Junction conditions and the propagation of isometries in general relativity

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A set of junction conditions is stated in terms of the Newman-Penrose variables (tetrad vectors and spin coefficients). It is shown that these conditions are equivalent to those of Darmois and Lichnerowicz. As an example we study the matching of the Schwarzschild metric with an axially and reflection-symmetric metric. For this particular example we study the propagation of the Killing vectors and show how the propagation is conditioned by the fulfillment of the junction conditions.

I. INTRODUCTION

The problem of matching two solutions of the Einstein equations belonging to two regions $(M,\bar{M})$ of the space-time, separated by a non-null hypersurface (S), is of greatest relevance in general relativity. Let us just recall the (trivial) fact that a solution of the Einstein equations in one of the regions (say M) can be considered as representing the source of the metric in the other $(\bar{M})$, only if the two metrics can be matched on the boundary of the source (S). (This last condition is, of course, not a sufficient one.) Thus the search for appropriate interior solutions, whose relevance is obvious, is a task demanding a clear understanding of the junction conditions.

In the past, researchers have used different sets of junction conditions indistinctly. The possible differences (or equivalences) between the different sets being a subject not well understood.

Recently, Bonnor and Vickers[1] (hereafter referred to as BV) studied in detail the three sets of junction conditions currently used in general relativity, namely, those of Darmois[2] (hereafter denoted by D), O'Brien and Synge[3] (hereafter OS), and Lichnerowicz[4] (hereafter L). Three important results emerge from BV.

(a) The conditions of D and L are equivalent.

(b) The conditions of OS are more restrictive than those of D and L.

(c) The conditions of OS exclude some physically interesting possibilities.

It should be noted, however, that since the conditions of OS (and also L) are not covariantly stated, (b) and (c) are true only for a given class of coordinate systems. It is in principle possible to find a coordinate system in which the conditions of OS and L (and D) are equivalent.[5,6]

In this paper we shall present yet another set of junction conditions, which, as will be seen below, are equivalent to the L (and D) conditions. Our set of junction conditions (hereafter HJ conditions) is stated in terms of the Newman-Penrose[7] variables (null tetrads and spin coefficients). This choice of junction conditions presents two main advantages.

(i) With respect to the L conditions: The junction conditions are partially stated in covariant form (the conditions on the spin coefficients).

(ii) With respect to the D conditions: The quantities on which the junction conditions are imposed (tetrad vectors and spin coefficients) are the dynamical variables in the Newman-Penrose formalism (beside the Riemann tensor components). Thus no additional calculations are needed in order to verify if the junction conditions are satisfied. (One does not need to calculate, for example, the second fundamental form.)

The paper is organized as follows. In Sec. II, we give the HJ junction conditions and show the equivalence between them and the L conditions. For the sake of completeness we also include in this section a very brief summary of the structure of the field equations in the Newman-Penrose formalism.

In Sec. III, we use the HJ conditions to match an axially and reflection-symmetric metric with the Schwarzschild metric. The propagation of the Killing vectors across the separating hypersurface is studied in Sec. IV. In Sec. V, we give some conclusions. The Killing equations in terms of the Newman-Penrose variables are given in the Appendix A.

II. THE FIELD EQUATIONS AND THE JUNCTION CONDITIONS

It is known that the field equations in the Newman-Penrose formalism[7] consist of three sets of first-order differential equations for the following three sets of variables.

(a) The components of the Riemann tensor decomposed in its irreducible parts.

(b) The spin coefficients.

(c) The field of tetrads, from which the metric tensor is built.

In order to treat a physical situation one must incorporate the Einstein condition

$$R_{\mu\nu}=K(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T) \qquad (1)$$

into the full system of equations mentioned above. Equation (1) is no longer considered as a second-order differential equation for the $g_{\mu\nu}$'s, but a relation linking the Ricci tensor (geometry) with the energy-momentum tensor (matter). The three sets of first-order differential equations with the condition (1) form the gravitational field equations in the Newman-Penrose formalism.

