Second-order contributions to relativistic time delay in the parametrized post-Newtonian formalism

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Using a parametrized expansion of the solar metric to second order in the Newtonian potential, we calculate the relativistic delay in the round-trip travel time of a radar signal reflected from a nearby planet. We find that one second-order contribution to the delay is on the order of ten nanoseconds, which is comparable to the uncertainties in present-day experiments involving the Viking spacecraft.

I. INTRODUCTION

In an earlier series of articles[1,2] we discussed light propagation in the solar system within the context of the parametrized post-Newtonian (PPN) formalism and presented a useful extension of the PPN form of the solar metric. We demonstrated that knowledge of light propagation to any given order requires knowledge of every component of the metric to that same order.[1] In contrast, to understand particle motion within the solar system to any given order requires that different components of the solar metric be known to different orders. For example, the PPN metric can be used to study particle motion to second order in $GM/c^2R \equiv \epsilon^2 \approx 2\times 10^{-6}$ (i.e., to order ϵ^4), where M and R are the mass and radius of the Sun, even though some components of the PPN metric are known only to first order (i.e., to order ϵ^2).

We extended the PPN metric until each component was given in a parametrized form to order ϵ^4 to form the parametrized post-linear (PPL) metric.[2] With the PPL metric we were able to calculate second-order contributions to gravitational light deflection.[2,3] Here, we will use the PPL metric to compute the second-order contributions to relativistic time delay.

To first order (i.e., order ϵ^2) the solar metric is linear in the Newtonian potential (and thus sometimes referred to as the linearized metric) and is given by

$$g_{00}=-1+2\frac{M}{r}\ , \qquad (1.1)$$

$$g_{0i}=0\ , \qquad (1.2)$$

$$g_{ij}=\delta_{ij}\left[1+2\gamma\frac{M}{r}\right]\ , \qquad (1.3)$$

where r is related to the usual PPN coordinates x, y, z by $r=(x^2+y^2+z^2)^{1/2}$, and where γ is a constant equal to one in general relativity. (Note that from this point on we will be working in geometrized units.) The corresponding round-trip travel time of a radar signal reflected from a nearby planet is (neglecting the orbital and rotational motions of the Earth) given to order ϵ^2 by

$$\Delta\tau=2(x_R-x_T)\left[1-\frac{M}{r_T}\right]+2(1+\gamma)M\ln\frac{x_R+r_R}{x_T+r_T}\ . \qquad (1.4)$$

Here x_R, r_R and x_T, r_T are the values of the coordinates x and r when evaluated at the positions of the reflector and transmitter, respectively. [The value of $\Delta\tau$ was first calculated by Shapiro.[4] His result appears in a different form than that of Eq. (1.4), however, because he used a different set of coordinates.]

There are higher-order contributions to the solar metric well known from the PPN form of the metric which are often formally treated as though they are of order ϵ^3.[5] However, a close examination of these contributions reveals that for any realistic model of the Sun such terms are never as large as they could be (ϵ^3) and, in fact, are never larger than ϵ^4.[1] Thus, the first post-linear contributions to light propagation are of order ϵ^4 and we must, unfortunately, systematically expand each component of the metric to that order. We have previously presented just such an expansion.[2] After choosing a specific gauge to work in and assuming the solar gravitational field to be stationary, the resulting parametrized post-linear (PPL) metric has the following components in the PPN coordinate frame:

$$g_{00}=-1+2\frac{M}{r}\left[1-J_2\frac{R^2}{r^2}\frac{3\cos^2\theta-1}{2}\right]-2\beta\frac{M^2}{r^2}\ , \qquad (1.5)$$

$$g_{0i}=-\frac{7\Delta_1+\Delta_2}{4}\frac{\epsilon_{ijk}J_jx_k}{r^3}\ , \qquad (1.6)$$

$$g_{ij}=\delta_{ij}\left[1+2\gamma\frac{M}{r}\left[1-J_2\frac{R^2}{r^2}\frac{3\cos^2\theta-1}{2}\right]+\frac{3}{2}\Lambda\frac{M^2}{r^2}+2\frac{A}{r}\right]\ . \qquad (1.7)$$

