Relativistic rotator. II. The simplest representation spaces

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As a continuation of the preceding paper in which the quantum relativistic rotator was described within the framework of quantum constrained Hamiltonian mechanics, here its simplest representation spaces are derived. They are shown to be discrete direct sums of irreducible representations of the Poincaré group, which shows that hadrons can be interpreted as different mass-spin levels of a quantum relativistic rotator.

I. INTRODUCTION

In the preceding paper[1] the quantum relativistic rotator (QRR) was defined as a constrained quantum system with a relativistic Hamiltonian that corresponded to those of well-known physical systems in the nonrelativistic, the classical, and the elementary limits. In this paper we will derive the simplest representation spaces of the QRR. Although the method we use is not specific to the rotator constraint and can be generalized to a large class of constraint relations and particle spectra, here we restrict our discussion to the particular model of Secs. I—III of paper I. In the Appendix we give a brief derivation of the representations of SO(3,2) which are used in this model.

In Sec. II of paper I we found that a rotating and translating quantum physical object has two different but intimately related internal angular momenta associated with it. The "intrinsic" angular momentum $S_{\mu\nu}$ corresponds classically to the angular momentum in the reference frame in which the particle position Q_μ is at rest, and the spin angular momentum $\Sigma_{\mu\nu}$ corresponds classically to the angular momentum in the reference frame in which the center-of-mass position Y_μ is at rest, which is the usual rest frame as for it $\vec{p}=0$. The intrinsic angular momentum operators $S_{\mu\nu}$ form the "intrinsic" homogeneous Lorentz group $SO(3,1)_{S_{\mu\nu}}$ of (I.2.2). The spin tensor $\Sigma_{\mu\nu}$ does not quite form a group but obeys (I.2.16). [In the rest frame $\Sigma_{ij}^{\rm rest}=S_{ij}^{\rm rest}=-{\rm sgn}\hat{p}_0\epsilon_{ijm}\hat{w}_m^{\rm rest}$ $(i,j,m=1,2,3)$ where $\hat{w}_\mu$ is the Pauli-Lubanski vector.]

Corresponding to these two angular momenta there are two basis systems of the representations of the Poincaré group: the canonical basis system

$$|\hat{p},m,s,s_3\rangle\ , \tag{1.1}$$

in which the spin $\hat{W}=\frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}$ is diagonal and has the eigenvalue $s(s+1)$, and the spinor basis system

$$|\hat{p},m,{}^{j}_{j_3}\}\ , \tag{1.2}$$

in which the intrinsic angular momentum, i.e., $\frac{1}{2}S_{\mu\nu}S^{\mu\nu}$ $(\mu,\nu=0,1,2,3)$ and $\frac{1}{2}S_{ij}S^{ij}$ $(i,j=1,2,3)$, is diagonal and in which $\frac{1}{2}S_{ij}S^{ij}$ $(i,j=1,2,3)$ has the eigenvalue $j(j+1)$.

The spinor basis has been in extensive use for the Dirac representation—the space of solutions of the Dirac equation—in which $SO(3,1)_{S_{\mu\nu}}$ has the representation[2] $(k_0=\frac{1}{2},c=\frac{3}{2})\oplus(k_0=\frac{1}{2},c=-\frac{3}{2})$, and also for a special type of finite-dimensional representations of $SO(3,1)_{S_{\mu\nu}}$ which remain irreducible under $SO(3)_{S_{ij}}$; these are the representations $(k_0,c=\pm(k_0+1))$ where k_0 is an integer or half-integer and the only value that j can take in these representations is $j=k_0$.[3] In the cases $(k_0,c=\pm(k_0+1))$, the value of s is the same as the value of j and is equal to k_0. The transformation between the canonical basis system (1) and the spinor basis system (2) is rather simple[4] in the representations $(k_0,c=\pm(k_0+1))$, but it is a little bit more complicated[5] in the Dirac case, which involves two $SO(3,1)_{S_{\mu\nu}}$ irreducible representations (irreps) with the same $j=\frac{1}{2}$. In order to go into the canonical basis in the Dirac case, one must first make a transformation from the eigenvectors of $\frac{1}{2}S_{\mu\nu}S^{\mu\nu}$ into eigenvectors of Γ_0, followed by the same transformation that transforms (2) into (1) in the cases $(k_0,c=\pm(k_0+1))$. In the Dirac case (where one more quantum number n, with the two eigenvalues of Γ_0, is needed) this transformation, i.e., the product of the two transformations described above, is known as the Foldy-Wouthuysen transformation.

In the unitary (Hermitian $S_{\mu\nu}$) and therefore infinite-dimensional representations of $SO(3,1)_{S_{\mu\nu}}$, the spectrum of j is infinite $j=k_0,k_0+1,k_0+2,\ldots,$ and one value of j (of the intrinsic angular momentum) no longer belongs to a single value of s (of the spin). Nevertheless, a transformation between the canonical and spinor bases can be found in complete analogy to the familiar cases, but now the transformation matrix is infinite dimensional and a canonical basis vector with a particular value of s (spin) is given as an infinite superposition of spinor basis vectors with different values of j (intrinsic angular momentum).

As discussed in the Appendix, the Majorana representations contain only one unitary irreducible representation of $SO(3,1)_{S_{\mu\nu}}$ for which ($k_0=0$, $c=\frac{1}{2}$) or ($k_0=\frac{1}{2}$, $c=0$), and in them the transformation between (1) and (2) is similar to the cases ($k_0,c=\pm(k_0+1)$).

In the Majorana cases, as in the four-dimensional Dirac case,[5] we also have an $SO(3,1)_{S_{\mu\nu}}$ vector operator Γ_μ which together with $S_{\mu\nu}$ forms an $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$.[6]

II. SPINOR BASIS

A semidirect product $\mathscr{P} \vdash S$ of the Poincaré group $\mathscr{P}$ with a semisimple group S which contains SO(3,1) as a subgroup [in our case $S=SO(3,2)$] has been called a relativistic symmetry.[7] The relativistic symmetry that we consider has the defining relations

$$[J_{\mu\nu},J_{\rho\sigma}]=-i(g_{\mu\rho}J_{\nu\sigma}+g_{\nu\sigma}J_{\mu\rho}-g_{\mu\sigma}J_{\nu\rho}-g_{\nu\rho}J_{\mu\sigma}) , \quad \text{(I.2.1a)}$$

$$[J_{\mu\nu},P_\rho]=-i(g_{\mu\rho}P_\nu-g_{\nu\rho}P_\mu) , \quad \text{(I.2.1b)}$$

$$[P_\mu,P_\nu]=0 , \quad \text{(I.2.1c)}$$

$$[S_{\mu\nu},S_{\rho\sigma}]=-i(g_{\mu\rho}S_{\nu\sigma}+g_{\nu\sigma}S_{\mu\rho}-g_{\mu\sigma}S_{\nu\rho}-g_{\nu\rho}S_{\mu\sigma}) , \quad \text{(I.2.2)}$$

$$[S_{\mu\nu},\Gamma_\rho]=-i(g_{\mu\rho}\Gamma_\nu-g_{\nu\rho}\Gamma_\mu) , \quad \text{(I.2.39)}$$

$$[\Gamma_\mu,\Gamma_\nu]=-iS_{\mu\nu} , \quad \text{(I.2.40)}$$

$$[J_{\mu\nu},S_{\rho\sigma}]=-i(g_{\mu\rho}S_{\nu\sigma}+g_{\nu\sigma}S_{\mu\rho}-g_{\mu\sigma}S_{\nu\rho}-S_{\nu\rho}S_{\mu\sigma}) , \quad (2.1)$$

$$[J_{\mu\nu},\Gamma_\rho]=-i(g_{\mu\rho}\Gamma_\nu-g_{\nu\rho}\Gamma_\mu) , \quad (2.2)$$

and

$$[\hat{P}_\mu,SO(3,2)_{S_{\mu\nu},\Gamma_\mu}]=0 , \quad (2.3)$$

For the purpose of finding the representations, we need not require that $J_{\mu\nu}$ have the particular form (I.2.3a) in terms of Q_μ and P_ν, but we need only require that

$$J_{\mu\nu}=M_{\mu\nu}+S_{\mu\nu} , \quad \text{(I.2.3a)}$$

where $M_{\mu\nu}$ fulfills

$$[M_{\mu\nu},S_{\rho\sigma}]=0,\ \ \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}=0 . \quad \text{(I.2.3b)}$$

