Relativistic rotator. I. Quantum observables and constrained Hamiltonian mechanics

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(Received 11 April 1983)

The model of the quantum relativistic rotator is defined by three correspondences: (1) the correspondence to a classical relativistic rotator when the quantum description goes over into the classical description (classical limit), (2) the correspondence to an elementary particle when the structure is ignored (elementary limit), and (3) the correspondence to a nonrelativistic quantum rotator (a rigid rotating string) in the nonrelativistic limit. The dynamics is given by a Hamiltonian which is obtained from a constraint relation that leads to a phenomenologically acceptable mass-spin trajectory relation. From the equation of motion it follows that the expectation value of the particle position spirals with approximately the velocity of light about the direction of the momentum, which is also the direction in which the center of mass propagates. The radius of this helical motion (i.e., the "size" of the rotator), as obtained from the phenomenological mass spectrum, is of the order of 10^{-13} cm.

I. INTRODUCTION

The nonrelativistic rotator is realized in nature by numerous quantum physical systems, particularly in the realms of molecular[1] and nuclear[2] physics. Because of this fact, it would also be desirable to see if hadrons can be described as rotators. It appears, therefore, to be an attractive endeavor to construct a model for a quantum relativistic rotator.

A. Definition of the quantum relativistic rotator

The quantum relativistic rotator (QRR) is an entity which can only be defined by correspondence with already well-established models. A new theory describing a new domain of physical reality is formulated in conjunction with an old familiar theory in such a way that the new theory, in a certain sense, "corresponds" to the old theory when the new domain of applicability is restricted to that of the old description. In this manner, the QRR will be specified by the use of three distinct correspondences:

(a) elementary limit$\to$elementary particle ,

(b) classical limit$\to$classical relativistic rotator , (1.1)

(c) nonrelativistic limit

$\to$nonrelativistic quantum rotator .

B. The elementary limit

In the elementary limit, the dynamics of the QRR should go over into the well-known dynamics of a point-like elementary particle (described in terms of the irreducible representations of the Poincaré group).[3] The constraint $P_\mu P^\mu - m^2 \approx 0$ defining the mass provides the relativistic Hamiltonian while further constraints may be imposed in order to fix the value of the spin.[4] Since the QRR possesses various mass and spin states, it is not considered as representing an elementary physical system[5] and is best thought of as an extended object.

One can easily visualize the elementary limit by considering the relativistic rotator as an extended object (in the same way as the nonrelativistic rotator is regarded as an extended object) characterized by an elementary length parameter R of the order of a fermi. Therefore, in the elementary limit this length parameter, being related to the radius of a micro-de Sitter space associated with the internal dynamics, is taken to infinity. Thereby, the observables that in this limit go into the momenta behave in such a manner that the representations of the de Sitter group go over into those of the Poincaré group. The Hamiltonian $\mathscr{H}$ for the QRR will, therefore, be given in terms of the generators of the de Sitter group yielding for $\mathscr{H}$ an expression which is a de Sitter constraint relation. In the limit $R\to\infty$, this $\mathscr{H}$ goes over into the Hamiltonian for the motion of a relativistic mass point characterized in terms of the Poincaré group.

C. The classical limit

Concerning the classical relativistic rotator, there exist numerous models and various formalisms.[6–10] The constrained Hamiltonian formalism is probably the most direct for entering the quantum domain for at least two reasons.

(1) A discrete, nontrivial mass spectrum (as it is experimentally observed) cannot be obtained from group-theoretical structures[11] alone and must, therefore, be imposed as a constraint relation which automatically provides the relativistic Hamiltonian.

(2) The constrained Hamiltonian formalism[12,13] clearly exhibits the correspondence between the classical and the quantum theories by juxtaposing the Poisson or Dirac brackets and the commutators.

The Hanson-Regge model was probably the first description to explore these possibilities in a systematic way. Unfortunately, their classical model used second-class constraints which led to several difficulties and, moreover, excluded the *Zitterbewegung* (i.e., in their model the velocity of the position of charge is proportional to the momentum). However, the *Zitterbewegung* appears to be a characteristic feature of a relativistic rotating object since for the electron its assumption immediately leads to the proper value for the g factor.[6,14]

A class of relativistic rotator models which uses only a single first-class (primary) constraint (the one needed for the mass formulas) is given by the "spinorial models" of Refs. 10 and 13. These models also have the appealing feature of using a Lorentz vector Γ_μ which together with the intrinsic part of the Lorentz generators $S_{\mu\nu}$ form an SO(3,2) algebra reminiscent of the SO(3,2) of the Dirac γ and σ matrices, which can serve as the starting point for the construction of current and transition operators. Most importantly, the equations of motion for this class of models do lead to the *Zitterbewegung*.

Therefore, we shall construct our quantum-mechanical relativistic rotator model using the classical spinorial models as a correspondence. This still leaves a large choice of possible Hamiltonians. The particular Hamilton operator that we shall choose (which will be determined by the correspondences given by the elementary and nonrelativistic limits) leads to equations of motion for the quantum-mechanical observables which result in a *Zitterbewegung* for their expectation values. Moreover, the constraint relation for this particular Hamiltonian leads to an experimentally acceptable mass spectrum.

Quantum-mechanical versions of constrained Hamiltonian mechanics for relativistic rotating objects have been discussed before[4,15] and those papers[9,10] that are mainly concerned with the classical models are also motivated by the desire to set up a quantum theory. Staunton's calculations are probably closest to a quantum theory; however, the Hamiltonian he uses originated from the Majorana equation (with all its problems). Furthermore, he did not use the interrelationship between the mass formula and the Hamiltonian. To obtain our quantum-mechanical equations of motion, we follow the prescription given by Dirac for the transition from the classical to the quantum level.[4,12] However, we shall use a different Hamilton operator which is consistent with the experimental mass-spin spectrum for hadrons and also leads to the *Zitterbewegung*.

D. The nonrelativistic limit

The nonrelativistic top is an extended object that can perform rotational as well as translational motions in three-dimensional space. It can be described by the position of the center of mass (a point $\vec{x}\in R_3$) and a rotating frame attached to this point (the body-fixed coordinate system describing the orientation of the extended object). Since the center of mass (c.m.) is described by a vector x^i and the rotating frame by a triad [a rotation matrix $R^i{}_j(\phi,\theta,\psi)$ which describes the orientation of the body-fixed frame with respect to a standard coordinate frame], the configuration of the top is described by the affine Euclidean frame $E=(x^i,R^i{}_j)$ in R_3. The characteristic mechanical parameters of the system are the mass and the three moments of inertia. The fact that the description of the nonrelativistic rotator is given in terms of the parameters of the affine group in Euclidean three-space suggests the use of an affine frame bundle over R_3 as the underlying geometric structure for the nonrelativistic description of a rotator formulated in terms of certain matter fields. Such a picture can, moreover, be carried over to a quantized description. From these remarks it is tempting to define a relativistic rotator as an object characterized in terms of the parameters of the Poincaré group.[16] In a quantized version this would lead to a gauge description for a rotator dynamics based on the geometry of an underlying affine frame bundle (Poincaré bundle) over Minkowski space-time M_4.[17,9] However, in this case the classical description already leads to the difficulties of the Hanson-Regge model.[9] Since there are some indications that hadrons are described by the de Sitter rather than the affine (i.e., Poincaré) bundle,[18] we will not follow this route of relativistic generalization. Instead, we shall attempt an extension to relativity using the algebra of observables for a much more specific type of top.

First, we shall assume that the top possesses a symmetry axis. Then rotations about this axis are irrelevant and the position of the top is described by only two angular variables: $(\phi,\theta,\psi=\phi)$. In this case it is more convenient to use the unit vector $\vec{a}$ (which points in the direction of the symmetry axis) instead of the matrix $R^i{}_j$.

The quantum-mechanical nonrelativistic symmetric top then possesses the following observables:

P_i $(i=1,2,3)$, the momentum generating the translational motion of the center of mass :

$$J_{ij}=L_{ij}+S_{ij}, \text{ the angular momentum, where} \tag{1.2a}$$

$$L_{ij}=Q_iP_j-Q_jP_i \text{ is the orbital angular momentum} \tag{1.2b}$$

and

S_{ij}, are the generators of "intrinsic" rotations of the extended object about the center of mass .

The S_{ij} obey the commutation relations of SO (3) and we call the group they generate $SO(3)_{S_{ij}}$. In general ,we will label a group by the observables that generate it.

Q_i, the nonrelativistic position operator (representing x^i) which is related to the generator of the proper Galilei transformation G_i by

$G_i = Q_i M$, where M is the nonrelativistic mass; (1.3)

P_i, J_{ij}, G_i, the central element M and the energy operator H are the generators of the extended (quantum mechanical) Galilei group whose defining commutation relations will be given in paper III.