Here J_2 is the dimensionless quadrupole moment parameter of the Sun, $\vec{J}$ is its total angular momentum, and θ is the angle between the symmetry axis of the Sun and the field point $\vec{r}$. Assuming the Sun to be symmetric about its angular momentum vector $\vec{J}$, we use[3]

$$\cos\theta=\frac{\vec{J}\cdot\vec{r}}{|\vec{J}|\,|\vec{r}|}\ . \qquad (1.8)$$

γ, β, Δ_1, Δ_2, and Λ are arbitrary parameters each of which is equal to one in general relativity. A is a quantity with units of centimeters given by

$$A \equiv \frac{1}{2}\left[\Upsilon_1 \int \rho_0(r)\Pi(r)d^3x + \Upsilon_2 \int \rho_0(r)U(r)d^3x + \Upsilon_3 \int P(r)d^3x + \Upsilon_4 \int \rho_0(r)\frac{1}{r}\left[\int_0^r \rho_0(r')d^3x'\right]d^3x\right] , \tag{1.9}$$

where Υ_1, Υ_2, Υ_3, and Υ_4 are arbitrary parameters each of which vanishes in general relativity and where ρ_0, Π, and P are the baryon mass density of the Sun, the specific internal energy density of the Sun, and the pressure within the Sun, respectively.

We will now use this PPL metric to calculate the relativistic time delay to order ϵ^4. We will calculate only the relativistic effect of the solar gravitational field. Thus, we will consider only an observer at rest with respect to the Sun. The motions of the Earth are, in fact, not negligible; however, they are quite difficult to deal with in full generality by analytic means. Furthermore, we will neglect contributions due to the solar corona, the Earth's gravitational field, etc. These effects can be and have been dealt with; we will not discuss them here.

II. THE BEAM PATH

Because the radar signal will travel along a null curve of the space-time, the beam path will satisfy

$$ds^2=0 . \tag{2.1}$$

When terms smaller than ϵ^4 are neglected this can immediately be rewritten in the form

$$0=g_{00}\left[\frac{dt}{dx}\right]^2+2g_{0x}\left[\frac{dt}{dx}\right]+g_{xx}+g_{yy}y'^2+g_{zz}z'^2 , \tag{2.2}$$

where

$$y'\equiv\frac{dy}{dx} , \tag{2.3}$$

$$z'\equiv\frac{dz}{dx} . \tag{2.4}$$

The calculation of the round-trip travel time involves little more than solving for dt/dx and integrating. However, because of the last two terms we must first find the equations of motion $y(x)$ and $z(x)$ of a photon in the radar beam. Fortunately, we need $y(x)$ and $z(x)$ only to order ϵ^2 to find dt/dx to order ϵ^4. Thus, we may calculate the photon trajectory using the linearized metric [expressions (1.1)—(1.3)].

To simplify the calculations let us first orient the PPN coordinate axes so that both the transmitter and reflector lie in the $z=0$ surface. Because the linearized metric is spherically symmetric, the photon trajectory will satisfy $z=0$. Next, further rotate the coordinate axes so that we have

$$y'_T\equiv\frac{dy}{dx}(x=x_T)=0 , \tag{2.5}$$

and such that the transmitter has coordinates x_T,y_T with $x_T<0$, $y_T>0$ and the reflector has coordinates x_R,y_R with $x_R>0$. [Note that this orientation of the axes is quite special. If the observer moves at all during the time that the beam is in transit to and from the reflector, then the return path of the beam will not coincide with the original path. Thus, the return beam would not satisfy Eq. (2.5), but would satisfy a time-dependent boundary condition.]

Now, to find the equation of motion of the photon we begin with the dynamical form of the Lagrangian,

$$L=\tfrac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} , \tag{2.6}$$

where λ is an affine parametrization of the trajectory. Because the metric is time independent, we immediately have

$$p_0=\frac{\partial L}{\partial\left[\dfrac{dx^0}{d\lambda}\right]}=\text{constant}\equiv E . \tag{2.7}$$

In addition, the fact that the trajectory is null gives us

$$0=g^{\alpha\beta}p_\alpha p_\beta . \tag{2.8}$$

By imposing these conditions [(2.7) and (2.8)], the equation of motion takes the form