After the constraint relation (I.2.37) has been imposed, (1.5) may no longer be fulfilled, whereas (2.1) and (2.2) will always be fulfilled in our case where $J_{\mu\nu}$ is given by (I.2.3). We will, however, postulate that instead of (2.3) we have in addition to the above relations (I.2.1), . . . , (2.2), the following relation[8] (Werle relation):

$$[\hat{P}_\mu,SO(3,2)_{S_{\mu\nu},\Gamma_\mu}]=0 ,$$
$$\hat{P}_\mu=P_\mu M^{-1}=\dot{Y}_\mu . \quad (2.4)$$

Although the constraint relation breaks the relativistic symmetry involving the momentum operator P_μ, the group structure that remains if (2.4) is fulfilled is the relativistic symmetry

$$\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}} \vdash SO(3,2)_{S_{\mu\nu},\Gamma_\mu} , \quad (2.5)$$

where $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}$ is the Poincaré group with the center-of-mass velocity operator $\hat{P}_\mu$ as a generator. The mathematical problem of finding the representations does not, of course, depend on the interpretation of the Poincaré groups so that the choice of $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}$ or $\mathscr{P}_{P_\mu,J_{\mu\nu}}$ is irrelevant to the representation theoretical problems of (2.5).

To find the irreducible representations of the QRR which fulfill relation (2.4), we will first derive the representations of $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}} \vdash SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$, and then we will define the physical momentum by $P_\mu=\hat{P}_\mu M$, where M is an operator obtained from the constraint relation.

Since

$$\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}} \vdash SO(3,2)_{S_{\mu\nu},\Gamma_\mu} = \mathscr{P}_{\hat{P}_\mu,M_{\mu\nu}} \otimes SO(3,2)_{S_{\mu\nu},\Gamma_\mu} , \quad (2.6)$$

we find the representations by taking the direct product of the representation spaces $\mathscr{H}^x$ of the "external" Poincaré group $\mathscr{P}_{\hat{P}_\mu,M_{\mu\nu}}$ and $\mathscr{H}^I$ of the "intrinsic" $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$. For $\mathscr{H}^I$ we restrict ourselves to one of the four irreducible Majorana representations of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ derived in the Appendix. Then a basis system in $\mathscr{H}^I$ is given by the vectors $|{}^{j}_{j_3}\}$ which have the property

$$U^I(\Lambda)|{}^{j}_{j_3}\}=\sum_{\substack{-j'\le j_3'\le j'\\ j'=k_0,k_0+1,\dots}} |{}^{j'}_{j_3'}\}D^{j'\,j}_{j_3'\,j_3}(\Lambda) \quad (2.7)$$

for $\Lambda\in SO(3,1)$.

In the irreducible representation space $\mathscr{H}^x$ of $\mathscr{P}_{\hat{P}_\mu,M_{\mu\nu}}$ we have $\hat{P}_\mu\hat{P}^\mu=1$ and $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu M^{\rho\sigma}=0$, thus $\mathscr{H}^x$ is equivalent to the mass$=1$ and spin$=0$ irrep space of the Poincaré group and we can use as a basis system in $\mathscr{H}^x$ the generalized eigenvectors

$$|\hat{p}\rangle \quad (2.8)$$

of the complete system of commuting operators[9]

$$\hat{P}_\mu\ \ (\mu=0,1,2,3),\ \ \hat{P}_\mu\hat{P}^\mu=1 . \quad (2.9)$$

These vectors have the properties

$$\hat{P}_\mu|\hat{p}\rangle=\hat{p}_\mu|\hat{p}\rangle , \quad \text{(2.10a)}$$

$$U^x(\Lambda)|\hat{p}\rangle=|\Lambda\hat{p}\rangle , \quad \text{(2.10b)}$$

where $U^x(\Lambda(\omega))=e^{i\omega^{\mu\nu}M_{\mu\nu}}$, and $\Lambda\in SO(3,1)$. We decide to "normalize" them in the following Lorentz-invariant way:

$$\langle\hat{p}'|\hat{p}\rangle=2|\hat{p}_0|\,\delta^3(\hat{p}-\hat{p}') . \quad (2.11)$$

There are two possibilities for $\mathscr{H}^x$ corresponding to the two possible signs of $\hat{p}_0=\pm(1+\vec{\hat{p}}^{\,2})^{1/2}$.

We obtain a basis system in the direct-product space $\mathscr{H}^x\otimes\mathscr{H}^I$ by defining the direct-product basis vectors

$$|\hat{p},{}^{j}_{j_3}\}=|\hat{p}\rangle\otimes|{}^{j}_{j_3}\} . \quad (2.12)$$

In this basis, the physical Lorentz group is represented in the direct-product form

$$U(\Lambda(\omega))=e^{i\omega^{\mu\nu}J_{\mu\nu}}=e^{i\omega^{\mu\nu}M_{\mu\nu}}\otimes e^{i\omega^{\mu\nu}S_{\mu\nu}}\ , \tag{2.13}$$

and the generators are given by

$$J_{\mu\nu}=M_{\mu\nu}\otimes 1+1\otimes S_{\mu\nu}\ . \tag{2.14}$$

Thus the basis (2.12) is adapted to the splitting (I.2.3a). Using the properties (2.7) and (2.10b), we have

$$U(\Lambda)\,|\hat{p},{}^{j}_{j_3}\}=\sum_{\substack{-j'\le j_3'\le j'\\ j'=k_0,k_0+1,\ldots}}|\Lambda\hat{p},{}^{j'}_{j_3'}\}D^{j'}_{j_3'j_3}(\Lambda)\ . \tag{2.15}$$

The basis vectors (2.12) are also (generalized) eigenvectors of $\hat{P}_\mu$ due to (2.10a):

$$\hat{P}_\mu\,|\hat{p},{}^{j}_{j_3}\}=\hat{p}_\mu\,|\hat{p},{}^{j}_{j_3}\}\ . \tag{2.16}$$

The "normalization" of (2.12) follows from (2.11) and (A28) and is given by

$$\{{}^{j'}_{j_3'}p'\,|p,{}^{j}_{j_3}\}=2\,|\hat{p}_0\,|\,\delta^3(\hat{p}-\hat{p}\,')\delta^{j'j}\delta_{j_3'j_3}\ . \tag{2.17}$$

The space $\mathscr{H}^x\otimes\mathscr{H}^I\equiv\mathscr{H}(\text{Maj})$ is thus an irreducible representation space of $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}\vdash \text{SO}(3,2)_{S_{\mu\nu},\Gamma_\mu}$ and, therewith, is a reducible representation space of the c.m.-velocity Poincaré group $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}$. It is characterized by the irreducible representation of $\text{SO}(3,2)_{S_{\mu\nu},\Gamma_\mu}$ (because $\mathscr{H}^x$ is totally specified by $\hat{P}_\mu\hat{P}^\mu=1$ and $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{P}^{\nu}M^{\rho\sigma}=0$, except for sgn $\hat{p}_0=+1$ or -1) which is indicated by Maj.