D_i, representing the body axis $\vec{a}$, is assumed to be a vector operator with respect to $SO(3)_{S_{ij}}$.

The commutation relation of D_i with D_j is not determined *a priori* but, since $\vec{a}$ points toward the center of charge, D_i is something like a dipole operator. Therefore, it is assumed that the D_i commute among themselves.[19] So the $S_i = \epsilon_{ijk} S_{jk}$ and the D_i together satisfy the commutation relations of the three-dimensional Euclidean group $E(3)_{D_i,S_i}$:

$$[S_i,S_j]=i\epsilon_{ijk}S_k,\quad [S_i,D_j]=i\epsilon_{ijk}D_k\ ,\qquad [D_i,D_j]=0\ . \tag{1.4}$$

This $E(3)_{D_i,S_j}$ is not related to the Euclidean subgroup of the physical Galilei group $E(3)_{P_i,J_{ij}}$ which describes rotations and translations. Instead, $E(3)_{D_i,S_i}$ is the spectrum-generating group[20] for the nonrelativistic rotator and the symmetric top.[21] The energy operator for this rotating and translating object with mass M and moments of inertia I_A and I_B is given by

$$H=\frac{\vec{P}^2}{2M}+\frac{1}{2I_B}\left[\vec{S}^2-\frac{I_A-I_B}{I_A}(D_i\cdot D_i)^{-2}(D_i\cdot S_i)^2\right]\ . \tag{1.5}$$

It is seen that the D_i do not commute with H and therefore describe transitions between different energy levels. This nonrelativistic symmetrical top is still not the model that we want to extend into a relativistic quantum theory. We further simplify the model to the rotator, which may be visualized as a dumbbell or as a one-dimensional rigid rod with $I_A = 0$ and $I_B = I_C = \mu R^2$, where μ is the reduced mass and R is the interparticle extension.

For a rotator, the angular momentum is always perpendicular to the figure axis so that

$$D_i \cdot S_i = 0\ , \tag{1.6}$$

and the energy operator becomes

$$H=\frac{\vec{P}^2}{2M}+\frac{\vec{S}^2}{2I_B}\ . \tag{1.7}$$

This restriction to the dumbbell or rigid-rod model avoids the multiplicity of the eigenvalues of $\vec{S}^2$ which is present for the symmetric top[21] [where one is not restricted to representations of E(3) that fulfill (1.6)]. For our relativistic model, we also want to restrict ourselves to a simple spin spectrum in order to facilitate the calculations. Therefore we impose a relation [the Majorana representation relation, Eq. (2.41) below] which, when the nonrelativistic limit is taken, results in Eq. (1.6). (The relaxation of this property probably does not lead to any principle difficulties.)

If one finds it helpful, one can—in analogy to the diatomic molecule—picture the nonrelativistic classical limit of the dumbbell as a diquark (or as a rigid array of three of more quarks since $I_A = 0$ and $I_B = I_C$ are the relevant assumptions and not the dumbbell nature). But, such a picture is really unimportant and certainly gets blurred when one goes to the relativistic and to the quantum domains, where the concepts of rigidity and trajectory, respectively, lose their usual meaning.

Thus, we will construct the model of the quantum relativistic rotator. We do not expect this model to provide a completely accurate description for a large number of hadron data because of its simplicity (i.e., no fine structure, no multiplicity of spin, no elasticity, etc.). But, the model will be consistent with the experimentally observed hadron spectrum and completely solvable theoretically. It provides us with a simple example of a relativistic quantum mechanics besides the trivial example given by the irreducible representations of the Poincaré group.

II. THE ALGEBRA OF OBSERVABLES OF THE QRR

A. The basic observables

In this section, we shall establish the quantum-mechanical observables for the QRR and their defining relations. They will be conjectured along the lines described in Sec. I. The result of such an inference is not, of course, uniquely determined. However, the reverse process, which is the limit from the new to the old model is uniquely determined. These limiting processes will be considered in paper III and it will be shown there that our conjectured model does obey the correspondences given in (1.1).

The Galilei group of the nonrelativistic model will have to be replaced by the Poincaré group in the relativistic case. Thus, there will be the observables P_μ and $J_{\mu\nu}$ with $\mu,\nu = 0,1,2,3$ which obey the commutation relations of the Poincaré group, $\mathscr{P}_{P_\mu,J_{\mu\nu}}$ $[g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)]$:

$$[J_{\mu\nu},J_{\rho\sigma}]=-i(g_{\mu\rho}J_{\nu\sigma}+g_{\nu\sigma}J_{\mu\rho}-g_{\mu\sigma}J_{\nu\rho}-g_{\nu\rho}J_{\mu\sigma})\ , \tag{2.1a}$$

$$[P_\mu,J_{\rho\sigma}]=-i(g_{\mu\sigma}P_\rho-g_{\mu\rho}P_\sigma)\ , \tag{2.1b}$$

$$[P_\mu,P_\nu]=0\ . \tag{2.1c}$$

In the relativistic case the three generators S_{ij} of intrinsic rotations, satisfying the SO(3) commutation relations, are generalized to the generators $S_{\mu\nu}$ satisfying the commutation relations of $SO(3,1)_{S_{\mu\nu}}$:

$$[S_{\mu\nu},S_{\rho\sigma}]=-i(g_{\mu\rho}S_{\nu\sigma}+g_{\nu\sigma}S_{\mu\rho}-g_{\mu\sigma}S_{\nu\rho}-g_{\nu\rho}S_{\mu\sigma}) . \quad (2.2)$$

The generators of the physical Lorentz transformations, $J_{\mu\nu}$, are now written in analogy to Eq. (1.2):

$$J_{\mu\nu}=Q_\mu P_\nu-Q_\nu P_\mu+S_{\mu\nu}\equiv M_{\mu\nu}+S_{\mu\nu} , \quad (2.3a)$$

where

$$[M_{\mu\nu},S_{\rho\sigma}]=0, \quad \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\mu M^{\rho\sigma}=0 . \quad (2.3b)$$

This splitting of the operators in the representation space of the Poincaré group will be qualified in paper II. The properties of the Q_μ operators are defined, in correspondence to the nonrelativistic limit, by the following commutation relations:

$$[J_{\mu\nu},Q_\rho]=i(g_{\nu\rho}Q_\mu-g_{\mu\rho}Q_\nu) , \quad (2.4)$$

$$[Q_\mu,P_\rho]=-ig_{\mu\rho}1 , \quad (2.5)$$

$$[Q_\mu,Q_\nu]=0 . \quad (2.6)$$

These relations also correspond to the Poisson and Dirac bracket relations for the classical position coordinates in the classical model which we have chosen as our classical limit.[10] They are, however, in disagreement with the Dirac bracket relations of the "position" for most other classical models. Although (2.5) and (2.6) define the Q_μ as straightforward generalizations of the nonrelativistic position operators, their equations of motion will show that they have quite unexpected properties.

At this stage (before the constraint relations are imposed) the Q_ρ and P_μ are assumed to commute with $S_{\mu\nu}$. $S_{\mu\nu}$ is *not* the spin (it is often called intrinsic angular momentum or the spin part of angular momentum).

We now define the following operator:

$$\hat{P}_\mu=P_\mu M^{-1}, \quad M=(P_\mu P^\mu)^{1/2} . \quad (2.7a)$$

This requires that $P_\mu P^\mu>0$. Below, we shall see that (2.7a) is a consequence of our constraint relation which serves as the Hamiltonian.

$\hat{P}_\mu$ is often called the four-velocity, although it is not identical to the velocity of the position for the QRR as will be shown in Sec. III below [see Eq. (3.19)]. It is proportional to the c.m. velocity and, for our particular model, it will turn out [see Eq. (3.21) below] to be identical to the c.m. velocity.

