge from the scattering region and exist asymptotically detached from all other fields. In fact we know that electrons never are separable from their Coulomb field nor, when moving, from the transverse photons required to flatten the Coulomb field and to constitute a magnetic field, as mandated by Lorentz covariance. Quite likely in all gauge theories, QED, Yang-Mills theory, and QCD, the "true" states determined by the constraints that define a physical subspace, are coherent superpositions of the Fock states implicit in the perturbative rules. There is a crucial difference, however, between non-Abelian theories, and the Abelian QED. In non-Abelian theories, the failure to correctly identify the "true" states, among them the vacuum, leads to serious errors in predicting long-range behavior. In QED the coupling of the charged particle and photon fields is such that scattering cross sections are immune to the failure of the perturbative theory to use "true" states, or to implement the subsidiary condition.[20]

Our plan for Yang-Mills theory is to employ a procedure we have used extensively in Abelian theories. It consists of implementing a constraint, in the form of a subsidiary condition, consistent with the time evolution dictated by a Hamiltonian; and requiring that Hamiltonian to imply the covariant perturbative rules that characterize the short-distance properties of the theory. We follow a pattern established for QED and reviewed in Sec. II of this paper: To construct a unitary operator U such that $U\Omega^i(\vec{k})U^{-1}=\Gamma^i(\vec{\kappa})$, and to transform the Hamiltonian for Yang-Mills theory by the same transformation, so that $UHU^{-1}=\widetilde{H}$. If that program is implementable, then $\Gamma^i(\vec{k})$ will itself be a spectrally pure operator in this representation, so that $[\widetilde{H},\Gamma^i(\vec{k})]+k\Gamma^i(\vec{k})=0$. Since $[H_0,\Gamma^i(\vec{k})]+k\Gamma^i(\vec{k})=0$ also holds, the interaction Hamiltonian in the new representation, $\widetilde{H}_1=\widetilde{H}-H_0$, would then commute with $\Gamma^i(\vec{k})$. If Yang-Mills theory follows the pattern that obtains in QED, $\widetilde{H}_1$ could contain the operators $a_Q^i(\vec{k})$ and $a_Q^{i*}(\vec{k})$ as well as $\Psi(1,2)$ and $\Psi^*(1,2)$, but not $a_R^i(\vec{k})$ and $a_R^{i*}(\vec{k})$ or $\phi(1,2)$ and $\phi^*(1,2)$ [given in Eq. (3.9)], which fail to commute with $\Gamma^i(\vec{k})$ or $\Gamma^{i*}(\vec{k})$. These restrictions inhibit, in this representation, ghost-mediated processes embodied in the Feynman rules, and require that explicit nonlocal interaction among quarks and transverse gluons be generated to take their place.

It is uncertain how far the program outlined above can be developed, but the construction of the operator $\Omega^i(\vec{k})$ can be a significant step toward clarifying long-range properties of Yang-Mills theory. It is easy to lose sight of the importance of this kind of subsidiary condition, and of how well it lends itself to specifying long-range behavior, because in QED, although it is possible to apply this procedure, it is never necessary to carry out the determination of long-range forces with the help of the subsidiary condition. In QED the long-range forces are known independently of the subsidiary condition, and the latter can be, and generally is, ignored in calculating S-matrix elements. But in non-Abelian theories we seem not to have this freedom, and this program may be of significant benefit in studying long-range behavior, and well worth pursuing.

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