The basis which has the property that the representative $U(\Lambda)$ of the Lorentz transformation Λ acts on it as a direct product (2.13) of one factor $U^x(\Lambda)=e^{i\omega^{\mu\nu}M_{\mu\nu}}$ acting only on the momenta as in (2.10b) and another factor $U^I(\Lambda)=e^{i\omega^{\mu\nu}S_{\mu\nu}}$ acting only on the discrete indices as in (2.7) is well known[9] for the representation $(k_0,c=\pm(k_0+1))$ and is called the spinor basis. Thus $|\hat{p},{}^{j}_{j_3}\}$ is the spinor basis for the infinite-dimensional case $(k_0=0,c=\frac{1}{2})$ (integer-angular-momentum Majorana representation) or $(k_0=\frac{1}{2},c=0)$ (half-integer—angular—momentum Majorana representation).

In addition to the Poincaré group transformations (2.15) and (2.16), we also have in the space $\mathscr{H}(\text{Maj})$ a representation of the vector operator Γ_μ; its action on the spinor basis $|\hat{p},{}^{j}_{j_3}\}$ follows from (A32) and is, e.g., given for Γ_3 by

$$\Gamma_3\,|\hat{p},{}^{j}_{j_3}\}=i\,\text{sgn}n[((j-j_3)(j+j_3))^{1/2}C_j\,|\hat{p},{}^{j-1}_{j_3}\}+((j-j_3+1)(j+j_3+1))^{1/2}C_{j+1}\,|\hat{p},{}^{j+1}_{j_3}\}]\ , \tag{2.18}$$

where C_j is given by (A31). From (A11) and (A17) it also follows that the spinor basis vectors are eigenvectors of $\vec{S}^2$, S_{12}, and $\frac{1}{2}S_{\mu\nu}S^{\mu\nu}$:

$$\vec{S}^2\,|\hat{p},{}^{j}_{j_3}\}=j(j+1)\,|\hat{p},{}^{j}_{j_3}\}\ , \tag{2.19a}$$

$$S_{12}\,|\hat{p},{}^{j}_{j_3}\}=j_3\,|\hat{p},{}^{j}_{j_3}\}\ , \tag{2.19b}$$

$$\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}\,|\hat{p},{}^{j}_{j_3}\}=-\tfrac{3}{4}\,|\hat{p},{}^{j}_{j_3}\}\ . \tag{2.19c}$$

Since $S_{\mu\nu}$ is the intrinsic angular momentum and not the spin (which is given by $\Sigma_{\mu\nu}$ or $\hat{w}_\mu$), j is not identical to the spin quantum number [except in the special case $(k_0,c=\pm(k_0+1))$, discussed in the literature]. The spin operator $\hat{W}=\frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}$ [cf. (I.2.49)] is in general not diagonal in the spinor basis and, as we shall discuss below, neither is the mass operator M if it is determined from $\hat{W}$ by the constraint relation (I.2.37).

III. TRANSFORMATION TO THE CANONICAL BASIS

As a representation space of the c.m.-velocity Poincaré group $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}$, $\mathscr{H}(\text{Maj})$ is reducible. In order to obtain the reduction of $\mathscr{H}(\text{Maj})$ into irreducible representations of $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}$ we make a basis transformation from the spinor basis into the canonical basis which consists of spin eigenvectors. [This will in general not yet guarantee that M^2 is diagonal, but we will see below that for our particular case of the Majorana representation and the constraint relation (I.2.37) this will indeed be the case.]

In exact analogy to the case $(k_0=j,c=\pm(j+1))$,[11] we define the canonical basis

$$|\hat{p},s,s_3\rangle=\sum_{\substack{-j\le j_3\le j\\ j=k_0,k_0+1,\ldots}}|\hat{p},{}^{j}_{j_3}\}D^{(k_0,c)js}{}_{j_3s_3}(L^{-1}(\hat{p}))\phi(s)\ , \tag{3.1}$$

where $L^{-1}(\hat{p})$ is the boost, i.e., the rotation-free Lorentz transformation with the property

$$L(\hat{p})\hat{p}=(\epsilon,0,0,0)\equiv\hat{p}_{\text{rest}}\ , \tag{3.2}$$

$\epsilon=\text{sgn}\hat{p}_0$, and $D^{(k_0,c)}$ is its matrix representation as in (2.7). $\phi(s)$ is a phase factor which can be chosen arbitrarily and which we fix as $\phi(s)=(-i)^s$. (Then the Naimark[2] phase is chosen for the matrices of Γ_i in the $|\hat{p}_{\text{rest}},s,s_3\rangle$ basis as well as for the matrices of S_{0i} in the $|{}^{j}_{j_3}\}$ basis.) In the finite-dimensional cases considered in the past, where $(k_0=j,c=\pm(j+1))$, the phase factor is trivial and is therefore omitted. Also, (3.1) is usually used for the generalized momentum eigenvectors and not for the eigenvectors of $\hat{P}_\mu$. But as the boost $L^{-1}(p)$ is a function of $\hat{p}=p/m$ only,[12] the transformation matrices for the momentum eigenvectors and for our $\hat{p}$ eigenvectors are equal $D^{(k_0,c)}(L^{-1}(p))=D^{(k_0,c)}(L^{-1}(\hat{p}))$. It is this circumstance that allows us to determine the representations for the case in which (2.4), instead of (2.3), is fulfilled in such close analogy to the conventional case in which (2.3) is fulfilled. This also means that, in addition to the theoretical and phenomenological[13] reasons, there is a very practical reason for using the Werle relation (2.4): without the Werle relation, while (2.3) does not hold, the whole structure would become virtually unmanageable with the presently available representation theoretical tools.

To show that (3.1) is indeed the canonical basis we calculate

$$U(\Lambda)|\hat{p},s,s_3\rangle = \sum_{\substack{-j\le j_3\le j\\ j=k_0,k_0+1,\ldots}} U(\Lambda)|\hat{p},{}^{j}_{j_3}\}D^{js}_{j_3s_3}(L^{-1}(\hat{p}))\phi(s)$$

$$= \sum_{\substack{-j\le j_3\le j\\ j=k_0,k_0+1,\ldots}} \sum_{\substack{-j'\le j'_3\le j'\\ j'=k_0,k_0+1,\ldots}} |\Lambda\hat{p},{}^{j'}_{j'_3}\}D^{j'j}_{j'_3j_3}(\Lambda)D^{js}_{j_3s_3}(L^{-1}(\hat{p}))\phi(s) . \quad (3.3)$$

Using the Wigner rotation

$$\mathscr{R}(\Lambda,\hat{p})=L(\Lambda\hat{p})\Lambda L^{-1}(\hat{p}) , \quad (3.4)$$

and the fact that the $D^{(k_0,c)}$ are representation matrices, one obtains

$$U(\Lambda)|\hat{p},s,s_3\rangle = \sum_{\substack{-j\le j_3\le j\\ j=k_0,k_0+1,\ldots}} \left[\sum_{\substack{-j'\le j'_3\le j'\\ j'=k_0,k_0+1,\ldots}} |\Lambda\hat{p},{}^{j'}_{j'_3}\}D^{j'j}_{j'_3j_3}(L^{-1}(\Lambda\hat{p})) \right] D^{js}_{j_3s_3}(\mathscr{R})\phi(s) . \quad (3.5)$$

For the term in brackets we use (3.1) with $\hat{p}$ replaced by $\Lambda\hat{p}$ and obtain

$$U(\Lambda)|\hat{p},s,s_3\rangle = \sum_{\substack{-j\le j_3\le j\\ j=k_0,k_0+1,\ldots}} |\Lambda\hat{p},j,j_3\rangle D^{(k_0,c)js}{}_{j_3s_3}(\mathscr{R})\phi^{-1}(j)\phi(s) . \quad (3.6)$$