In addition to the $\hat{P}_\mu$, other dimensionless quantities are also very convenient to work with.[22] Therefore, we also define

$$\hat{Q}_\mu=Q_\mu M \quad (\neq MQ_\mu) . \quad (2.7b)$$

Immediate consequences of (2.5) are the following relations, which will be needed below:

$$\begin{aligned}
&[M,Q_\mu]=i\hat{P}_\mu ,\\
&[M^{-1},Q_\mu]=-iP_\mu M^{-3} ,\\
&[\hat{Q}_\rho,Q_\mu]=iQ_\rho\hat{P}_\mu ,\\
&[\hat{Q}_\rho,P_\nu]=-ig_{\rho\nu}M ,\\
&[\hat{Q}_\rho,\hat{P}_\nu]=-i(g_{\rho\nu}-\hat{P}_\rho\hat{P}_\nu) ,\\
&[\hat{Q}_\mu,\hat{Q}_\nu]=i(\hat{Q}_\mu\hat{P}_\nu-\hat{Q}_\nu\hat{P}_\mu) ,\\
&[\hat{P}_\sigma,Q_\mu]=i(g_{\sigma\mu}-\hat{P}_\mu\hat{P}_\sigma)M^{-1} .
\end{aligned} \quad (2.8)$$

Not as obvious as the preceding assumptions is the choice of what to use as the spectrum-generating group in place of the nonrelativistic $E(3)_{D_i,S_i}$. In particular, it is not clear what the relativistic generalization of the D_i should be. We now define the following operators as the relativistic replacement for the D_i:

$$d_\mu=S_{\mu\nu}\frac{\hat{P}^\nu}{M} . \quad (2.9a)$$

In the nonrelativistic limit, these operators go over into the D_i and, in the classical case, correspond to the vector from the position of the particle (position of charge) x^μ to the c.m. y^μ.[23] In the c.m. frame, the classical analog of d^i $(i=1,2,3)$ is, therefore, proportional to the vector along the dumbbell axis $(-a^i)$.

Again, as in Eq. (2.7), we will define the dimensionless quantity[22]

$$\hat{d}_\mu=d_\mu M=S_{\mu\nu}\hat{P}^\nu . \quad (2.9b)$$

By a straightforward calculation using $[J_{\mu\nu},S_{\rho\sigma}]=[S_{\mu\nu},S_{\rho\sigma}]$ given by Eq. (2.2), one can verify that $\hat{d}_\mu$ is a Lorentz vector operator:

$$[\hat{d}_\mu,J_{\rho\sigma}]=i(g_{\mu\rho}\hat{d}_\sigma-g_{\mu\sigma}\hat{d}_\rho) . \quad (2.10)$$

However, it is not a vector operator with respect to $SO(3,1)_{S_{\mu\nu}}$:

$$[\hat{d}_\mu,S_{\rho\sigma}]=i(g_{\mu\rho}\hat{d}_\sigma-g_{\mu\sigma}\hat{d}_\rho)-i(S_{\mu\rho}\hat{P}_\sigma-S_{\mu\sigma}\hat{P}_\rho) . \quad (2.11)$$

The commutator of two $\hat{d}$'s is found to be

$$[\hat{d}_\mu,\hat{d}_\rho]=-i(S_{\mu\rho}+\hat{d}_\rho\hat{P}_\mu-\hat{d}_\mu\hat{P}_\rho) . \quad (2.12)$$

This result suggests the definition of a new operator[24]:

$$\Sigma_{\mu\rho}=S_{\mu\rho}-\hat{d}_\mu\hat{P}_\rho+\hat{d}_\rho\hat{P}_\mu=S_{\mu\rho}-d_\mu P_\rho+d_\rho P_\mu . \quad (2.13)$$

Therefore, the commutation relation (2.12) can be written as

$$[\hat{d}_\mu,\hat{d}_\rho]=-i\Sigma_{\mu\rho} , \quad (2.14a)$$

or

$$[d_\mu,d_\rho]=-i\frac{1}{M^2}\Sigma_{\mu\rho}=-i\frac{1}{M^2}(S_{\mu\rho}+d_\rho P_\mu-d_\mu P_\rho) . \quad (2.14b)$$

Using the above results and definitions, the following commutation relations are established:

$$[\hat{d}_\mu,\Sigma_{\rho\sigma}]=i(g_{\mu\rho}\hat{d}_\sigma-g_{\mu\sigma}\hat{d}_\rho)-i(\Sigma_{\rho\sigma}-S_{\rho\sigma})\hat{P}_\mu\ , \quad (2.15)$$

$$[\Sigma_{\mu\nu},\Sigma_{\rho\sigma}]=-i(g_{\mu\rho}-\hat{P}_\mu\hat{P}_\rho)\Sigma_{\nu\sigma}-i(g_{\nu\sigma}-\hat{P}_\nu\hat{P}_\sigma)\Sigma_{\mu\rho}+i(g_{\mu\sigma}-\hat{P}_\mu\hat{P}_\sigma)\Sigma_{\nu\rho}+i(g_{\nu\rho}-\hat{P}_\nu\hat{P}_\rho)\Sigma_{\mu\sigma}\ . \quad (2.16)$$

The commutation relation (2.16) was already given in Ref. 15 for the intrinsic spin tensor, which $\Sigma_{\mu\nu}$ will turn out to represent.

From the definition (2.9) for d_μ and (2.13) for $\Sigma_{\mu\nu}$, one finds

$$d_\mu P^\mu=0,\ \ \hat{d}_\mu\hat{P}^\mu=0\ , \quad (2.17)$$

and

$$P^\mu\Sigma_{\mu\nu}=0\ . \quad (2.18)$$

We note at this point that (2.17) and (2.18) are consequences of the definitions and resulting commutation relations. They are not constraint relations like those of Pryce,[25] i.e., additionally imposed conditions to eliminate certain components of his spin tensor. We shall not, for instance, impose conditions such as $P^\mu S_{\mu\nu}=0$. Our $S_{\mu\nu}$ is not the spin, but is the "spin part of the angular momentum" best understood in analogy to the $\sigma_{\mu\nu}$ matrices in the theory of the Dirac equation (the $\sigma_{\mu\nu}$ are representation matrices of the $S_{\mu\nu}$ in a very particular representation not used here). $\Sigma_{\mu\nu}$ is the spin tensor and is not identical with $S_{\mu\nu}$. In contrast, Ref. 9 demanded the existence of a primary constraint which resulted in $\Sigma_{\mu\nu}$ and $S_{\mu\nu}$ becoming (classically) equal, and which led, therefore, to some unwelcome consequences.

The correct value for the gyromagnetic ratio of structureless particles, $g_e=2$, follows from the assumption that velocity and momentum are not parallel.[26] Therefore, one has to distinguish between two types of "rest frames": the ordinary rest frame in which the c.m. is at rest, i.e., $\hat{P}_\mu=(1,0,0,0)$, and another in which the particle position is at rest, i.e., $\dot{x}_\mu=(1,0,0,0)$ for the velocity.[27] The latter will not correspond to inertial frames. As will be seen in Sec. I of paper III, in particular (5.27), Eq. (2.13) can be interpreted in two distinct ways: (i) It implies that the angular momentum with respect to the center of mass, i.e., the spin $\Sigma_{\mu\nu}$, is the intrinsic angular momentum $S_{\mu\nu}$ plus the orbital angular momentum of the particle position with respect to the c.m. ($-d_m$ is the vector from the c.m. to the particle position); or (ii) resolved for $S_{\mu\nu}$, Eq. (2.13) implies that the angular momentum with respect to the particle position, i.e., intrinsic angular momentum, is the spin plus the orbital angular momentum of the c.m. with respect to the particle position.

Inserting the definition (2.13) of $\Sigma_{\mu\nu}$ into Eq. (2.3), one can write the angular momentum operator as

$$J_{\mu\nu}=Y_\mu P_\nu-Y_\nu P_\mu+\Sigma_{\mu\nu}\ , \quad (2.19)$$

where Y_μ is defined as

$$Y_\mu=Q_\mu+d_\mu\ . \quad (2.20)$$

Y_μ is the operator corresponding to the c.m. position of the classical relativistic rotator.[28]

B. The SO(4,1) Hamiltonian

In analogy to Eqs. (2.7) and (2.9b), we define

$$\hat{Y}_\mu=\hat{Q}_\mu+\hat{d}_\mu=\hat{Q}_\mu+S_{\mu\nu}\hat{P}^\nu\ . \quad (2.21)$$

Using (2.7b) and (2.21), the operators $J_{\mu\nu}$ can be written as

$$J_{\mu\nu}=\hat{Q}_\mu\hat{P}_\nu-\hat{Q}_\nu\hat{P}_\mu+S_{\mu\nu}=\hat{Y}_\mu\hat{P}_\nu-\hat{Y}_\nu\hat{P}_\mu+\Sigma_{\mu\nu}\ . \quad (2.22)$$

$\hat{Q}_\mu$ is obviously a $J_{\mu\nu}$-vector operator, i.e., with $J_{\mu\nu}$ it obeys the same commutation relations (2.4) as the Q_μ do. Furthermore, since Q_μ and M commute with $S_{\mu\nu}$ (M at least before the constraint relation has been imposed), one immediately obtains

$$[\hat{Q}_\rho,S_{\mu\nu}]=0\ , \quad (2.23)$$

$$[Q_\rho,S_{\mu\nu}]=0\ . \quad (2.24)$$

A direct calculation, using the commutation relations (2.8), also shows that the $\hat{Q}_\rho$ are $M_{\mu\nu}$ vector operators. Therefore, (2.23) must be satisfied even in the more general situation where M does not commute with $S_{\mu\nu}$. Also, since Q_ρ is an $M_{\mu\nu}$ vector operator, (2.24) must also be fulfilled in this more general case.