We introduce the null tetrad $(l,n,m,\bar{m})$ (an overbar stands for complex conjugation).

The first set of equations (metric equations) reads

$$Dn^\mu-\Delta l^\mu=-(\gamma+\bar{\gamma})l^\mu+(\pi+\bar{\tau})m^\mu+(\tau+\bar{\pi})\bar{m}^\mu-(\epsilon+\bar{\epsilon})n^\mu , \quad (2)$$

$$\delta l^\mu-Dm^\mu=(\bar{\alpha}+\beta-\bar{\pi})l^\mu-(\bar{\rho}+\epsilon-\bar{\epsilon})m^\mu-\sigma\bar{m}^\mu+\kappa n^\mu , \quad (3)$$

$$\delta n^\mu-\Delta m^\mu=-\bar{\nu}l^\mu+(\mu-\gamma+\bar{\gamma})m^\mu+\lambda\bar{m}^\mu+(\tau-\bar{\alpha}-\beta)n^\mu , \quad (4)$$

$$\delta\bar{m}^\mu-\bar{\delta}m^\mu=(\mu-\bar{\mu})l^\mu+(\bar{\beta}-\alpha)m^\mu+(\bar{\alpha}-\beta)\bar{m}^\mu+(\rho-\bar{\rho})n^\mu , \quad (5)$$

where

$$D\equiv l^\mu\frac{\partial}{\partial x^\mu},\quad \Delta\equiv n^\mu\frac{\partial}{\partial x^\mu} ,$$
$$\delta\equiv m^\mu\frac{\partial}{\partial x^\mu},\quad \bar{\delta}\equiv\bar{m}^\mu\frac{\partial}{\partial x^\mu} ,$$
$$l^\mu n_\mu=-\bar{m}^\mu m_\mu=1 ,$$
$$m^\mu m_\mu=l^\mu l_\mu=n^\mu n_\mu=1^\mu m_\mu=n^\mu m_\mu=0 , \quad (6)$$

and the lower-case Greek letters stand for the spin coefficients. We shall not display the second and the third set of equations because we do not need them here.

Now we shall state the HJ junctions conditions as follows.

Two regions M and $\bar{M}$ separated by a non-null hypersurface S are said to match across S if

(1) there exists a coordinate system such that the tetrad vectors are continuous across S;

(2) the spin coefficients, taken with respect to the continuous tetrad whose existence is being asserted, are continuous across S.

We shall now prove that these conditions are equivalent to the L (and consequently to the D) conditions.

First of all, let us recall that the L conditions demand the existence of a coordinate system such that the metric components and their first derivatives are continuous across S. Next, the continuity of the metric components is implied by the continuity of the tetrad vectors, as can be seen at once from the expression

$$g_{\alpha\beta}=2n_{(\alpha}l_{\beta)}-2m_{(\alpha}\bar{m}_{\beta)} \quad (7)$$

(the brackets stand for symmetrization). Conversely, it can be shown that the continuity of the tetrad vectors is implied by the continuity of the metric components. In fact, given the metric tensor, the tetrad field is determined up to (a) a null rotation about l_μ; (b) a boost in the $l^\mu-n^\mu$ plane, and a spatial rotation in the $m^\mu-\bar{m}^\mu$ plane; (c) a null rotation about n_μ.

Thus if there exists a coordinate system for which the metric components are continuous, for that same coordinate system we can demand the tetrad vectors to be continuous.