$D^{(k_0,c)js}{}_{j_3s_3}(\mathscr{R}(\Lambda,\hat{p}))$ is the matrix representation of the rotation $\mathscr{R}$ in the representation (k_0,c) of $SO(3,1)_{S_{\mu\nu}}$. But the representation (k_0,c) reduces with respect to the rotation subgroup into the direct sum

$$D^{(k_0,c)}(\mathscr{R})=\mathscr{D}^{(k_0)}(\mathscr{R})\oplus\mathscr{D}^{(k_0+1)}(\mathscr{R})\oplus\cdots , \quad (3.7)$$

where $\mathscr{D}^{(j)}_{j_3s_3}(\mathscr{R})$ are the representation matrices of the irreducible representation (j) of the rotation group. Thus the $SO(3,1)_{S_{\mu\nu}}$ representation matrix of the rotation $\mathscr{R}$ is given by

$$D^{(k_0,c)js}{}_{j_3s_3}(\mathscr{R})=\delta^{js}\mathscr{D}^{(j)}_{j_3s_3}(\mathscr{R}) \quad (3.8)$$

for every value of s that occurs in the representation (k_0,c), otherwise it is zero. Thus for every value

$$s=k_0,k_0+1,k_0+2,\ldots , \quad (3.9)$$

(3.6) takes the form

$$U(\Lambda)|\hat{p},s,s_3\rangle = \sum_{-s\le j_3\le s} |\Lambda\hat{p},s,j_3\rangle\mathscr{D}^{(s)}_{j_3s_3}(\mathscr{R}(\Lambda(\hat{p}))) . \quad (3.10)$$

This is exactly the transformation property of the canonical basis vectors of the $(\hat{p}^2=1,s)$ irreps of the Poincaré group. Thus s is the value of the spin

$$\hat{W}|\hat{p},s,s_3\rangle=\tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}|\hat{p},s,s_3\rangle$$
$$=s(s+1)|\hat{p},s,s_3\rangle , \quad (3.11)$$

and we have shown that the representation space $\mathscr{H}(\mathrm{Maj})$ reduces[14] into the following direct sum of irreps of $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}$:

$$\mathscr{H}(\mathrm{Maj}) \underset{\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}}{\rightleftarrows} \sum_{s=k_0,k_0+1,\ldots} \oplus\mathscr{H}(1,s) . \quad (3.12)$$

The canonical basis vectors defined by (3.1) are clearly generalized eigenvectors of the operators $\hat{P}_\mu$ since the $|\hat{p},{}^{j}_{j_3}\}$ are so according to (2.16). The normalization of the canonical basis vectors follows from the normalization (2.17) of the spinor basis vectors and a straightforward calculation gives

$$\langle s'_3,s',\hat{p}'|\hat{p},s,s_3\rangle=2|\hat{p}_0|\delta^3(\hat{p}-\hat{p}')\delta^{s's}\delta_{s'_3s_3} . \quad (3.13)$$

Also, as $L^{-1}(\hat{p}_{\mathrm{rest}})=1$ [by the definition (3.2) of the boost] it follows from (3.1) that in the rest frame

$$|\hat{p}_{\mathrm{rest}},s,s_3\rangle=|\hat{p}_{\mathrm{rest}},{}^{j=s}_{j_3=s_3}\}\phi(s) . \quad (3.14)$$

Thus $j=s$ for the rest states. This is the quantum number form of the statement made in Sec. II of paper I that in the rest frame the intrinsic angular momentum (which has the quantum number j) is equal to the spin (which has the quantum number s).

The canonical basis vectors are eigenvectors of $\hat{P}_\mu\Gamma^\mu$ with the eigenvalue $+(s+\frac{1}{2})$ or $-(s+\frac{1}{2})$. To show this we use the fact that since $\hat{P}_\mu$ and Γ_μ are vector operators under $SO(3,1)_{J_{\mu\nu}}$, $\hat{P}_\mu\Gamma^\mu=\Gamma_\mu\hat{P}^\mu$ is a Lorentz invariant:

$$\Gamma_\mu\hat{P}^\mu U(\Lambda)=U(\Lambda)\Gamma_\mu\hat{P}^\mu . \quad (3.15)$$

Then with (3.10), (3.14), (A30), and $\Lambda=L^{-1}(\hat{p})$ for the particular Lorentz transformation in (3.15), we calculate

$$\begin{aligned}\hat{P}_\mu\Gamma^\mu\,|\hat{p},s,s_3\rangle &= \Gamma_\mu\hat{P}^\mu\,|\hat{p},s,s_3\rangle = \Gamma_\mu\hat{P}^\mu U(L^{-1}(\hat{p}))\,|\hat{p}_{\rm rest},s,s_3\rangle \\ &= U(L^{-1}(\hat{p}))\Gamma_\mu\hat{P}^\mu\,|\hat{p}_{\rm rest},s,s_3\rangle = U(L^{-1}(\hat{p}))\Gamma_0\epsilon\,|\hat{p}_{\rm rest},s,s_3\rangle \\ &= U(L^{-1}(\hat{p}))\Gamma_0\epsilon\,|\hat{p}_{\rm rest},{}^{s}_{s_3}\}\phi(s) \\ &= U(L^{-1}(\hat{p}))\,{\rm sgn}\,n(s+\tfrac{1}{2})\epsilon\,|\hat{p}_{\rm rest},{}^{s}_{s_3}\}\phi(s) \\ &= {\rm sgn}\,n(s+\tfrac{1}{2})\epsilon U(L^{-1}(\hat{p}))\,|\hat{p}_{\rm rest},s,s_3\rangle\ . \end{aligned} \tag{3.16}$$

Thus

$$\hat{P}_\mu\Gamma^\mu\,|\hat{p},s,s_3\rangle = \epsilon{\rm sgn}\,n(s+\tfrac{1}{2})\,|\hat{p},s,s_3\rangle\ . \tag{3.17}$$

$\epsilon={\rm sgn}\hat{p}_0=\pm 1$ depends on whether we choose the representation of $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}$ with $\{{}^{\rm positive}_{\rm negative}\}$ energy, and ${\rm sgn}n=\pm 1$ depends on whether we choose a Majorana representation with $\{{}^{{\rm spectrum}\Gamma_0>0}_{{\rm spectrum}\Gamma_0<0}\}$. Thus

$${\rm sgn}({\rm spectrum}\ \hat{P}_\mu\Gamma^\mu)={\rm sgn}({\rm spectrum}\ \hat{P}_0){\rm sgn}({\rm spectrum}\ \Gamma_0)=\pm 1\ .$$

From this we see that even if we restrict ourselves to $\epsilon=+1$, as is usually done following Wigner, we can still obtain representations with ${\rm sgn}({\rm spectrum}\ \hat{P}_\mu\Gamma^\mu)=+1$ or -1, and the same statement holds for $\epsilon=-1$.

In contrast to the spinor basis vectors, the canonical basis vectors are not eigenvectors of Γ_0 and the matrix elements of Γ_i, which are simple for the spinor basis, e.g., (2.18), are complicated for the canonical basis except for those at rest, for which—in the Majorana case—one has (3.14). From Eqs. (2.18) and (3.14) with $\phi(s)=(-i)^s$, one obtains for the action of Γ_3, and similarly for the other Γ_i,

$$\Gamma_3\,|\hat{P}_{\rm rest},s,s_3\rangle={\rm sgn}\,n\{[(s-s_3)(s+s_3)]^{1/2}C_s\,|\hat{p}_{\rm rest},s-1,s_3\rangle-[(s+s_3+1)(s-s_3+1)]^{1/2}C_{s+1}\,|\hat{p}_{\rm rest},s+1,s_3\rangle\}\ , \tag{3.18}$$

where C_s is given by (A31). Thus the spinor basis is much easier to use for practical calculations, which is the reason it has found wider use in the four-dimensional Dirac case[15] in spite of the fact that the canonical basis is better suited for hadron states; hadron states are (wave packets formed with) the canonical basis vectors.