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left[\frac{\partial g_{xx}}{\partial y}+\frac{\partial g_{00}}{\partial y}\right] \tag{2.9}$$

to order ϵ^2. When we substitute the linearized form of the metric [Eqs. (1.1) and (1.3)], we obtain (to order ϵ^2)

$$\frac{d^2y}{dx^2}=-(1+\gamma)M\frac{y}{r^3} . \tag{2.10}$$

The solution of this equation to order ϵ^2 which satisfies the appropriate boundary conditions is

$$y=y_T+(1+\gamma)\frac{M}{y_T}[(x_T^2+y_T^2)^{1/2}-(x^2+y_T^2)^{1/2}] + (1+\gamma)\frac{M}{y_T}(x-x_T)\frac{x_T}{(x_T^2+y_T^2)^{1/2}} . \tag{2.11}$$

We thus have

$$y'=(1+\gamma)\frac{M}{y_T}\left[\frac{x_T}{(x_T^2+y_T^2)^{1/2}}-\frac{x}{(x^2+y_T^2)^{1/2}}\right] \tag{2.12}$$

for use in Eq. (2.2).

Before we leave the subject of the trajectory let us point out that use of Eq. (2.11) in $r=(x^2+y^2)^{1/2}$ gives

$$r=(x^2+y_T^2)^{1/2}\left[1+2(1+\gamma)\frac{M}{(x^2+y_T^2)^{1/2}}f(x) + (1+\gamma)^2\frac{M^2}{y_T^2}f^2(x)\right]^{1/2} , \tag{2.13}$$

where

$$f(x)\equiv\frac{1}{(x^2+y_T^2)^{1/2}}\left[(x_T^2+y_T^2)^{1/2}-(x^2+y_T^2)^{1/2} + (x-x_T)\frac{x_T}{(x_T^2+y_T^2)^{1/2}}\right] . \tag{2.14}$$

It is not difficult to prove that $f(x)$ is a negative, monotonically decreasing function of x for $x>x_T$, which asymptotically approaches a finite value as $x\to\infty$. In fact, $f(x)$ satisfies

$$|f(x)|\le 2 \tag{2.15}$$

for all $x>x_T$. Because of this finite upper bound on $|f(x)|$, and because

$$2(1+\gamma)\frac{M}{(x^2+y_T{}^2)^{1/2}}\lesssim\epsilon^2 , \tag{2.16}$$

$$(1+\gamma)^2\frac{M^2}{y_T{}^2}\lesssim\epsilon^4 , \tag{2.17}$$

we have

$$r=(x^2+y_T{}^2)^{1/2}\left[1+(1+\gamma)\frac{M}{(x^2+y_T{}^2)^{1/2}}f(x)+O(\epsilon^4)\right] . \tag{2.18}$$

This relation will be very useful in the calculations to come. As further consequences of Eq. (2.11) we point out that

$$\frac{y}{r}=\frac{y_T}{r}+O(\epsilon^2) , \tag{2.19}$$

and that

$$\frac{R^2}{y_R{}^2}\left[1-\frac{x_R}{r_R}\right]=\frac{R^2}{y_T{}^2}\left[1-\frac{x_R}{r_R}\right]+O(\epsilon^2) . \tag{2.20}$$

These facts will also be used later.

III. PROPER-TIME CALCULATION

Now we wish to calculate τ_{TR}, the lapse of proper time (as measured by the stationary observer) between transmission and reflection of the radar signal. First, we will calculate t_{TR}, the lapse of coordinate time between transmission and reflection. To do this we solve Eq. (2.2) for dt/dx. After substituting the PPL form of the metric [Eqs. (1.5)—(1.7)], using Eq. (2.19) and expanding, the result is given to order ϵ^4 by

$$\frac{dt}{dx}=1+(1+\gamma)U+[\tfrac{1}{2}(1+\gamma)(3-\gamma)-\beta+\tfrac{3}{4}\Lambda]\frac{M^2}{r^2}+\frac{A}{r}+\frac{7\Delta_1+\Delta_2}{4}\frac{J_z y_T}{r^3}+\frac{1}{2}y'^2 , \tag{3.1}$$

where U is the Newtonian potential to order ϵ^4,

$$U=\frac{M}{r}\left[1-J_2\frac{R^2}{r^2}\frac{3\cos^2\theta-1}{2}\right] . \tag{3.2}$$