Let us now consider the metric equations (2)—(5). Using (6) we get

$$l_\mu(Dn^\mu-\Delta l^\mu)=-(\epsilon+\bar{\epsilon}) , \quad (8)$$
$$n_\mu(Dn^\mu-\Delta l^\mu)=-(\gamma+\bar{\gamma}) , \quad (9)$$
$$m_\mu(Dn^\mu-\Delta l^\mu)=-(\tau+\bar{\pi}) , \quad (10)$$
$$\bar{m}_\mu(Dn^\mu-\Delta l^\mu)=-(\pi+\bar{\tau}) , \quad (11)$$
$$l_\mu(\delta l^\mu-Dm^\mu)=\kappa , \quad (12)$$
$$n_\mu(\delta l^\mu-Dm^\mu)=(\bar{\alpha}+\beta-\bar{\pi}) , \quad (13)$$
$$m_\mu(\delta l^\mu-Dm^\mu)=\sigma , \quad (14)$$
$$\bar{m}_\mu(\delta l^\mu-Dm^\mu)=(\bar{\rho}+\epsilon-\bar{\epsilon}) , \quad (15)$$
$$l_\mu(\delta n^\mu-\Delta m^\mu)=(\tau-\bar{\alpha}-\beta) , \quad (16)$$
$$n_\mu(\delta n^\mu-\Delta m^\mu)=-\bar{\nu} , \quad (17)$$
$$m_\mu(\delta n^\mu-\Delta m^\mu)=-\bar{\lambda} , \quad (18)$$
$$\bar{m}_\mu(\delta n^\mu-\Delta m^\mu)=-(\mu-\gamma+\bar{\gamma}) , \quad (19)$$
$$l_\mu(\delta\bar{m}^\mu-\bar{\delta}m^\mu)=(\rho-\bar{\rho}) , \quad (20)$$
$$n_\mu(\delta\bar{m}^\mu-\bar{\delta}m^\mu)=(\mu-\bar{\mu}) , \quad (21)$$
$$m_\mu(\delta\bar{m}^\mu-\bar{\delta}m^\mu)=-(\bar{\alpha}-\beta) , \quad (22)$$
$$m_\mu(\delta\bar{m}^\mu-\bar{\delta}m^\mu)=-(\alpha-\bar{\beta}) . \quad (23)$$

Thus, if the first derivatives of the tetrad vectors (metric components)[8] are continuous across S, we get from (12), (14), (17), and (18) that κ, σ, ν, and λ are continuous across S. Combining (19), (21), and (9), it follows that γ and μ are also continuous. The continuity of ρ and ϵ can be seen from (20), (15), and (8). Finally, the continuity of π, τ, α, and β follows from (22), (16), (13), and (11).

Thus if the tetrad vectors and their first derivatives are continuous, all the spin coefficients are continuous, which proves that the L (and therefore D) conditions imply the HJ conditions.

Let us now show that the HJ conditions imply the L (or D) conditions. For simplicity we shall work in this part with the D instead of the L conditions.

We recall that the D conditions demand that both the first fundamental form

$$(g_{\alpha\beta}dx^\alpha dx^\beta)_S \quad (24)$$

and the second fundamental form

$$(P_{\mu;\nu}dx^u dx^\nu)_S \quad (25)$$

be continuous across S [in (25) P_μ is a unit vector normal to S]. Now, from (7) it is evident that the continuity of the tetrad vectors across S implies the continuity of the first fundamental form. Thus the existence of a coordinate system where the tetrad vectors are continuous across S guarantees the continuity of the first fundamental form.

Next, let us show that the continuity of the spin coefficients, taken with respect to the continuous tetrad whose existence is being asserted, implies the continuity of (25). The unit vector P^μ can be written as

$$P^\mu=fl^\mu+gn^\mu+qm^\mu+\bar{q}\,\bar{m}^\mu , \quad (26)$$

where

$$fg - q\bar{q} = \tfrac{1}{2} \, .$$

Without changing the coordinate system, we have the freedom to perform a "boost" in the $l^\mu - n^\mu$ plane:

$$l'^\mu = \lambda l^\mu \, , \quad m'^\mu = m^\mu \, , \quad n'^\mu = \lambda^{-1} n^\mu \, , \tag{27}$$

and a null rotation about the vector n'^μ:

$$n''^\mu = n'^\mu \, , \quad l''^\mu = l'^\mu + b\bar{m}'^\mu + \bar{b} m'^\mu + b\bar{b} n'^\mu \, , \quad m''^\mu = m'^\mu + b n'^\mu \tag{28}$$