We have obtained above a representation of the c.m.-velocity Poincaré group $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}$ which contains the spins $s=k_0,k_0+1,\ k_0+2,\ldots$, etc., and of the vector operator Γ_μ, which transforms between different values s of the spin (and between different values j of the intrinsic angular momentum). Whereas the group $\mathscr{P}_{\hat{P}_\mu,J_{\mu\nu}}$ was not block diagonal in the spinor basis, it is block diagonal in the canonical basis [due to (3.12)]. Further, the canonical basis vectors are also eigenvectors of the other constant of motion for the quantum relativistic rotator, $\hat{P}_\mu\Gamma^\mu$. For the Majorana representations this is obvious because then (I.2.50) holds, but the proof (3.16) also carries through in the more general case whenever one chooses the canonical basis vectors at rest to be eigenvectors of Γ_0.

IV. THE MASS AND SPIN SPECTRUM

In order to obtain a representation of the physical Poincaré group $\mathscr{P}_{P_\mu,J_{\mu\nu}}$ we have to define the momentum operators P_μ, which have the dimension GeV or cm^{-1}, because physical translations are measured in cm. We can therefore define for any positive real number m with dimension cm^{-1} an operator

$$P_\mu=\hat{P}_\mu m \tag{4.1}$$

in the space $\mathscr{H}$(Maj). In this way $\mathscr{H}$(Maj) becomes a representation space of the Poincaré group $\mathscr{P}_{P_\mu,J_{\mu\nu}}$, which is then characterized by Maj (specifying the spin spectrum) and in addition by the parameter m. The translation by a vector a^μ, the length of which is measured in cm, is described by

$$U(a)=e^{ia^\mu m\hat{P}_\mu}\ . \tag{4.2}$$

With this we obtain in $U(a)$ and $U(\Lambda)$ a representation of the physical Poincaré group for each value of the parameter m. The spectrum of s is given for these representations by (3.9). The most general Majorana representation would then be obtained by taking the continuous direct sum over all m^2 with $0\le m^2<\infty$. Instead of choosing one number m to get a representation of the physical translation group, we can also choose an operator M with dimension cm^{-1} to fix the scale for the displacement. The momentum operator is then defined by

$$P_\mu=\hat{P}_\mu M\ , \tag{4.3}$$

and the operator $U(a)$, representing displacement by a distance a^μ in cm, can be written as

$$U(a)=e^{ia^\mu P_\mu}\ . \tag{4.4}$$

$U(a)$ acts on the basis vectors $|\hat{p},{}^{j}_{j_3}\}$ according to

$$U(a)\,|\hat{p},{}^{j}_{j_3}\}=e^{ia^\mu M\hat{p}_\mu}\,|\hat{p},{}^{j}_{j_3}\}\ . \tag{4.5}$$

As the spinor basis vector $|\hat{p},{}^{j}_{j_3}\}$ is in general not an eigenvector of M, its transformation property under translation is very complicated and depends upon the constraint relation which defines M in terms of other operators. For the elementary particle constraint (I.2.36), i.e., if M is a multiple m of the unit operator, $|\hat{p}^{j}_{,j_3}\}$ is a generalized eigenvector of P_μ.

If our constraint relation

$$\Phi = P_\mu P^\mu + \tfrac{5}{2}\lambda^2 - \lambda^2(\hat{P}_\rho \Gamma^\rho)^2 - \lambda^2\alpha^2 = 0 \tag{4.6}$$

[cf. (I.2.56)] is imposed with $\alpha^2 = \text{const}$ [$\geq \frac{9}{4}$ as required by the principal series representation of $SO(4,1)_{\hat{B}_\mu, J_{\mu\nu}}$], then P_μ defined by (4.3) is diagonal in the basis in which $\hat{P}_\mu \Gamma^\mu$ is diagonal because the constraint (4.6) relates the operator $P_\mu P^\mu = M^2$ to the operator $\hat{P}_\rho \Gamma^\rho$. This basis is the canonical basis (3.1) and not the spinor basis (2.12). $\hat{P}_\rho \Gamma^\rho$ has the following nontrivial spectrum in $\mathcal{H}(\text{Maj})$ [cf. (3.17)]:

$$\text{spect}(\hat{P}_\mu \Gamma^\mu)^2 = (s + \tfrac{1}{2})^2 ,$$

$$s = \begin{cases} 0,1,2,3,\ldots \text{ for integer Maj reps} \\ \text{or} \\ \frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots \text{ for half-integer Maj reps .} \end{cases} \tag{4.7}$$

Thus the spectrum of the operator M^2 for the representation space of the physical Poincaré group, obtained in the space $\mathcal{H}(\text{Maj})$ from the constraint relation, is by (4.6)

$$\text{spect } M^2 \equiv m^2(s) = \lambda^2\alpha^2 - \lambda^2\tfrac{9}{4} + \lambda^2 s(s+1) . \tag{4.8}$$

The representative of the physical translation group and its generator, the momentum operator, act on the canonical basis vectors in the following way:

$$U(a)\,|\hat{p},s,s_3\rangle = e^{ia^\mu M \hat{P}_\mu}\,|\hat{p},s,s_3\rangle$$
$$= e^{ia^\mu \hat{P}_\mu m(s)}\,|\hat{p},s,s_3\rangle \tag{4.9}$$

and

$$P_\mu\,|\hat{p},s,s_3\rangle = \hat{P}_\mu m(s)\,|\hat{p},s,s_3\rangle . \tag{4.10}$$

Thus for every value of s in $\mathcal{H}(\text{Maj})$, which means on every subspace $\mathcal{H}(1,s)$ contained in (3.12), we can define a momentum operator $P_\mu^{(s)}$ by

$$P_\mu^{(s)} = \hat{P}_\mu m(s) , \tag{4.11}$$

and with this operator we can define an irreducible representation of the physical Poincaré group by $U^{(s)}(a) = e^{ia^\mu P_\mu^{(s)}}$ and by $U^{(s)}(\Lambda)$, which is the restriction of (3.10) to $\mathcal{H}(1,s)$. Therewith, we obtain in $\mathcal{H}(1,s)$ the irreducible representation $(m(s),s)$ of the physical Poincaré group $\mathcal{P}_{P_\mu, J_{\mu\nu}}$, and in $\mathcal{H}(\text{Maj})$ we obtain the representation

$$\sum_{s=k_0,k_0+1,\ldots} \oplus (m(s),s) \tag{4.12}$$

of $\mathcal{P}_{P_\mu, J_{\mu\nu}}$.

The above construction can be done for any given value of $\alpha^2(\alpha^2 \geq \frac{9}{4})$ in the constraint relation, so that, as a representation space of $\mathcal{P}_{P_\mu, J_{\mu\nu}}$, the space $\mathcal{H}(\text{Maj})$ should also be labeled by the value of the parameter α: $\mathcal{H}^\alpha(\text{Maj})$. This value of α will characterize the physical system that is described by $\mathcal{H}^\alpha(\text{Maj})$ in the same way as the value of m characterizes the elementary particle. (In fact, see paper III, in the elementary limit $\lambda^2\alpha^2$ goes into m^2.) As a representation space of $\mathcal{P}_{P_\mu, J_{\mu\nu}}$, the space $\mathcal{H}^\alpha(\text{Maj})$ reduces into a discrete direct sum of irreducible representations

$$\mathcal{H}^\alpha(\text{Maj}) \underset{\mathcal{P}_{P_\mu, J_{\mu\nu}}}{\Rightarrow} \sum_{s=k_0,k_0+1,\ldots} \oplus \mathcal{H}^\alpha(m(s),s) , \tag{4.13}$$

which follows immediately from (3.12).