Now we use form (1.8) of $\cos\theta$, substitute expressions (2.12) for y' and (3.2) for U into Eq. (3.1) and use Eqs. (2.18) and (2.19) to rewrite every function of r as a function of x. The result is, to order ϵ^4,

$$\begin{aligned}\frac{dt}{dx}={}&1+(1+\gamma)\frac{M}{(x^2+y_T{}^2)^{1/2}}-(1+\gamma)^2M^2\left[\frac{y_T{}^2}{(x_T{}^2+y_T{}^2)^{1/2}}\frac{1}{(x^2+y_T{}^2)^{3/2}}-\frac{1}{x^2+y_T{}^2}+\frac{x_T}{(x_T{}^2+y_T{}^2)^{1/2}}\frac{x}{(x^2+y_T{}^2)^{3/2}}\right]\\&-\tfrac{1}{2}(1+\gamma)J_2MR^2\left[3\frac{J_x{}^2}{J^2}\frac{x^2}{(x^2+y_T{}^2)^{5/2}}+6\frac{J_xJ_y}{J^2}\frac{y_Tx}{(x^2+y_T{}^2)^{5/2}}+3\frac{J_y{}^2}{J^2}\frac{y_T{}^2}{(x^2+y_T{}^2)^{5/2}}-\frac{1}{(x^2+y_T{}^2)^{3/2}}\right]\\&+[\tfrac{1}{2}(1+\gamma)(3-\gamma)-\beta+\tfrac{3}{4}\Lambda]\frac{M^2}{x^2+y_T{}^2}+\frac{A}{(x^2+y_T{}^2)^{1/2}}+\frac{7\Delta_1+\Delta_2}{4}\frac{J_zy_T}{(x^2+y_T{}^2)^{3/2}}\\&+\tfrac{1}{2}(1+\gamma)^2\frac{M^2}{y_T{}^2}\left[\frac{x_T{}^2}{x_T{}^2+y_T{}^2}-2\frac{x_T}{(x_T{}^2+y_T{}^2)^{1/2}}\frac{x}{(x^2+y_T{}^2)^{1/2}}+\frac{x^2}{x^2+y_T{}^2}\right] .\end{aligned} \tag{3.3}$$

Next, we integrate and simplify, using Eqs. (2.18)—(2.20), to obtain, to order ϵ^4,

$$\begin{aligned}t_{TR}={}&x_R-x_T+(1+\gamma)M\ln\frac{x_R+r_R}{x_T+r_T}+[2(1+\gamma)-\beta+\tfrac{3}{4}\Lambda]\frac{M^2}{y_T}\left[\arctan\frac{x_R}{y_T}-\arctan\frac{x_T}{y_T}\right]\\&+\tfrac{1}{2}(1+\gamma)^2\frac{M^2}{y_T{}^2}\left\{(x_R-x_T)\left[\left[1-\frac{x_T}{r_T}\right]^2+2\frac{x_Rx_T-y_T{}^2}{r_Rr_T}\right]+2(r_T-r_R)\left[\frac{x_T}{r_T}+\frac{x_R}{r_R}-1\right]\right\}\\&+A\ln\frac{x_R+r_R}{x_T+r_T}+\frac{7\Delta_1+\Delta_2}{4}\frac{J_z}{y_T}\left[\frac{x_R}{r_R}-\frac{x_T}{r_T}\right]\\&+\tfrac{1}{2}(1+\gamma)J_2\frac{MR^2}{y_T{}^2}\left\{\left[\left[\frac{J_x}{J}\right]^2-\left[\frac{J_y}{J}\right]^2\right]\left[\frac{x_T{}^3}{r_T{}^3}-\frac{x_R{}^3}{r_R{}^3}\right]+2\frac{J_xJ_y}{J^2}\left[\frac{y_T{}^3}{r_R{}^3}-\frac{y_T{}^3}{r_T{}^3}\right]+\left[1-3\left[\frac{J_y}{J}\right]^2\right]\left[\frac{x_R}{r_R}-\frac{x_T}{r_T}\right]\right\} .\end{aligned} \tag{3.4}$$