One can now compute the commutation relation of Y_μ with Y_ν. Using (2.5), (2.14), and

$$[Q_\mu,d_\nu]=i(S_{\mu\nu}+2\hat{d}_\nu\hat{P}_\mu)M^{-2} \quad (2.25)$$

[which follows from the definition (2.9) and the commutation relations (2.8)], one obtains

$$[Y_\mu,Y_\nu]=i\Sigma_{\mu\nu}M^{-2}\ . \quad (2.26)$$

Furthermore, one finds

$$[\Sigma_{\mu\nu},\hat{Y}_\rho]=i(\Sigma_{\nu\rho}\hat{P}_\mu-\Sigma_{\mu\rho}\hat{P}_\nu)\ , \quad (2.27a)$$

and (at least before the constraint is imposed)

$$[\Sigma_{\mu\nu},Y_\rho]=i(\Sigma_{\nu\rho}P_\mu-\Sigma_{\mu\rho}P_\nu)\ . \quad (2.27b)$$

The commutation relation for two $\hat{Y}_\mu$'s can now be deduced from (2.26) and (2.8); the result is

$$[\hat{Y}_\mu,\hat{Y}_\nu]=iJ_{\mu\nu}\ . \quad (2.28)$$

The commutation relation of $\hat{Y}_\mu$ and $J_{\mu\nu}$ is obtained by use of Eqs. (2.10) and (2.4) and is given by

$$[J_{\mu\nu},\hat{Y}_\rho]=i(g_{\nu\rho}\hat{Y}_\mu-g_{\mu\rho}\hat{Y}_\nu)\ . \quad (2.29)$$

Therefore, $\hat{Y}_\rho$ is seen to be a Lorentz vector operator. Together with Eqs. (2.28) and (2.1a), this means that the operators $\hat{Y}_\rho$, together with $J_{\mu\nu}$, generate a non-Hermitian representation of the Lie algebra of $SO(4,1)_{J_{\mu\nu},\hat{Y}_\nu}$.

As mentioned above, $Y_\mu=\hat{Y}_\mu M^{-1}$ is the operator for the center of mass of the relativistic rotator. As will be shown in Sec. III, the expectation values of these noncommuting operators follow a straight line in the direction of the momentum. The operators $\hat{Y}_\mu$, which are not Hermitian, derive their meaning from this property. $\hat{Y}_\mu$ is not the center operator of Finkelstein,[29] which is $(M^{-1}\hat{b}_\mu)=\hat{b}_\mu M^{-1}$ where $\hat{b}_\mu$ is defined by

$$\hat{b}_\mu=\tfrac{1}{2}\{J_{\mu\nu},\hat{P}^\nu\}\equiv\tfrac{1}{2}\{\hat{Q}_\mu\hat{P}_\nu-\hat{Q}_\nu\hat{P}_\mu,\hat{P}^\nu\}+\hat{d}_\mu \ . \tag{2.30}$$

$J_{\mu\nu}$ and $\hat{b}_\mu$ also generate a representation of the Lie algebra of $SO(4,1)_{J_{\mu\nu},\hat{b}_\mu}$, i.e.,

$$[\hat{b}_\mu,\hat{b}_\nu]=iJ_{\mu\nu} \ . \tag{2.31}$$

The difference between the representation of $SO(4,1)_{J_{\mu\nu},\hat{b}_\mu}$ and the representation of $SO(4,1)_{J_{\mu\nu},\hat{Y}_\mu}$ is that the $\hat{b}_\mu$ are Hermitian and, more importantly, are defined in terms of the generators of a unitary representation of the Poincaré group $\mathscr{P}_{J_{\mu\nu},P_\mu}$.[30]

If one adds to the $\hat{b}_\mu$ any multiple of P_μ, i.e., $(1/\lambda)P_\mu$, where λ is a constant of dimension MeV (or a Poincaré-invariant operator), one will obtain another representation of SO(4,1) in terms of the Poincaré generators. Therefore, we define the more general operator

$$\hat{B}_\mu=\frac{1}{\lambda}P_\mu-\hat{b}_\mu=\frac{1}{\lambda}P_\mu+\tfrac{1}{2}\{J_{\nu\mu},\hat{P}^\nu\} \ . \tag{2.32}$$

The minus sign in $\hat{b}_\mu$ is insignificant and originates from conventions which are, perhaps, awkward. (Note that $-d_\mu$ is the operator from the c.m. to the particle position and $\hat{b}_\mu$ was defined accordingly.) As will be discussed in paper III in the nonrelativistic limit $\hat{B}_m$ will go over into (-1) times the nonrelativistic mass multiplied by the c.m. position operator, i.e., only in the nonrelativistic limit is $\hat{B}_m=\hat{Y}_m$.

The SO(4,1) generators, $\hat{B}_\mu$, have another significance (also discussed in detail in paper III) which is best seen if one uses the dimensional form

$$B_\mu=\lambda\hat{B}_\mu\equiv P_\mu+\frac{\lambda}{2}\{J_{\nu\mu},\hat{P}^\nu\} \ , \tag{2.33}$$

with the commutation relation

$$[B_\mu,B_\nu]=i\lambda^2J_{\mu\nu} \ . \tag{2.34}$$

The difference between $\hat{B}_\mu$ and B_μ is that the $\hat{B}_\mu$ are the generators of an SO(4,1) rotation by an angle, say ϵ, whereas the B_μ are the generators of motion along a (4,1) de Sitter sphere of radius $R=1/\lambda$ by a distance ϵR measured in cm (the inverse of units of λ or of P_μ). In the elementary limit, $\lambda\to 0$, the de Sitter group contracts to the Poincaré group (the group of Lorentz rotations and translations by a distance measured in cm) and the B_μ go over into the momenta P_μ. Therefore, the $\hat{B}_\mu$ play a dual role: in the nonrelativistic limit they (in dimensionalized form) go over into the c.m. position and in the elementary limit they go over into the momenta.

This $SO(4,1)_{B_\mu,J_{\mu\nu}}$, first introduced in Ref. 31, will play the central role for the quantum relativistic rotator. In the same way as the relativistic mass point is characterized by the eigenvalue m^2 of the Casimir operator $P_\mu P^\mu$ of the Poincaré group $\mathscr{P}_{P_\mu,J_{\mu\nu}}$, the relativistic rotator will be characterized by the eigenvalue $\lambda^2\alpha^2$ of the $SO(4,1)_{B_\mu,J_{\mu\nu}}$ Casimir operator

$$\lambda^2C=B_\mu B^\mu-\frac{\lambda^2}{2}J_{\mu\nu}J^{\mu\nu} \ . \tag{2.35}$$

Our principle postulate is that the constraint relation for the relativistic mass point (elementary particle)[32]

$$P_\mu P^\mu-m^2\approx 0 \tag{2.36}$$

is replaced, for the relativistic rotator, by the constraint relation

$$\Phi\equiv B_\mu B^\mu-\frac{\lambda^2}{2}J_{\mu\nu}J^{\mu\nu}-\lambda^2\alpha^2\approx 0 \ , \tag{2.37}$$

and that the relativistic Hamiltonian for the quantum relativistic rotator must be [according to the rules suggested by constrained Hamiltonian mechanics (see Refs. 12 and 13), and in the spirit of Ref. 4]

$$\begin{aligned}\mathscr{H}&=\phi\left[B_\mu B^\mu-\frac{\lambda^2}{2}J_{\mu\nu}J^{\mu\nu}-\lambda^2\alpha^2\right]\\&=\phi(\lambda^2C-\lambda^2\alpha^2) \ ,\end{aligned} \tag{2.38}$$

where ϕ is an "unknown velocity" or Lagrange multiplier.[33] ϕ will be determined by the choice of the parameter τ (proper c.m. time), with respect to which one has equations of motion, but for the moment we will leave it arbitrary.

When we introduced the definition (2.7), we remarked that P_μ was timelike. Here we see that, indeed, this can be achieved. Equation (2.33) gives a relation between a representation of the $SO(4,1)_{J_{\mu\nu},B_\mu}$ Lie algebra and a unitary representation of the Poincaré group.[34] If the eigenvalue of the SO(4,1) Casimir operator is chosen to be one of the values for the principal series representation, then the Poincaré group representation, related to it by Eq. (2.33), has $P_\mu P^\mu>0$. The form of the Hamiltonian [see Eq. (2.55) below] will make this more obvious.