The above representation of $\mathcal{P}_{P_\mu, J_{\mu\nu}}$ can be extended to include parity P. Due to the hypothesis that Γ_μ is a proper vector operator,

$$P\Gamma_\mu P = -\Gamma_\mu , \tag{4.14}$$

and the property (3.18), it follows that in each $\mathcal{H}(m(s),s)$ the parity is represented by

$$P = \eta(-1)^s , \tag{4.15}$$

where either $\eta = +1$ (normal) for the whole space $\mathcal{H}^\alpha(\text{Maj})$ or $\eta = -1$ (abnormal) for the whole space $\mathcal{H}^\alpha(\text{Maj})$.

The most general Majorana representation, previously obtained as the continuous direct sum over all $m^2(0 \leq m^2 < \infty)$ of (4.2), is now obtained as a continuous direct sum of the $\mathcal{H}^\alpha(\text{Maj})$ with α^2 extending over the values $\frac{9}{4} \leq \alpha^2 < \infty$ of the principal series representations of $SO(4,1)_{\hat{B}_\mu, J_{\mu\nu}}$. The irreducible representation (4.13) can then be understood as the subspace in which the constraint relation (4.6) is fulfilled for a given fixed value of α^2. We have called this subspace the physical subspace in paper I; it describes the QRR characterized by the value α^2.

We have, therewith, constructed a representation space for a physical system for which the hadrons with spin-parity s^P and mass $m(s)$ are just different substates. We have used the particular de Sitter constraint relation (4.6) with the Majorana condition (I.2.50) for the QRR, but it is clear that the construction is not restricted to this particular case. For a different constraint relation one can in an analogous way construct a representation space which is a discrete direct sum of irreducible representations of the Poincaré group $\mathcal{P}_{P_\mu, J_{\mu\nu}}$, but the mass spectrum will then be different depending upon the particular constraint relation.

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APPENDIX: IRREDUCIBLE MAJORANA REPRESENTATIONS OF $SO(3,2)_{S_{\mu\nu}, \Gamma_\mu}$

In this appendix we will find all of the irreducible representations of $SO(3,2)_{S_{\mu\nu}, \Gamma_\mu}$ which satisfy, in addition to the commutation relations, the relation

$$\{\Gamma_\mu,\Gamma^\rho\}+\{S_{\mu\nu},S^{\rho\nu}\}=-\delta_\mu{}^\rho \ . \tag{A1}$$

These representations we will call the Majorana representations, or, following Dirac, the remarkable representations. They are isomorphic to the representations of SO(3,2) for which the generators are realized by a degenerate pair of boson operators.[16] They can be derived without making use of this realization in terms of creation and annihilation operators if one makes use of Eq. (A1) and reduces them into representations of $SO(3,1)_{S_{\mu\nu}}$. We will present this derivation here assuming knowledge of the properties of the representations of SO(3,1).[2]

Our analysis will rely on the following subgroup reduction sequence of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$:

$$SO(3,2)_{S_{\mu\nu},\Gamma_\mu}\supset SO(3,1)_{S_{\mu\nu}}\supset SO(3)_{S_{ij}} \ , \tag{A2}$$

where $SO(3,1)_{S_{\mu\nu}}$ is one of the two mathematically equivalent but physically different Lorentz subgroups of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ [the other one is $SO(3,1)_{S_{ij},\Gamma_i}$ $(i,j=1,2,3)$], and $SO(3)_{S_{ij}}$ is the rotation group. The linear irreducible representations of $SO(3,1)_{S_{\mu\nu}}$ are characterized by (k_0,c), where k_0 is an integer or half-integer and c is a complex number. [Note that the representations $(k_0,c),(-k_0,-c)$ are equivalent.] The values of the Casimir operators of $SO(3,1)_{S_{\mu\nu}}$ in an irreducible representation (k_0,c) are given by

$$C_1\equiv\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}=k_0{}^2+c^2-1 \tag{A3}$$

and

$$C_2\equiv\tfrac{1}{8}\epsilon^{\mu\nu\rho\sigma}S_{\mu\nu}S_{\rho\sigma}=ik_0c \ . \tag{A4}$$

The commutation relations (I.2.2), (I.2.39), and (I.2.40) of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ may be written in the alternate form

$$[S_{AB},S_{CD}]=-i(g_{AC}S_{BD}+g_{BD}S_{AC}-g_{AD}S_{BC}-g_{BC}S_{AD}) \ , \tag{A5}$$

where A,B, etc. $=0,1,2,3,5,g_{55}=1$, and $\Gamma_\mu=S_{\mu5}$ $(\mu=0,1,2,3)$. An equivalent form of Eq. (A1) is then

$$\{S_{AB},S^{CB}\}=-\delta_A^C \ . \tag{A6}$$

Setting $C=A$ in Eq. (A6) gives

$$\{S_{AB},S^{AB}\}=-5 \ , \tag{A7}$$

which yields the value

$$D_1\equiv\tfrac{1}{2}S_{AB}S^{AB}=\Gamma_\mu\Gamma^\mu+\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}=-\tfrac{5}{4} \tag{A8}$$

for the second-order Casimir operator D_1 of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$. At the same time, setting $\rho=\mu$ in Eq. (A1) gives

$$\{\Gamma_\mu,\Gamma^\mu\}+\{S_{\mu\nu},S^{\mu\nu}\}=-4 \ , \tag{A9}$$

or

$$\Gamma_\mu\Gamma^\mu+S_{\mu\nu}S^{\mu\nu}=-2 \ . \tag{A10}$$

Equations (A8) and (A10) together yield the value

$$C_1\equiv\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}=S_{0i}S^{0i}+\tfrac{1}{2}S_{ij}S^{ij}=-\tfrac{3}{4} \tag{A11}$$

for the first Casimir operator of $SO(3,1)_{S_{\mu\nu}}$, and the value

$$\Gamma_\mu\Gamma^\mu=-\tfrac{1}{2} \tag{A12}$$

for the Lorentz scalar $\Gamma_\mu\Gamma^\mu$. It also follows from Eq. (A1) that the second Casimir operator of $SO(3,1)_{S_{\mu\nu}}$ is zero:

$$C_2\equiv\tfrac{1}{8}\epsilon^{\mu\nu\rho\sigma}S_{\mu\nu}S_{\rho\sigma}=0 \ . \tag{A13}$$

Because of Eqs. (A11) and (A13) the only two irreducible representations of $SO(3,1)_{S_{\mu\nu}}$ that may occur in the irreducible representation space $\mathscr{H}^I$ of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ are $(k_0=0,c=\frac{1}{2})$ and $(k_0=\frac{1}{2},c=0)$. This can easily be seen by comparing the values of the Casimir operators given in Eqs. (A11) and (A13) for the special Majorana representations with their values given in Eqs. (A3) and (A4) for the general irreducible representations (k_0,c). The properties of the irreducible representation spaces $\mathscr{H}(k_0,c)$ are well known[2]; both $\mathscr{H}(k_0=0,c=\frac{1}{2})$ and $\mathscr{H}(k_0=\frac{1}{2},c=0)$ are unitary, infinite-dimensional irreducible representation spaces and they reduce with respect to $SO(3)_{S_{ij}}$ into direct sums of the irreducible representation spaces $\mathscr{R}^{(j)}$ according to

$$\mathscr{H}(k_0,c)\underset{SO(3)_{S_{ij}}}{\rightrightarrows}\sum_{j=k_0,k_0+1,\ldots}\oplus\mathscr{R}^{(j)} \ . \tag{A14}$$

Since there is no operator among the generators $S_{\mu\nu},\Gamma_\mu$ that transforms like a half-integer representation of the rotation group, the value of j cannot be changed by a half-integer. This means that $\mathscr{H}(k_0=0,c=\frac{1}{2})$ and $\mathscr{H}(k_0=\frac{1}{2},c=0)$ cannot both be contained in the same irreducible representation space of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ (Ref. 17); however, $\mathscr{H}^I$ may still contain either $\mathscr{H}(k_0=0,c=\frac{1}{2})$ or $\mathscr{H}(k_0=\frac{1}{2},c=0)$ with a multiplicity.