(both of which, of course, leave the metric unchanged).[9,10]

Since λ is an arbitrary real function and b an arbitrary complex function, we may choose

$$q = \bar{b} \, , \quad f = \lambda \, .$$

Doing so we get for P^μ (we omit the primes)

$$P^\mu = l^\mu + \frac{n^\mu}{2} \, , \tag{29}$$

where we have taken into account the condition

$$P^\mu P_\mu = 1 \, .$$

If we now calculate (25) using (29), then since the covariant derivatives of the tetrad vectors are given in terms of the spin coefficients and the tetrad vectors, the continuity of those implies the continuity of (25). We have shown here the equivalence between the HJ and D (and L) conditions.[11]

III. MATCHING THE SCHWARZSCHILD METRIC WITH AN AXIALLY AND REFLECTION-SYMMETRIC METRIC

Let us consider a nonstatic distribution of matter which is axially and reflection symmetric. In radiation coordinates[12] the metric takes the form

$$ds^2 = \left[\frac{V}{r} e^{2b} - U^2 r^2 e^{2g} \right] du^2 + 2e^{2b} du\, dr + 2Ur^2 e^{2g} du\, d\theta - r^2 (e^{2g} d\theta^2 + e^{-2g} \sin^2\theta\, d\phi^2) \, , \tag{30}$$

where U, V, g, and b are functions of u, θ, and r. Here $u \equiv x^0$ is the timelike coordinate, $r \equiv x^1$ is a null coordinate (not an affine parameter), and θ and ϕ are the usual angular coordinates.

The associated tetrad may be chosen to be

$$l^\mu = e^{-2b} \delta_1^\mu , \quad n^\mu = \delta_0^\mu - \frac{V}{2r} \delta_1^\mu + U \delta_2^\mu \, , \quad m^\mu = \frac{1}{r\sqrt{2}} (e^{-g} \delta_2^\mu + i e^{g} \csc\theta\, \delta_3^\mu) \, , \tag{31}$$

or in covariant components

$$l_\mu = \delta_\mu^0 , \quad n_\mu = \frac{Ve^{2b}}{2r} \delta_\mu^0 + e^{2b} \delta_\mu^1 \, , \quad m_\mu = \frac{r}{\sqrt{2}} (U e^{g} \delta_\mu^0 - e^{g} \delta_\mu^2 - i e^{-g} \sin\theta\, \delta_\mu^3) \, . \tag{32}$$

For the spin coefficients we get

$$\begin{aligned}
&\kappa = \epsilon = 0 , \quad \rho = -\frac{e^{-2b}}{r} , \quad \sigma = -e^{-2b} g_1 \, , \\
&\gamma = \frac{e^{-2b}}{2} \left[\left[\frac{Ve^{2b}}{2r} \right]_1 - (e^{2b})_0 - U (e^{2b})_2 \right] , \\
&\alpha = \frac{1}{4} \left[\frac{re^{g} e^{-2b}}{\sqrt{2}} U_1 - \frac{\sqrt{2} e^{-g}}{r} b_2 - \frac{\sqrt{2} e^{-g}}{r} (\cot\theta - g_2) \right] , \\
&\beta = \frac{1}{4} \left[\frac{re^{g} e^{-2b}}{\sqrt{2}} U_1 - \frac{\sqrt{2} e^{-g}}{r} (b_2 + g_2 - \cot\theta) \right] , \\
&\tau = \frac{e^{-2b}}{2\sqrt{2}} \left[re^{g} U_1 - \frac{2e^{-g} e^{2b}}{r} b_2 \right] , \\
&\pi = \frac{e^{-2b}}{2\sqrt{2}} \left[re^{g} U_1 + \frac{2e^{-g} e^{2b}}{r} b_2 \right] , \\
&\lambda = U \left[g_2 - \frac{\cot\theta}{2} \right] + g_0 + \tfrac{1}{2} \left[U_2 - \frac{V}{r} g_1 \right] , \\
&\mu = \frac{U}{2} \cot\theta - \frac{V}{2r^2} + \frac{U_2}{2} \, , \\
&\nu = \frac{e^{2b} V_2 e^{-g}}{2\sqrt{2} r^2}
\end{aligned} \tag{33}$$