A justification for the choice of Eq. (2.38) as the Hamiltonian for a quantum relativistic rotator is possible only by correspondence. In paper III we shall show in detail that in the elementary limit $(\lambda\to 0,\alpha\to\infty)$, when the extended relativistic rotator goes over into the structureless relativistic mass point, the Hamilton operator (2.38) reduces to the Hamilton operator for the relativistic mass point and that in the nonrelativistic limit $(1/c\to 0, mc\to\infty)$ (note that when c is not set equal to 1 the eigenvalue of the Casimir operator $P_\mu P^\mu$ of $\mathscr{P}$ is c^2m^2—see also paper III) when the relativistic rotator goes over into the nonrelativistic rotator, the Hamilton operator (2.38) reduces to the Hamilton operator for the nonrelativistic rotator. In the correspondence to classical physics, the Hamilton operator (2.38) corresponds to a classical Hamiltonian which is a special case of the general classical model of Ref. 13.

C. The infinite Γ's

We have already mentioned that the spin is not to be constrained to a fixed value. This means that we will need an operator which transforms between different irreducible representations of the Poincaré group thus describing an observable that performs the transitions between different rotator levels. Such a transition operator (the remnant of the current operator for our simple model) must in the nonrelativistic correspondence be related to the time derivative of the dipole operator d_μ. But, since it is supposed to transform between different irreducible representations of the Poincaré group, $\mathscr{P}$, it cannot be constructed in terms of the algebra of $\mathscr{P}$. In analogy to the Dirac γ matrices, which fulfill an analogous purpose for the theory of the electron, we choose for this operator a Hermitian vector operator Γ_μ, which together with the $S_{\mu\nu}$ (which are generalizations of the $\sigma_{\mu\nu}$) form the simplest unitary (infinite-dimensional) representation of SO(3,2). These Γ_μ and $S_{\mu\nu}$ are defined by the commutation relations (2.2),

$$[S_{\mu\nu},\Gamma_\rho]=i(g_{\nu\rho}\Gamma_\mu-g_{\mu\rho}\Gamma_\nu)\ , \qquad (2.39)$$

and

$$[\Gamma_\mu,\Gamma_\nu]=-iS_{\mu\nu}\ , \qquad (2.40)$$

along with the additional representation relation

$$\{\Gamma_\rho,\Gamma_\sigma\}+\{S_{\rho\mu},S_\sigma{}^\mu\}=-g_{\rho\sigma}\ . \qquad (2.41)$$

The relation (2.41) specifies, of the many irreducible representations[35] of the commutation relations (2.2), (2.39), and (2.40) integrating to $\mathrm{SO}(3,2)_{\Gamma_\mu,S_{\mu\nu}}$, the four Majorana representations whose main feature is that they contain only one irreducible representation of the $\mathrm{SO}(3,1)_{S_{\mu\nu}}$ subgroup. [Equation (2.41) is the analog of the relation $\{\Gamma_\rho,\Gamma_\sigma\}=\frac{1}{2}g_{\rho\sigma}$ for the four-dimensional Dirac case.]

Some of the many consequences of the representation-fixing relation (2.41) are

$$\omega^\mu\equiv\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}S_{\nu\rho}\Gamma_\sigma=0\ , \qquad (2.42)$$

or, equivalently,

$$\epsilon_{\mu\rho\sigma\kappa}\omega^\mu=S_{\rho\sigma}\Gamma_\kappa+S_{\sigma\kappa}\Gamma_\rho+S_{\kappa\rho}\Gamma_\sigma=0\ . \qquad (2.43)$$

The supplementary condition

$$\{\Gamma^\mu,S_{\mu\nu}\}=0\ , \qquad (2.44)$$

which in the classical analog ensures that the electric moment in the velocity rest frame is zero,[36] is also a consequence of the representation relation (2.41). Equation (2.44) follows immediately from

$$\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}=-\tfrac{3}{4}\ , \qquad (2.45)$$

which in turn follows from (2.41).[37]

The Pauli-Lubanski vector and the second Casimir operator of $\mathscr{P}$ can be recast in terms of the $S_{\mu\nu}$ or $\Sigma_{\mu\nu}$ and, for the special case of the Majorana representation, also in terms of the Γ_μ. First, for the general case one can use instead of the spin tensor $\Sigma_{\mu\nu}$ obeying the three supplementary conditions (2.18) the vector operator

$$\hat{w}_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu J^{\rho\sigma}\ . \qquad (2.46)$$

This four-vector can be rewritten, using $\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu M^{\rho\sigma}=0$ and $\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu(d^\rho P^\sigma-d^\sigma P^\rho)=0$, as

$$\hat{w}_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu S^{\rho\sigma}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu\Sigma^{\rho\sigma}\ , \qquad (2.47)$$

and it obeys the supplementary relation

$$P_\mu\hat{w}^\mu=0\ .$$

Therefore,

$$\hat{W}\equiv-\hat{w}_\mu\hat{w}^\mu=\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}-\hat{P}^\rho\hat{P}^\sigma S_{\rho\mu}S_\sigma{}^\mu =\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}-\hat{d}_\mu\hat{d}^\mu\ , \qquad (2.48)$$

or

$$\hat{W}\equiv-\hat{w}_\mu\hat{w}^\mu=\tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}\ . \qquad (2.49)$$

With the Majorana representation relation (2.41), Eqs. (2.49), and (2.47) can be written in the forms

$$\hat{W}=\hat{P}^\rho\hat{P}^\sigma\Gamma_\rho\Gamma_\sigma-\tfrac{1}{4} \qquad (2.50)$$

and

$$\hat{w}_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu\hat{d}^\rho\Gamma^\sigma(\hat{P}\cdot\Gamma)^{-1}\ , \qquad (2.51)$$

respectively. So, in the Majorana representation one finds

$$\hat{d}_\mu\hat{d}^\mu=-\tfrac{3}{4}-\hat{W}=-\tfrac{1}{2}-(\hat{P}^\rho\Gamma_\rho)^2\ . \qquad (2.52)$$

Using the definitions (2.9a) and (2.9b) of $\hat{d}_\mu$, and Eq. (2.43), one can show by a straightforward calculation that

$$\hat{d}_\mu\Gamma_\nu-\hat{d}_\nu\Gamma_\mu=S_{\mu\nu}(\hat{P}\cdot\Gamma)\ . \qquad (2.53)$$

This relation will be important for the interpretation of the motion of the relativistic rotator.

As a consequence of the commutation relation (2.39), one immediately obtains the commutation relation of $\hat{d}_\mu$ with Γ_ρ,

$$[\hat{d}_\mu,\Gamma_\rho]=i(\Gamma_\mu\hat{P}_\rho-g_{\mu\rho}(\hat{P}_\sigma\Gamma^\sigma))\ . \qquad (2.54)$$

By definition, Γ_μ as well as $S_{\mu\nu}$ commute with the operators P_μ, $M_{\mu\nu}$, and Q_μ (at least before the constraint relation has been imposed). Therefore, as we shall discuss in paper II, the representation space will be the direct product of the representation space of $\mathscr{P}_{P_\mu,M_{\mu\nu}}$ and of $\mathrm{SO}(3,2)_{S_{\mu\nu},\Gamma_\nu}$.[38]

To conclude this section, we shall establish a simpler form of the Hamiltonian (2.38) which is valid for the case of the Majorana representation. This will be carried out in two steps. First, we insert (2.33) into (2.38) and use (2.46) and (2.48). Since this does not make use of the Γ_μ's and the Majorana representation, the resulting expression is generally valid. A straightforward, but lengthy, calculation gives

$$\mathscr{H}=\phi(P_\mu P^\mu+\tfrac{9}{4}\lambda^2-\lambda^2\hat{W}-\lambda^2\alpha^2)\ . \qquad (2.55)$$

The form of $\mathscr{H}$ in the Majorana case is then obtained by inserting (2.50) into (2.55) yielding

$$\mathscr{H}=\phi(P_\mu P^\mu+\tfrac{5}{2}\lambda^2-\lambda^2(\hat{P}_\rho\Gamma^\rho)^2-\lambda^2\alpha^2)\ . \qquad (2.56)$$

It is this form of the Hamiltonian which we shall use in Sec. III in deriving the equations of motion.