Therewith we have shown that as a consequence of Eq. (A1) the irreducible representation space $\mathscr{H}^I$ is given either by

$$\mathscr{H}^I\underset{SO(3,1)_{S_{\mu\nu}}}{\rightrightarrows}\sum_\xi\oplus\mathscr{H}_\xi(k_0=0,c=\tfrac{1}{2}) \tag{A15a}$$

or by

$$\mathscr{H}^I\underset{SO(3,1)_{S_{\mu\nu}}}{\rightrightarrows}\sum_\xi\oplus\mathscr{H}_\xi(k_0=\tfrac{1}{2},c=0) \ , \tag{A15b}$$

where ξ labels the multiplicity. Thus the basis system in $\mathscr{H}^I$ is

$$|{}^{j}_{j_3},\xi\} \text{ with } \begin{array}{l} j=k_0,k_0+1,k_0+2,\ldots \\ j_3=-j,-j+1,\ldots,j \ . \end{array} \tag{A16}$$

These basis vectors are eigenvectors of $\vec{S}^2=\frac{1}{2}S_{ij}S^{ij}$ and $S_3=S_{12}$:

$$\vec{S}^2|{}^{j}_{j_3},\xi\}=j(j+1)|{}^{j}_{j_3},\xi\} \ , \tag{A17a}$$

$$S_3|{}^{j}_{j_3},\xi\}=j_3|{}^{j}_{j_3},\xi\} \ . \tag{A17b}$$

Another consequence of Eq. (A1) is that the matrix elements of Γ_0 are determined up to a sign by the matrix elements of $\vec{S}^2$. To see this we set $\rho=\mu=0$ in Eq. (A1) and obtain

$$2\Gamma_0{}^2+2S_{0i}S^{0i}=-1\ . \tag{A18}$$

Then Eqs. (A18) and (A11) together yield

$$\Gamma_0{}^2=\vec{S}^2+\tfrac{1}{4}\ . \tag{A19}$$

This equation relates the operator Γ_0 to the Casimir operator $\vec{S}^2$ of $SO(3)_{S_{ij}}$ and operating with it on the basis vectors $|{}^{j}_{j_3},\xi\}$ gives

$$\begin{aligned}\Gamma_0{}^2|{}^{j}_{j_3},\xi\} &= [j(j+1)+\tfrac{1}{4}]\,|{}^{j}_{j_3},\xi\}\\ &=(j+\tfrac{1}{2})^2|{}^{j}_{j_3},\xi\}\ .\end{aligned} \tag{A20}$$

The matrix elements of Γ_0 may then be written as

$$\Gamma_0|{}^{j}_{j_3},\xi\}=n\,|{}^{j}_{j_3},\xi'\}$$

and (A21)

$$\Gamma_0|{}^{j}_{j_3},\xi'\}=n\,|{}^{j}_{j_3},\xi\}\ ,$$

where

$$n=+(j+\tfrac{1}{2})\ \text{or}\ n=-(j+\tfrac{1}{2})\ . \tag{A22}$$

We now have to distinguish two cases: (1) if $\xi'=\xi$, then

$$\Gamma_0|{}^{j}_{j_3},\xi\}=\operatorname{sgn} n\,(j+\tfrac{1}{2})\,|{}^{j}_{j_3},\xi\}$$

and (2) if $\xi'\neq\xi$, then we define vectors

$$|{}^{j}_{j_3},\xi^{\pm}\}\equiv\frac{1}{\sqrt{2}}(\,|{}^{j}_{j_3},\xi\}\pm|{}^{j}_{j_3},\xi'\}\,)$$

with the property

$$\Gamma_0|{}^{j}_{j_3},\xi^{\pm}\}=\pm n\,|{}^{j}_{j_3},\xi^{\pm}\}\ .$$

Thus in either case we obtain eigenvectors of Γ_0 with the eigenvalues $(j+\frac{1}{2})$ or $-(j+\frac{1}{2})$. We label these vectors by $|{}^{j}_{j_3},\pm,|\xi|\,\}$ with the property

$$\Gamma_0|{}^{j}_{j_3},\pm,|\xi|\,\}=\pm(j+\tfrac{1}{2})\,|{}^{j}_{j_3},\pm\,|\xi|\,\}\ , \tag{A23}$$

where $|\xi|$ now labels the (possible) multiplicity, in addition to the multiplicity expressed by $\pm$, with which $\mathscr{H}(k_0=0,c=\frac{1}{2})$ or $\mathscr{H}(k_0=\frac{1}{2},c=0)$ may occur in the direct sums of Eq. (A15). $S_{\mu\nu}$ cannot transform out of a given $\mathscr{H}(k_0,c)$ and, therefore, cannot change the multiplicity label $(\pm,|\xi|)$; Γ_0 also does not change $(\pm,|\xi|)$ as can be seen from Eq. (A23). Further, since Γ_i is given in terms of S_{0i} and Γ_0 by

$$\Gamma_i=i[S_{0i},\Gamma_0]\ , \tag{A24}$$

there is no operator among $S_{\mu\nu},\Gamma_\mu$ which changes the multiplicity label $(\pm,|\xi|)$. Thus, for the irreducible Majorana representations the multiplicity label is redundant and, therefore, $\mathscr{H}^I$ contains either $\mathscr{H}(k_0=0,c=\frac{1}{2})$ or $\mathscr{H}(k_0=\frac{1}{2},c=0)$ exactly once. But as representations of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ we also have to distinguish the two cases:

$$\operatorname{sgn}(\text{spectrum}\,\Gamma_0)=+1\ \text{and}\ \operatorname{sgn}(\text{spectrum}\,\Gamma_0)=-1\ . \tag{A25}$$

Therewith we conclude that there are four inequivalent irreducible Majorana representations of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$:

	values of j	spectrum Γ_0
$(k_0=0,\quad c=\frac{1}{2},\quad \operatorname{sgn}n=+)$	$0,1,2,\ldots$	$\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ldots$
$(k_0=0,\quad c=\frac{1}{2},\quad \operatorname{sgn}n=-)$	$0,1,2,\ldots$	$-\frac{1}{2},\ -\frac{3}{2},\ -\frac{5}{2},\ldots$
$(k_0=\frac{1}{2},\quad c=0,\quad \operatorname{sgn}n=+)$	$\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ldots$	$1,2,3,\ldots$
$(k_0=\frac{1}{2},\quad c=0,\quad \operatorname{sgn}n=-)$	$\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ldots$	$-1,\ -2,\ -3,\ldots\ .$

(A26)