The proper time τ of the observer is related to the coordinate time t by

$$-d\tau^2 = g_{\alpha\beta} dx^\alpha dx^\beta \,, \tag{3.5}$$

which gives (to order ϵ^4)

$$\frac{d\tau}{dt} = (-g_{00} - g_{ij} v^i v^j)^{1/2} \,, \tag{3.6}$$

where the metric is to be evaluated along the world line of the observer and where

$$v^i \equiv \frac{dx^i}{dt} \,. \tag{3.7}$$

Using the PPL metric [Eqs. (1.5) and (1.7)], we can rewrite this to order ϵ^4 as

$$\frac{d\tau}{dt} = \left[1 - 2U + 2\beta \frac{M^2}{r^2} - \left[1 + 2\gamma \frac{M}{r}\right] v^2\right]^{1/2} \,. \tag{3.8}$$

Note that if the Earth were in the Sun's equatorial plane ($\theta = \pi/2$) and in a circular orbit with

$$v = \omega r \,, \tag{3.9}$$

$$r = \text{constant} \,, \tag{3.10}$$

the right-hand side of Eq. (3.8) would simply be a constant. However, the radial motion of the Earth is nonnegligible as a simple calculation makes clear. Since

$$\frac{\text{radius of Earth at aphelion} - \text{radius of Earth at perihelion}}{\frac{1}{2}\text{year}} \approx 10^{-6} c \,, \tag{3.11}$$

we have

$$\frac{\text{time delay due to Earth's radial motion}}{\text{total round-trip travel time}} \sim 10^{-6} \approx \epsilon^2 \,. \tag{3.12}$$

It is because we cannot neglect the complexities of the Earth's orbit that we have restricted our consideration to an observer at rest with respect to the Sun. For such an observer we simply have, from Eq. (3.6),

$$\tau_{TR} = \sqrt{-g_{00}}\,|_T\, t_{TR} \,. \tag{3.13}$$

Thus, to order ϵ^4, τ_{TR} is given by

$$\begin{aligned}
\frac{\tau_{TR}}{R} = \frac{x_R - x_T}{R} \Bigg\{ & 1 - \frac{M}{R}\frac{R}{r_T} + \tfrac{1}{2}(1+\gamma)^2 \frac{M^2}{R^2}\frac{R^2}{{y_T}^2} \left[\left[1 - \frac{x_T}{r_T}\right]^2 + 2\frac{x_R x_T - {y_T}^2}{r_R r_T} \right] + \tfrac{1}{2}(2\beta - 1)\frac{M^2}{R^2}\frac{R^2}{{r_T}^2} \Bigg\} \\
& + \left[(1+\gamma)\frac{M}{R}\left[1 - \frac{M}{R}\frac{R}{r_T}\right] + \frac{A}{R} \right] \ln \frac{x_R + r_R}{x_T + r_T} + [2(1+\gamma) - \beta + \tfrac{3}{4}\Lambda] \frac{M^2}{R^2}\frac{R}{y_T} \left[\arctan\frac{x_R}{y_T} - \arctan\frac{x_T}{y_T} \right] \\
& + (1+\gamma)^2 \frac{M^2}{R^2}\frac{R}{y_T}\frac{r_T}{y_T}\left[1 - \frac{r_R}{r_T}\right]\left[\frac{x_T}{r_T} + \frac{x_R}{r_R} - 1\right] + \frac{7\Delta_1 + \Delta_2}{4}\frac{J_z}{R^2}\frac{R}{y_T}\left[\frac{x_R}{r_R} - \frac{x_T}{r_T}\right] \\
& + \tfrac{1}{2}(1+\gamma) J_2 \frac{M}{R}\frac{R^2}{{y_T}^2} \Bigg\{ \left[\left[\frac{J_x}{J}\right]^2 - \left[\frac{J_y}{J}\right]^2 \right] \left[\frac{{x_T}^3}{{r_T}^3} - \frac{{x_R}^3}{{r_R}^3} \right] + 2\frac{J_x J_y}{J^2}\left[\frac{{y_T}^3}{{r_R}^3} - \frac{{y_T}^3}{{r_T}^3}\right] \\
& + \left[1 - 3\left[\frac{J_y}{J}\right]^2\right]\left[\frac{x_R}{r_R} - \frac{x_T}{r_T}\right] \Bigg\} \,,
\end{aligned} \tag{3.14}$$

where we have dropped one term proportional to

$$J_2 \frac{M}{R}\frac{R^3}{{r_T}^3}\frac{x_R - x_T}{R} \ll J_2 \frac{M}{R} \sim \epsilon^4 \,. \tag{3.15}$$

The total round-trip travel time $\Delta\tau$ is just twice the magnitude of τ_{TR}.