(differentiations with respect to u, r, and θ are denoted by subscripts 0, 1, and 2, respectively).

Now we shall match metric (30) with the Schwarzschild metric on a hypersurface S defined by the equation

$$r = r_1 \, , \tag{34}$$

where r_1 may be a constant or a function of u.

For the Schwarzschild metric the expressions for the spin coefficients and the metric functions are

$$\begin{aligned}
&\sigma_S = \kappa_S = \lambda_S = \nu_S = \pi_S = \tau_S = \epsilon_S = 0 \, , \\
&\rho_S = -\frac{1}{r} ; \ \alpha_S = -\beta_S = -\frac{\cot\theta}{2\sqrt{2} r} ; \ \mu_S = \frac{-1}{2r} \left[1 - \frac{2m}{r} \right] , \\
&\gamma_S = \frac{m}{2r^2} \, ,
\end{aligned} \tag{35}$$

$$b_S = g_S = U_S = 0 \, , \quad V_S = r - 2m \tag{36}$$

(here the subscript S stands for Schwarzschild).

We start by demanding the continuity of all spin coefficients. Thus, from the continuity of ρ and σ we obtain

$$(g_1)_{r=r_1\pm 0}=0\ , \tag{37}$$

$$(b)_{r=r_1\pm 0}=0\ . \tag{38}$$

Next, from the continuity of b [which is implied by (38)] and the continuity of α and β, we get

$$\frac{1}{4}\left[\frac{re^{g}}{\sqrt{2}}U_1-\frac{\sqrt{2}e^{-g}}{r}b_2-\frac{\sqrt{2}e^{-g}}{r}(\cot\theta-g_2)\right]_{r=r_1\pm 0}=-\left[\frac{\cot\theta}{2\sqrt{2}r}\right]_{r=r_1\pm 0}, \tag{39}$$

$$\frac{1}{4}\left[\frac{re^{g}}{\sqrt{2}}U_1-\frac{\sqrt{2}e^{-g}b_2}{r}+\frac{\sqrt{2}e^{-g}}{r}(\cot\theta-g_2)\right]_{r=r_1\pm 0}=\left[\frac{\cot\theta}{2\sqrt{2}r}\right]_{r=r_1\pm 0}. \tag{40}$$

Using

$$(b_2)_{r=r_1\pm 0}=0\ , \tag{41}$$

$$(e^{g}U_1)_{r=r_1\pm 0}=0\ , \tag{42}$$

which result from the continuity of b and τ, respectively, we get from (39) and (40)

$$(g)_{r=r_1\pm 0}=0\ , \tag{43}$$

$$(g_2)_{r=r_1\pm 0}=0\ ,\quad (U_1)_{r=r_1\pm 0}=0\ . \tag{44}$$

Finally from the continuity of λ and μ we have

$$\left[-\frac{U}{2}\cot\theta+g_0+\tfrac{1}{2}U_2\right]_{r=r_1\pm 0}=0\ , \tag{45}$$

$$\left[\frac{U}{2}\cot\theta-\frac{V}{2r^2}+\frac{U_2}{2}\right]_{r=r_1\pm 0}=-\left[\frac{1}{2r}\left[1-\frac{2m}{r}\right]\right]_{r=r_1\pm 0}. \tag{46}$$

Observe that g_0 could be discontinuous across $r=r_1$, because r_1 may be a function of u. However g_1 is continuous by virtue of (37), it follows that g_0 is continuous too.