The basis of $\mathscr{H}^I$ is given by the basis of $\mathscr{H}(k_0=0,c=\frac{1}{2})$ or by the basis of $\mathscr{H}(k_0=\frac{1}{2},c=0)$:

$$|{}^{j}_{j_3}\}\ \text{with}\ \begin{array}{l} j=k_0,k_0+1,k_0+2,\ldots \\ j_3=-j,-j+1,\ldots,j\ ,\end{array} \tag{A27}$$

and we normalize the basis vectors according to

$$\{{}^{j'}_{j_3'}|{}^{j}_{j_3}\}=\delta^{j'j}\delta_{j_3'j_3}\ . \tag{A28}$$

Choosing Naimark's[2] phase convention for the matrices of the operators $S_{\mu\nu}$ in this basis, we have the following matrix elements:

$$S_{12}|{}^{j}_{j_3}\}=j_3\,|{}^{j}_{j_3}\}\ , \tag{A29a}$$

$$(S_{23}\pm iS_{31})|{}^{j}_{j_3}\}=[(j\pm j_3+1)(j\mp j_3)]^{1/2}\,|{}^{j}_{j_3\pm1}\}\ , \tag{A29b}$$

$$S_{03}|{}^{j}_{j_3}\}=[(j-j_3)(j+j_3)]^{1/2}C_j\,|{}^{j-1}_{j_3}\}-[(j+j_3+1)(j-j_3+1)]^{1/2}C_{j+1}\,|{}^{j+1}_{j_3}\}\ , \tag{A29c}$$

$$(S_{01}\pm iS_{02})|{}^{j}_{j_3}\}=\pm[(j\mp j_3)(j\mp j_3-1)]^{1/2}C_j\,|{}^{j-1}_{j_3\pm1}\}\pm[(j\pm j_3+1)(j\pm j_3+2)]^{1/2}C_{j+1}\,|{}^{j+1}_{j\pm1}\}\ , \tag{A29d}$$

$$\Gamma_0 \,|\,{}^{j}_{j_3}\} = \operatorname{sgn} n (j+\tfrac{1}{2})\,|\,{}^{j}_{j_3}\} \,, \tag{A30}$$

where

$$C_j = \begin{cases} 0, & \text{for } j=0,\tfrac{1}{2} \\ \dfrac{i}{2j}, & \text{for } j=1,\tfrac{3}{2},2,\tfrac{5}{2},\ldots \,. \end{cases} \tag{A31}$$

Using Eqs. (A24), (A29c), and (A30) we obtain, for example,

$$\Gamma_3 \,|\,{}^{j}_{j_3}\} = i \operatorname{sgn} n [((j-j_3)(j+j_3))^{1/2} C_j \,|\,{}^{j-1}_{j_3}\} + ((j-j_3+1)(j+j_3+1))^{1/2} C_{j+1} \,|\,{}^{j+1}_{j_3}\}] \,. \tag{A32}$$

The vectors $|\,{}^{j}_{j_3}\}$ have the following transformation property under an element Λ of the Lorentz group $SO(3,1)_{S_{\mu\nu}}$:

$$U(\Lambda)\,|\,{}^{j}_{j_3}\} = \sum_{\substack{-j' \le j_3' \le j' \\ j'=k_0,k_0+1,\ldots}} |\,{}^{j'}_{j_3'}\} D^{(k_0,c)j'j}_{j_3' j_3}(\Lambda) \,, \tag{A33}$$

where

$$U(\Lambda(\omega)) = e^{i\omega^{\mu\nu} S_{\mu\nu}} \tag{A34}$$

is the operator which represents Λ in the space $\mathscr{H}(k_0,c)$, and

$$D^{(k_0,c)j'j}_{j_3' j_3}(\Lambda) = \{{}^{j'}_{j_3'} \,|\, U(\Lambda)\,|\,{}^{j}_{j_3}\} \tag{A35}$$

is the matrix representation of the operator $U(\Lambda)$ in the basis $|\,{}^{j}_{j_3}\}$.

[1]R. R. Aldinger, A. Bohm, P. Kielanowski, M. Loewe, P. Magnollay, N. Mukunda, W. Drechsler, and S. R. Komy, preceding paper, Phys. Rev. D 28, 3020 (1983). We refer to this paper as paper I, and to equations in it by equation numbers like Eq. (I.2.2).

[2]For the representations of SO(3,1) we use the notation and conventions of M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, New York, 1964). A much shorter derivation of these representations can be found in the Appendix to Sec. V.3 of A. Bohm, *Quantum Mechanics* (Springer, New York, 1979).

[3]More generally, finite-dimensional irreps are given by $(k_0, c=\pm(k_0+l+1))$ where $k_0=0,\frac{1}{2},1,\frac{3}{2},\ldots$ and $l=0,1,2,\ldots$. In the physics literature the finite-dimensional irreps $(k_0=j, c=j+1)$ and $(k_0=j, c=-j-1)$ are usually denoted by $(0,j)$ and $(j,0)$, respectively.

[4]S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1967), Chap. 4.

[5]A. Bohm and G. B. Mainland, Fortschr. Phys. 18, 275 (1970).

[6]The only difference here is that instead of the representation relation $\{\Gamma_\mu, \Gamma_\rho\} = \frac{1}{2} g_{\mu\rho}$, or

$$\{\gamma_\mu, \gamma_\rho\} = 2 g_{\mu\rho} \tag{*}$$

for the representation matrices $\gamma_\mu = 2\langle\,|\Gamma_\mu|\,\rangle$, one now has the representation relation (I.2.41). Thus, whereas in the usual generalizations of the four-dimensional Dirac γ matrices one retains (*) and, perhaps, gives up the commutation relations, we retain the commutation relations (I.2.2), (I.2.39), and (I.2.40) of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ and give up the representation relation (*).

[7]P. Budini and C. Fronsdal, Phys. Rev. Lett. 14, 968 (1965); V. Ottoson, A. Kihlberg, and J. S. Nilsson, Phys. Rev. 137, B658 (1965).

[8]This relation was first suggested by J. Werle (unpublished). Applied to SO(3,2) it led to the concept of relativistic spectrum-generating groups [A. Bohm, in *Lectures in Theoretical Physics*, edited by A. O. Barut (Gordon and Breach, New York, 1967), Vol. 10B, p. 483; A. Bohm, Phys. Rev. 175, 1767 (1968)] and the concept of dynamical stability group [H. van Dam and L. C. Biedenharn, Phys. Lett. 62B, 190 (1976); Phys. Rev. D 14, 405 (1976)].

[9]Although we are not concerned in this paper with the mathematical aspects, we mention that the existence of the complete system of generalized eigenvectors is guaranteed by the nuclear spectral theorem.

[10]H. Joos, Fortschr. Phys. 10, 65 (1962); S. Weinberg, Phys. Rev. 133, B1318 (1964); and Ref. 4.

[11]See Eq. (4.62) of Ref. 4.

[12]See Eq. (3.13) of Ref. 10, first citation.

[13]A. Bohm and P. Kielanowski, Phys. Rev. D 27, 166 (1983); A. Bohm and R. B. Teese, *ibid.* 26, 1103 (1982).

[14]The symbol $\underset{G}{\Rightarrow}$ means that the spaces are equal when the transformations are restricted to the subgroup G.

[15]The form factors are conventionally referred to the spinor basis so that a transformation from the canonical basis to the spinor basis becomes necessary; this transformation involves the Dirac spinors $u(p)$ and $v(p)$, which are nothing other than $D(L^{-1}(p))\langle c\,|\,n\rangle$ where D is the representation matrix of the boost in the representation $(k_0+\frac{1}{2}, c=\frac{3}{2}) \oplus (k_0=\frac{1}{2}, c=-\frac{3}{2})$, and $\langle c\,|\,n\rangle$ is the representation matrix from eigenvectors of $\frac{1}{2}S_{\mu\nu}S^{\mu\nu}$ to eigenvectors of Γ_0. See Ref. 5.

[16]P. A. M. Dirac, J. Math. Phys. 4, 901 (1963), or see also Chap. II, Sec. 2 of N. Mukunda, H. van Dam, and L. C. Biedenharn, in *Relativistic Models of Extended Hadrons Obeying a Mass-Spin Trajectory Constraint* (Springer, New York, 1982).

[17]One can however introduce such a spinor operator and then combine representations with integer and half-integer angular momentum.