Now let us estimate the size of each term in $\Delta\tau$. For simplicity, assume the transmitter and reflector lie in the equatorial plane of the Sun. Then,

$$J_z = \pm J \,, \tag{3.16}$$

$$J_x = J_y = 0 \,. \tag{3.17}$$

Now consider a radar beam that just grazes the limb of the Sun:

$$y_T \approx R \,. \tag{3.18}$$

Then we have

$$x_R \approx r_R \left[1 - \frac{1}{2}\frac{R^2}{{r_R}^2}\right] \,, \tag{3.19}$$

$$x_T \approx -r_T \left[1 - \frac{1}{2}\frac{R^2}{{r_T}^2}\right] \,. \tag{3.20}$$

Since reflectors have historically varied from Mercury, with [in astronomical units (AU)]

$$r \geq 0.3 \text{ AU} , \tag{3.21}$$

to Mars, with

$$r \leq 1.7 \text{ AU} , \tag{3.22}$$

we will use

$$r_R \approx r_T \equiv r = 1 \text{ AU} . \tag{3.23}$$

Then $\Delta\tau$ becomes

$$\frac{\Delta\tau}{R} = 4\frac{r}{R} - 4\frac{M}{R} + 4(1+\gamma)^2\frac{M^2}{R^2}\frac{r}{R} + 2(2\beta-1)\frac{M^2}{R^2}\frac{R}{r} + 2(1+\gamma)\frac{M}{R}\ln\frac{4r^2}{R^2} - 2(1+\gamma)\frac{M^2}{R^2}\frac{R}{r}\ln\frac{4r^2}{R^2}$$
$$+ 2\frac{A}{R}\ln\frac{4r^2}{R^2} + 4[2(1+\gamma)-\beta+\tfrac{3}{4}\Lambda]\frac{M^2}{R^2}\arctan\frac{r}{R} \pm (7\Delta_1+\Delta_2)\frac{J}{R^2} + 2(1+\gamma)J_2\frac{M}{R} . \tag{3.24}$$

Table I lists the magnitude of each term in expression (3.24) in general relativity.

It is worthwhile to point out at this time that since we have calculated a proper-time interval, the result is independent of the gauge choice we made to put the metric in the form of expressions (1.5)—(1.7). Of course, in a different gauge the functional form of $\Delta\tau$ would look different, but its magnitude would remain the same.

IV. CONCLUSIONS

Note that the largest of the second-order contributions to $\Delta\tau$ is on the order of ten nanoseconds. This is already as large as the uncertainties in measurements of $\Delta\tau$ using the Viking spacecraft, according to a recent report by Shapiro.[6] *Any* improvements in the measurement of time delays (see Shapiro's article for a discussion of the possibilities for improvements) will make it necessary to consider second-order contributions in such experiments.

Shapiro[6] incorrectly states that second-order contributions to the time delay are only as large as some ten picoseconds. This is clearly contradicted by our calculations. The largest second-order contribution is on the order of 10^{-11} times the total round-trip travel time: $10^{-11}\times 10^3$ sec $\sim 10^{-8}$ sec. This term accounts for the fact that the gravitational deflection of light will increase the length of the trajectory by forcing the light beam to follow a curved path. The total one-way path length is

$$\int_T^R (dx^2+dy^2)^{1/2} = \int_T^R (1+y'^2)^{1/2}dx \approx \int_T^R (1+\tfrac{1}{2}y'^2)dx . \tag{4.1}$$

Thus, we see that the transit time $x_R - x_T$ is increased by an amount

$$(x_R - x_T)\times \text{average value of } \tfrac{1}{2}y'^2 \sim (x_R - x_T)\times 10^{-11} . \tag{4.2}$$

The second-order terms which arise due to the variable coordinate speed of the light signal are smaller than this length increase by factors of nearly 100 or more.