Thus, from (45) we have

$$\left[-\frac{U}{2}\cot\theta+\tfrac{1}{2}U_2\right]_{r=r_1\pm 0}=0\ . \tag{47}$$

To satisfy (47) we have two possible choices.

(a) If U is a continuous function across $r=r_1$, then by virtue of (36)

$$(U)_{r=r_1\pm 0}=U_S=0\ .$$

(b) U is of the form

$$(U)_{r=r_1-0}=f(u,r)\sin\theta\ , \tag{48}$$

$$(U)_{r=r_1+0}=0\ . \tag{49}$$

With the first choice, we obtain from (46)

$$(V)_{r=r_1\pm 0}=(r-2m)_{r=r_1\pm 0}\ , \tag{50}$$

which implies the continuity of V.

If we consider the second choice, then, because of (46) and (47) and the continuity of ν, we get

$$[f(u,r)]_{r=r_1\pm 0}=0\ .$$

Thus U and V should be continuous across $r=r_1$.

It remains to study the continuity of γ. We have, using the previous results,

$$\frac{1}{2}\left[\frac{V_1}{2r}+\frac{Vb_1}{2r}-\frac{V}{2r^2}-2b_0\right]_{r=r_1\pm 0}=\left[\frac{m}{2r^2}\right]_{r=r_1\pm 0}. \tag{51}$$

Again, observe that b_0 may be discontinuous across $r=r_1$, because r_1 may be a function of u. Unlike g_1, we do not need to demand b_1 to be continuous. Thus b_0 may be also discontinuous. At $r=r_1+0$, (51) is automatically satisfied because

$$(V)_{r=r_1+0}\equiv(V_S)_{r=r_1+0}\ ,$$

$$(b_0)_{r=r_1+0}=(b_1)_{r=r_1+0}=0\ .$$

To study (51) at $r=r_1-0$, let us generalize V inside the matter as follows[13]:

$$V=e^{2b}[r-2m(r)]\ . \tag{52}$$

Obviously

$$(V)_{r=r_1}\equiv V_S$$

if $m(r_1)\equiv m$; feeding back (52) into (51) we obtain

$$\left[b_1\left[1-\frac{2m}{r}\right]-\frac{m_1}{2r}-b_0\right]_{r=r_1-0}=0\ . \tag{53}$$

Parenthetically, (53) is the same condition used in Ref. 13 to avoid the presence of δ functions in the pressure when studying the contraction of gravitating spheres. In that reference, this condition appears by inspection of the field equations. Here we obtain it just by demanding the appropriate junction conditions.

Interestingly enough, the continuity of the metric components (tetrad vectors) appears automatically as a consequence of the continuity of the spin coefficients.

Finally, observe that if we choose to use the L instead of the HJ conditions, in the example above, we have to make a coordinate transformation since in the coordinate system we have used, some derivatives of the metric components (b_0,b_1) are not continuous.

IV. THE PROPAGATION OF THE KILLING VECTORS

We shall now show how the results of the previous section may be applied to the problem of propagation of Killing vectors across the hypersurface separating the two regions M and $\bar{M}$.

As before, let the metric of the (inner) region M be given by (30), and the metric of the (outer) region, $\bar{M}$, be the Schwarzchild metric.

In radiation coordinates, as before, let the separating hypersurface S, be defined by $r=r_1$.

We shall now consider the following problem: Let us assume that there exist in M a Killing vector (say ξ^α), besides the Killing vector of the axial symmetry, such that on the separating hypersurface $r=r_1$, it coincides with one of the Killing vectors of $\bar{M}$ (say ξ^α_S). We want to know under which conditions ξ^α defines inside M the same kind of symmetry as defined by ξ^α_S in $\bar{M}$.