The constraint relation $\Phi=0$ taken between the Wigner (canonical) basis vectors $|pss_3\rangle$ (which, as we shall see in paper II, will form a basis of the space of physical states with s having either the spectrum $s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$ or $s=0,1,2,\ldots$) leads to the mass formula

$$m^2=\lambda^2(\alpha^2-\tfrac{9}{4})+\lambda^2 s(s+1)\ . \tag{2.57}$$

III. RELATIVISTIC QUANTUM DYNAMICS OF THE QRR

In this section, we derive the equations of motion for the observables of the QRR which have been introduced in Sec. II. The expectation values of the quantum observables will turn out to be in a well-defined correspondence with the classical observables of the classical relativistic rotator described in Ref. 6. Particularly, the expectation value of the particle position operator Q_μ will be found to perform a *Zitterbewegung* about the expectation value of the center-of-mass position operator Y_μ.

In order to determine the time development [here time refers to proper time or any other Lorentz-invariant parameter τ labeling events along a world line for the rotator; its physical meaning will be fixed later on by using the freedom in choosing the unknown velocity, ϕ, in the Hamiltonian (2.56)] we will use the exact analog of classical constrained Hamiltonian mechanics[4,9,10,12] with the Poisson brackets replaced by $-i[\ ,\ ]$. However, for the Hamiltonian in our specific model, we shall use the SO(4,1) Hamiltonian, i.e., Eqs. (2.38) or (2.56).

Therefore, in the spirit of Ref. 4 we calculate the time derivatives of the observables $\mathscr{O}$ by using the defining relations of Sec. II for the observables prior to imposing the constraint $\Phi=0$ [Eq. (2.37)]:

$$\frac{d\mathscr{O}}{d\tau}=\dot{\mathscr{O}}=\frac{1}{i}[\mathscr{O},\mathscr{H}]\ .$$

The equalities of the defining relations and those of the time derivatives correspond, therefore, to the "weak" equalities of classical constrained Hamiltonian mechanics. Only after the derivatives have been computed is the constraint imposed by demanding that $\Phi=0$ be fulfilled. This constraint relation constitutes an additional defining relation for the elements of the algebra of observables which changes their properties. The new properties can no longer be satisfied in the whole space but only in the "physical subspace" which is selected out by the relation $\Phi=0$. In this subspace, the original relations are retained but in addition the operators have now acquired new properties imposed upon them by the application of the constraint relation.

The canonical Hamiltonian is zero for the relativistic rotator so that the full Dirac Hamiltonian is given by the constraint relation:

$$\mathscr{H}=\phi\Phi=\phi(P_\mu P^\mu+\tfrac{5}{2}\lambda^2-\lambda^2(\hat{P}_\rho\Gamma^\rho)^2-\lambda^2\alpha^2)\ . \tag{3.1}$$

A simpler and more obvious Hamiltonian for the constraint relation would be[4,15]

$$\Phi'=\nu(P_\rho\Gamma^\rho-\kappa)\ .$$

However, this constraint is unattainable from a Lagrangian formalism for the corresponding classical relativistic rotator models of Ref. 10. More importantly, this simpler expression does not lead to an experimentally correct mass-spin relationship and can, at best, be used for a single mass-spin level to which, in fact, Ref. 15 is restricted. Moreover, the justification for (3.1) which follows from the correspondence of the elementary and nonrelativistic limits, cannot be given for the simpler constraint Φ'.

First, we shall compute the time derivatives of the "intrinsic observables": $S_{\mu\nu}$, Γ_μ, and $\hat{d}_\mu$. For these observables, only the third term, $-\phi\lambda^2(\hat{P}_\rho\Gamma^\rho)^2$, in the Hamiltonian is relevant since the first term and, obviously, the second and fourth terms commute with them. The calculation is straightforward and the result is

$$\dot{S}_{\mu\nu}=\frac{1}{i}[S_{\mu\nu},\mathscr{H}]$$
$$=-\phi\lambda^2\{\hat{P}^\rho\Gamma_\rho,\hat{P}_\nu\Gamma_\mu-\hat{P}_\mu\Gamma_\nu\}\ . \tag{3.2}$$

Here, the commutation relation (2.39) has been used. Furthermore, one finds

$$\dot{\hat{d}}_\mu=\dot{S}_{\mu\nu}\hat{P}^\nu=-\phi\lambda^2\{\hat{P}^\rho\Gamma_\rho,\Gamma_\mu-(\Gamma_\nu\hat{P}^\nu)\hat{P}_\mu\}\ , \tag{3.3}$$

where the first equality in (3.3) follows since

$$\dot{\hat{P}}_\mu=0\ . \tag{3.4}$$

Further, we have

$$\frac{d}{d\tau}(\hat{P}_\rho\Gamma^\rho)=0;\ \ \dot{P}_\mu=0\ . \tag{3.5}$$

Using the commutation relation (2.40) and the definition (2.9) we obtain

$$\dot{\Gamma}_\mu=+\phi\lambda^2\{\hat{P}^\rho\Gamma_\rho,\hat{d}_\mu\}\ . \tag{3.6}$$

From the definition (2.13), and from (3.2), (3.3), and (3.4) above, it can immediately be seen that

$$\dot{\Sigma}_{\mu\nu}=0\ , \tag{3.7}$$

i.e., the spin tensor is a constant of the motion.

The second derivative of $\hat{d}_\mu$ is now calculated using Eqs. (3.5) and (3.6):

$$\ddot{\hat{d}}_\mu=-(\phi\lambda^2)^2\{\hat{P}^\rho\Gamma_\rho,\{\hat{P}^\sigma\Gamma_\sigma,\hat{d}_\mu\}\}\ . \tag{3.8}$$

This can also be written, using (2.52), as

$$\ddot{\hat{d}}_\mu=-(\phi\lambda^2)^2 2((-\tfrac{1}{2}-\hat{d}_\rho\hat{d}^\rho)\hat{d}_\mu+(\hat{P}_\rho\Gamma^\rho)\hat{d}_\mu(\hat{P}_\sigma\Gamma^\sigma))\ . \tag{3.9}$$

These equations can now be compared with the equations for the corresponding classical observables appearing in Ref. 13. The quantum dynamical equations (2.2), (2.3), and (2.6) are brought into agreement with the classical equations of motion (8.3.1), (8.3.3), and (7.3.3) of Ref. 13, respectively, applying the following correspondence:

$$\underset{v\alpha'\cdot}{\text{classical model}} \leftrightarrow \underset{-\phi\lambda^2\frac{1}{M}\{\hat{P}^\rho\Gamma_\rho,\cdot\}}{\text{quantum model}} . \qquad (3.10)$$

Here v and α' are quantities defined in Ref. 13 and the center dot on the left-hand side represents an expression in terms of the classical observables whereas the center dot on the right-hand side represents the corresponding expression in terms of the quantum observables. In order to make this comparison, note the change in notation between Ref. 13 and the present conventions:

$$\begin{aligned} &-g_{\rho\mu}\leftrightarrow+g_{\rho\mu},\ \ V_\mu\leftrightarrow\Gamma_\mu,\ \ -P^2\leftrightarrow M^2 ,\\ &-d_\mu M\leftrightarrow\hat{d}_\mu,\ \ P\cdot V\leftrightarrow-P\cdot\Gamma . \end{aligned} \qquad (3.11)$$

Equation (3.8) is the operator equation for a simple harmonic motion. To see this explicitly and to calculate the frequency of rotation for the dipole operator $\hat{d}_\mu$, we take the expectation value of (3.8) between physical states. For the simple case of the Majorana representation we will show, in paper II, that the canonical basis vectors $|pss_3\rangle$ are the basis vectors for the space of physical states, the only difference being that now, the spin s is not a fixed number. For the (two) half-integer Majorana representations, the values of s are $s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots,$ and for the (two) integer Majorana representations, the values of s are $s=0,1,2,\ldots.$ Furthermore, for the integer as well as the half-integer Majorana representations one has

$$\hat{P}_\mu\Gamma^\mu\,|pss_3\rangle=\pm\epsilon(s+\tfrac{1}{2})\,|pss_3\rangle , \qquad (3.12)$$

where the Poincaré-invariant quantity $\epsilon=\text{sgn}p_0$ is usually chosen to be $+1$. For the expectation values of (3.8) using these basis vectors,[39] one obtains the result

$$\langle\,|\ddot{\hat{d}}_\mu|\,\rangle=-(\phi\lambda^2)^2 4(s+\tfrac{1}{2})^2\langle\,|\hat{d}_\mu|\,\rangle . \qquad (3.13)$$

One thus sees that the expectation value of the dipole operator performs rotations with an angular frequency given by

$$\omega=\pm\phi\lambda^2 2(s+\tfrac{1}{2}) . \qquad (3.14)$$

This can be completely specified only after we have fixed the meaning of ϕ (or of the world-line parameter τ).