Finally, let us point out that there is one term, that proportional to J_z, which varies in sign on opposite sides of the Sun. This is apparently the only contribution to the round-trip time delay with this feature. This unique variation may make this term accessible to measurement sooner than others. Even an upper limit on the magnitude would be useful in placing an upper limit on the size of the total solar angular momentum.

As mentioned earlier, the results presented here would take a different form if another coordinate system were used. Thus, it is clearly undesirable to leave our results in a coordinate-dependent form. Expressing the delay in terms of measurable quantities would be far more useful, as was pointed out by Ross and Schiff soon after the original calculations of the first-order delay.[7] However, attempts to eliminate all reference to coordinate quantities in the expression for the delay would be very complicated analytically because of the complicated nature of the orbit of the Earth. It would not be difficult to express the delay in terms of observable quantities if the Earth were in a circular orbit and if the reflector happened to lie in the plane of that orbit. However, neither of these simplifying assumptions can be applied to any real experiment. [Equation (3.12) makes it clear that we cannot neglect the radial

TABLE I. To estimate the size of contributions to the time of propagation, we assume source and reflector at $r \simeq 1$ AU on opposite sides of the Sun, and a grazing photon orbit past the Sun. Then the time-delay expression simplifies as in Eq. (3.24). This table lists the individual terms appearing in (3.24), and estimates their size in the case of general relativity.

Absolute value of term in expression (3.24)	Value in general relativity of corresponding term in $\Delta\tau$ (seconds)
$4\frac{r}{R}$	2000
$4\frac{M}{R}$	2.0×10^{-5}
$4(1+\gamma)^2\frac{M^2}{R^2}\frac{r}{R}$	3.6×10^{-8}
$2(2\beta-1)\frac{M^2}{R^2}\frac{R}{r}$	9.7×10^{-14}
$2(1+\gamma)\frac{M}{R}\ln\frac{4r^2}{R^2}$	2.4×10^{-4}
$2(1+\gamma)\frac{M^2}{R^2}\frac{R}{r}\ln\frac{4r^2}{R^2}$	2.4×10^{-12}
$2\frac{A}{R}\ln\frac{4r^2}{R^2}$	0
$4[2(1+\gamma)-\beta+\frac{3}{4}\Lambda]\frac{M^2}{R^2}\arctan\frac{r}{R}$	2.5×10^{-10}
$(7\Delta_1+\Delta_2)\frac{J}{R^2}$	$\leq 3.0\times 10^{-10}$ [a]
$2(1+\gamma)J_2\frac{M}{R}$	$\leq 4.9\times 10^{-10}$ [a]

[a]Here the values listed are for Dicke's model of the Sun. The values would be smaller for a uniformly rotating Sun.

motion of the Earth. Furthermore, angular inclinations between the orbits of the transmitter and reflector on the order of a degree, as is typical in the solar system, are not negligible even at first order.] Because of the motion of the transmitter, we would also be required to study the propagation of the radar signal along two independent paths. Nonetheless, the results presented here are a useful first step in the study of second-order effects for several reasons. For one thing, it has become standard practice to study relativistic experiments in the solar system using isotropic coordinates. Having our results in terms of isotropic coordinates will be the most convenient form for further work if the orbital elements of the transmitter and reflector can be expressed in terms of those same coordinates. Furthermore, we have demonstrated that there is at least one physical effect (the increase in path length due to the bending of the beam path) that will have to be considered in future experiments. This important conclusion follows from our results even though those results may not be written in a form that is immediately accessible to experiment.

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[1]R. A. Matzner and G. W. Richter, Astrophys. Space Sci. 79, 119 (1981).

[2]G. W. Richter and R. A. Matzner, Phys. Rev. D 26, 1219 (1982).

[3]G. W. Richter and R. A. Matzner, Phys. Rev. D 26, 2549 (1982).

[4]I. I. Shapiro, Phys. Rev. Lett. 13, 789 (1964).

[5]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[6]I. I. Shapiro, in *General Relativity and Gravitation, One Hundred Years After the Birth of Albert Einstein,* edited by A. Held (Plenum, New York, 1980), Vol. 2.

[7]D. K. Ross and L. I. Schiff, Phys. Rev. 141, 1215 (1966).