To reproduce the *Zitterbewegung* for Q_μ, we must show that $\dot{\hat{Y}}\sim\hat{P}_\mu$. That this is indeed the case can easily be seen from Eq. (2.19). The time derivative of this equation gives

$$\begin{aligned} \dot{J}_{\mu\nu}&=\dot{\hat{Y}}_\mu\hat{P}_\nu-\dot{\hat{Y}}_\nu\hat{P}_\mu+\dot{\Sigma}_{\mu\nu}\\ &=\dot{Y}_\mu P_\nu-\dot{Y}_\nu P_\mu+\dot{\Sigma}_{\mu\nu} . \end{aligned} \qquad (3.15)$$

From the form (2.38) of $\mathscr{H}$ it immediately follows that

$$\dot{J}_{\mu\nu}=0 , \qquad (3.16)$$

and also that

$$\dot{B}_\mu=0 , \qquad (3.17)$$

because $\lambda^2 C$ is the $SO(4,1)_{B_\mu,J_{\mu\nu}}$ Casimir operator. Equations (3.16) and (3.7) then show that $\dot{Y}_\mu$ must be parallel to P_μ (and that $\dot{\hat{Y}}_\mu$ is parallel to $\hat{P}_\mu$). This, then, establishes the *Zitterbewegung:* As τ proceeds, the expectation value $\langle\,|Y_\mu|\,\rangle$ of the center-of-mass operator follows a straight world line in a direction parallel to $\langle\,|P_\mu|\,\rangle$, and the position $\langle\,|Q_\mu|\,\rangle=\langle\,|Y_\mu|\,\rangle-\langle\,|d_\mu|\,\rangle$ performs, according to Eq. (3.13), a helical motion about this world line with a rotational frequency given by Eq. (3.14). See Fig. 1 for a complete classical description of the motion.

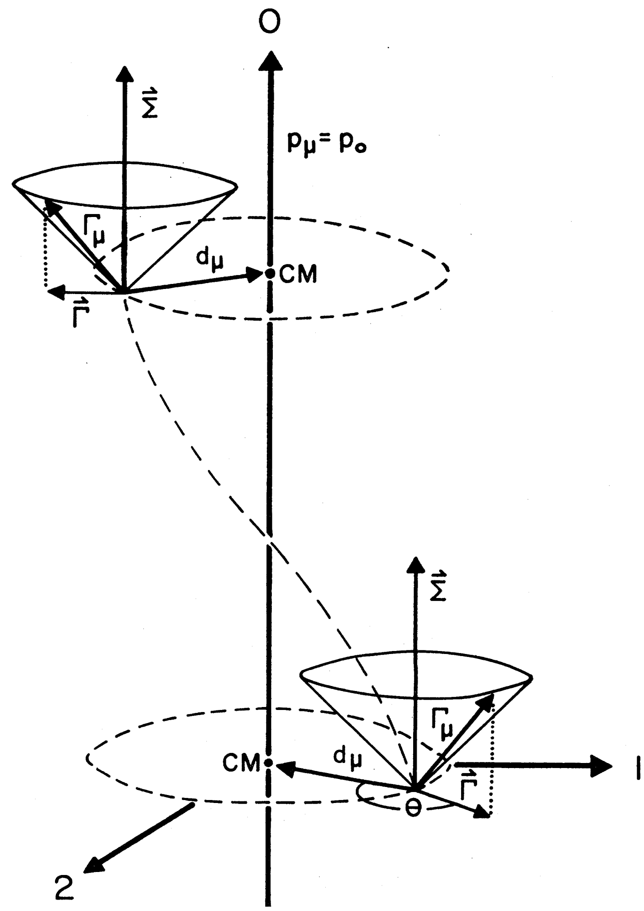

FIG. 1. Classical description of the motion for states at rest. In the Majorana representation, both $\vec{d}^{\,2}$ and $\vec{\Gamma}^{\,2}$ are constants of the motion for states at rest ($d_0=0$ and Γ_0=constant). Restricting Eq. (2.53) to spatial indices, we have $\vec{d}\times\vec{\Gamma}=\vec{S}(P_0\Gamma^0)=\vec{\Sigma}(P_0\Gamma^0)$ (where we have used the fact that for states at rest, $\vec{S}=\vec{\Sigma}$). From Eqs. (3.5) and (3.7) it follows that $\vec{d}\times\vec{\Gamma}$ is a constant of the motion and therefore, $|\vec{d}\times\vec{\Gamma}|=|\vec{d}|\,|\vec{\Gamma}|\sin\theta$=constant. Choosing the spatial components such that $d_3=0$ and using Eq. (3.13) we find that d_μ and, therefore, $\vec{\Gamma}$ rotate around the world line in the $1-2$ plane with an angular frequency ω keeping the angle θ between them a constant. Also depicted in the figure is the spin angular momentum three-vector $\vec{\Sigma}$ where, for convenience, we have chosen its third spatial component to point in the 0th direction.

In the elementary limit, $\lambda\to 0$ (see paper III) $\dot{\hat{d}}$ and $\ddot{\hat{d}}$ go to zero. That is, a relativistic mass point does not perform *Zitterbewegung.*

Now, we will explicitly compute the velocity of the center of mass. First we calculate $\dot{Q}_\mu=-i[Q_\mu,\mathscr{H}]$. Using (2.5), (2.8), and the fact that at this stage

$$[\Gamma_\sigma,Q_\mu]=0\ , \tag{3.18}$$

we obtain the result

$$\dot{Q}_\mu=-2\phi P_\mu+\phi\lambda^2\{\hat{P}\cdot\Gamma,\Gamma_\mu-(\hat{P}\cdot\Gamma)\hat{P}_\mu\}M^{-1}\ . \tag{3.19}$$

From Eq. (3.3)

$$\dot{d}_\mu=\dot{\hat{d}}_\mu M^{-1}=\phi\lambda^2\{\hat{P}\cdot\Gamma,-\Gamma_\mu+(\hat{P}\cdot\Gamma)\hat{P}_\mu\}M^{-1}\ . \tag{3.20}$$

Adding Eqs. (3.19) and (3.20) yields

$$\dot{Y}_\mu=-2\phi P_\mu\ . \tag{3.21}$$

This is the desired result: the velocity of the center of mass is parallel and proportional to the momentum P_μ, i.e., although the Y_μ do not commute with each other, the operators $\dot{Y}_\mu$ do.

We can now fix the parameter τ by demanding that

$$\dot{Y}_\mu\dot{Y}^\mu=1\ . \tag{3.22}$$

We can regard this condition as a way of defining proper time in the quantum theory even though, strictly speaking, there is no well-defined world line for the center of mass in quantum theory [cf. also (2.26)]. From this relation, we obtain

$$\phi=\pm\tfrac{1}{2}M^{-1}\ , \tag{3.23}$$

for which, according to (3.21), the $(-)$ sign is the correct choice:

$$\dot{Y}_\mu=P_\mu/M=\hat{P}_\mu\ . \tag{3.23a}$$

Thus, the unknown velocity is (up to the factor $-\frac{1}{2}$) given by the mass operator, which is a constant of the motion. That it also does not depend upon other constants of the motion such as s is not an obvious result.

Inserting (3.23) into (3.14) gives the expression for the angular frequency of the spiral motion

$$\omega=\frac{\lambda^2}{m}(s+\tfrac{1}{2})=\lambda\left[\frac{1}{1+\dfrac{\alpha^2-\frac{10}{4}}{(s+\frac{1}{2})^2}}\right]^{1/2}\ , \tag{3.24}$$

where for the second quality, we have used the mass formula (2.57). For large values of spin s, Eq. (3.24) implies that

$$\omega\to\lambda\ . \tag{3.25}$$

As will be discussed briefly in paper II, the empirical value of α^2 for mesons is $\alpha^2_{\rm meson}\approx\frac{9}{4}$, and for baryons (proton) $\alpha^2_{\rm baryon}\approx\frac{18}{4}$. Therefore,

$$\omega^{\rm baryon}\approx\lambda\left[\frac{1}{1+\dfrac{2}{(s+\frac{1}{2})^2}}\right]^{1/2}<\lambda\ , \tag{3.26}$$

and

$$\omega^{\rm meson}\approx\lambda\left[\frac{1}{1-\dfrac{1}{4(s+\frac{1}{2})^2}}\right]^{1/2}>\lambda\ . \tag{3.27}$$

Therewith, we see from Eqs. (3.19), (3.23a), and (3.13) that the particle position performs a helical motion with a frequency ω given by (3.24) around the center of mass, which in turn moves on a straight world line parallel to P_μ.

We can also obtain an idea of the "size" of the QRR, i.e., the radius of the spiral given by $\sqrt{\vec{d}^{\,2}}$. From (2.52), taken between rest states $|p=0ss_3\rangle$, it follows that

$$d^2\equiv\langle s_3sp=0|-d_\mu d^\mu|0ss_3\rangle$$
$$=\langle s_3s0|\vec{d}^{\,2}|0ss_3\rangle \tag{3.28}$$

has the spectrum

$$d^2=[\tfrac{3}{4}+s(s+1)]\frac{1}{m^2}\ . \tag{3.29}$$

With the mass formula (2.57), one obtains

$$d=\frac{1}{\lambda}\left[\frac{1}{1+\dfrac{\alpha^2-3}{(s+\frac{1}{2})^2+\frac{1}{2}}}\right]^{1/2}\to\frac{1}{\lambda}\ , \tag{3.30}$$

which is seen to be of the order of $1/\lambda$ and approaches $1/\lambda$ for large values of s. $1/\lambda=R$ was the radius of the de Sitter space in which $SO(4,1)_{B_\mu,J_{\mu\nu}}$ acts as the group of motion. Empirically, the value of λ can be obtained by comparing the mass formula with the experimental data and is found to be (see paper III)

$$\lambda\approx0.53\ \text{GeV}\ ;$$

therefore

$$1/\lambda=R\approx0.37\times10^{-13}\ \text{cm}\ . \tag{3.31}$$

Thus, it is the *Zitterbewegung* which causes the hadronic size and modifies the affine (Poincaré) description for the extensionless top into that of the de Sitter description[18] which is a de Sitter space of radius $R=0.37$ fermi, i.e., an extended object.

Too great an emphasis should not be put on the specific forms of (3.24) and (3.30) since they came from our particular model assumptions which can only represent a rough approximation to reality. In particular, the choice of the Majorana representation and the use of a constant value for λ is probably too crude of an approximation and fine-structure effects will probably require the replacement of λ by an operator that is a function of the constants of motion. In this case the form of Eqs. (3.24), (3.30), and (2.57) will also undergo minor changes. However, our general conclusion that the size of a hadron (being of the order of 10^{-13} cm) is the radius of the helical motion that the particle position executes with a speed $\omega\cdot d$ (of the order of the velocity of light) around the center of mass will probably remain unaffected by these fine-structure effects.

From Eqs. (3.14), (3.29), and (2.57) one can obtain the velocity with which the expectation value of the particle position circles in the center-of-mass frame:

$$\omega\cdot d=\frac{\lambda^2}{m^2}(s+\tfrac{1}{2})[\tfrac{1}{2}+(s+\tfrac{1}{2})^2]^{1/2}$$
$$=\frac{(s+\frac{1}{2})[(s+\frac{1}{2})^2+\frac{1}{2}]^{1/2}}{[\alpha^2-\frac{10}{4}+(s+\frac{1}{2})^2]}\;.$$

For large values of s [when (2.57) is probably no longer a good approximation] $\omega\cdot d$ goes to 1 (the velocity of light), and for $\alpha^2\geq\frac{11}{4}$, $\omega\cdot d$ is always less than 1. For baryons, the phenomenological value of α^2 is approximately $\frac{18}{4}$ or larger. Thus the velocity in the c.m. frame is, for all spins, a fraction of the velocity of light and approaches it for $s\rightarrow\infty$.

For mesons, the phenomenological value for $\alpha^2\approx\frac{9}{4}$ or $\frac{10}{4}$, for which the value of $\omega\cdot d>1$. If one tries to trace the origin of this disturbing result, one finds that it is a direct consequence of the value $-\frac{3}{4}$ in Eq. (2.52) which in turn follows from (2.45), i.e., it directly follows from our choice of the Majorana representation. Equation (2.52) is not valid for the classical model where $d^2=\hat{W}/m^2$. For the general case, (2.52) must be replaced by (2.48) where $\frac{1}{2}S_{\mu\nu}S^{\mu\nu}$ is not diagonal in the Wigner basis (3.12). Then d^2, defined by Eq. (3.29), is no longer an eigenvalue as it is in Eq. (3.28) and the above-mentioned difficulty can probably be avoided for any phenomenologically reasonable value of α^2.

In the elementary limit $\lambda\rightarrow 0$ and $\alpha\rightarrow\infty$ (see paper III), we see from Eq. (3.19) that

$$\dot{Q}_\mu\rightarrow\dot{Q}_\mu^{\,\text{elem}}\equiv-2\phi P_\mu=\dot{Y}_\mu\;.$$

There is no *Zitterbewegung* for the relativistic mass point.

In the first part of this paper we defined the observables by their algebraic relations and calculated immediate consequences thereof. In order to obtain numbers we already made use of some of the properties of the representations of the observables. In a following paper we will derive these representations and, although it will not contain many intuitively appealing results, it will be the core of this paper since it demonstrates that such an algebra really does exist as an algebra of operators in a space which will turn out to be a precisely specified (by the Majorana representation relation) direct sum of irreducible representation spaces of the Poincaré group. It is this result that establishes the already anticipated fact that hadrons are different mass-spin levels of the QRR.

ACKNOWLEDGMENTS

Support from the Department of Energy is gratefully acknowledged. This work was begun while one of the authors (A.B.) was a guest of the Max Planck Institute für Physik in Munich and the Institut für Physik Universität Würzburg. He would like to express his gratitude to Professor H. P. Dürr, Professor K. Kraus, and the Alexander von Humboldt Foundation for the hospitality extended to him. We gratefully acknowledge valuable discussions on the subject of this paper with L. C. Biedenharn and H. van Dam.

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[20]The concept of spectrum-generating groups was first introduced by A. O. Barut and A. Bohm, Phys. Rev. 139, B1107 (1965) (under the name dynamical groups) and by Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Rev. Lett. 15, 1041 (1965).

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[23]See Chap. VIII of Ref. 13, or Chap. II, Sec. 6 of Ref. 6.

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[26]Compare Sec. I.1 of Ref. 6 and Sec. IX.3 of Ref. 14.

[27]Compare Sec. II.8 of Ref. 6.

[28]See Chap. VIII of Ref. 13. This operator is called the mean center of mass in Ref. 7.

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$$-d_\mu = x_\mu - \tfrac{1}{2}M^{-1}\{\hat{P}_\mu\{P^\nu, x_\nu\}\} - b_\mu \ .$$

They are related to the Hermitized boosted "generalized space coordinate" $\tilde{x}_\mu(x)$ considered in H. Matsumoto, G. Semenoff, and H. Umezawa, University of Alberta report, 1983 (unpublished); O. Steinmann, Z. Phys. C 6, 139 (1980); H. Matsumoto, G. Semenoff, and H. Umezawa, Prog. Theor. Phys. 69, 1631 (1983).

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[32]We use the symbol $\approx$ because this constraint has nonvanishing commutators as will be discussed in paper III. For the elementary particle with a fixed value of spin, one has in addition to the constraint (2.36) further constraints fixing the value of spin, cf., e.g., Ref. 4. Since we do not want to fix the value for the spin, we do not use those additional constraints.

[33]See, e.g., Chap. V of Ref. 13 or Ref. 12.

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[36]See Ref. 6, Chap. II.7.

[37]In Ref. 6, the classical analog of the quantity $(-\frac{1}{2}S_{\mu\nu}S^{\mu\nu})^{1/2}$ is called "intrinsic spin," i.e., angular momentum of the system in which the particle (charge) velocity is zero. It is a very particular feature of the Majorana representation that $\frac{1}{2}S_{\mu\nu}S^{\mu\nu}$ is an invariant. For the more complicated representations of SO(3,2), which probably give a better picture of reality, the spectrum of $\frac{1}{2}S_{\mu\nu}S^{\mu\nu}$ is nontrivial (and discrete for some of them, Ref. 35).

[38]Before the constraint is imposed, this is a special case of a "relativistic symmetry " $\mathscr{P}_{P_\mu, J_{\mu\nu}} \sim$SO(3,2). P. Budini and C. Fronsdal, Phys. Rev. Lett. 14, 968 (1965); V. Ottoson, A. Kihlberg, and J. S. Nilsson, Phys. Rev. 137, B658 (1965).

[39]In actuality, the physical states are given by

$$\phi = \int |pss_3\rangle \phi(p)[d^3p/2E(p)] \ ,$$

where $|\phi(p)|^2$ is a narrow momentum distribution. Thus, Eq. (3.13) involves singular expressions since the $|pss_3\rangle$ are generalized eigenvectors. However, properly interpreted, this will not give rise to any problems.