Let us start by considering the timelike Killing vector of the Schwarzschild metric,

$$\xi^\mu_S=\tfrac{1}{2}\left[1-\frac{2m}{2}\right]l^\mu_S+n^\mu_S \ , \tag{54}$$

where the vectors l^μ_S, n^μ_S together with m^μ_S form the tetrad associated to the Schwarschild metric. They can be obtained at once by putting

$$b=g=U=0,\quad V=r-2m$$

into (31) or (32). We get

$$l^\mu_S=\delta^\mu_1,\quad n^\mu_S=\delta^\mu_0-\tfrac{1}{2}\left[1-\frac{2m}{r}\right]\delta^\mu_1 \ ,$$
$$m^\mu_S=\frac{1}{r\sqrt{2}}(\delta^\mu_2+i\csc\theta\delta^\mu_3) \ . \tag{55}$$

In general, any vector ξ^μ may be written as

$$\xi^\mu=Al^\mu+Bn^\mu+Cm^\mu+\bar{C}\bar{m}^\mu \ , \tag{56}$$

where A, B, and C are functions of the coordinates. Thus the Killing vector ξ^μ_S may be written as

$$\xi^\mu_S=Al^\mu_S+Bn^\mu_S$$

with

$$B=1,\quad A=\tfrac{1}{2}\left[1-\frac{2m}{r}\right] .$$

Next, assuming that ξ^μ as given by (56) is a Killing vector in M such that

$$A(u,r_1,\theta,\phi)=\tfrac{1}{2}\left[1-\frac{2m}{r_1}\right] , \tag{57}$$

$$B(u,r_1,\theta,\phi)=1 \ , \tag{58}$$

$$C(u,r_1,\theta,\phi)=0 \ , \tag{59}$$

we obtain from (A11) and (58)

$$B(u,r,\theta,\phi)=1 \ . \tag{60}$$

From (A12) we get the equations

$$\frac{\partial}{\partial r}(\mathrm{Re}C)=\frac{\mathrm{Re}C}{r}+g_1\mathrm{Re}C-\frac{re^g}{\sqrt{2}}U_1 \ ,$$
$$\frac{\partial}{\partial r}(\mathrm{Im}C)=\frac{\mathrm{Im}C}{r}-g_1\mathrm{Im}C \ ,$$

whose solutions are

$$\mathrm{Re}C(u,r,\theta)=\frac{r}{\sqrt{2}}e^g[U(u,r_1,\theta)-U(u,r,\theta)] \ , \tag{61}$$

$$\mathrm{Im}C(u,r,\theta)=0 \ . \tag{62}$$

Feeding back (61) and (62) in (A13), we get

$$\begin{aligned}A&=\frac{r}{2}e^{2b}\left[\frac{V}{r^2}-U\cot\theta-U_2\right]\\&\quad+\frac{r}{2}(g_2-\cot\theta)e^{2b}[U(u,r_1,\theta)-U(u,r,\theta)]\\&\quad-\frac{r}{2}e^{2b}e^{-g}\{e^g[U(u,r_1,\theta)-U(u,r,\theta)]\}_2\\&=\frac{re^{2b}}{2}\left[\frac{V}{r^2}-\cot\theta U(u,r_1,\theta)-U_2(u,r_1,\theta)\right] .\end{aligned}$$

Using (57),

$$\left[\frac{e^{2b}}{r}V-\left[1-\frac{2m}{r}\right]-re^{2b}(U\cot\theta+U_2)\right]_{r=r_1}=0 \ ,$$

or equivalently, as can be seen from (33),

$$\left[\frac{\mu}{\rho}\right]_{r=r_1}=\tfrac{1}{2}\left[1-\frac{2m}{r_1}\right] . \tag{63}$$

Condition (63) is automatically satisfied by virtue of the continuity of the spin coefficients across S.

Next, let us use the Killing equations which have not been used yet. From (A8)—(A10) and (A14) and the previous results, we have

$$\begin{aligned}&\frac{r}{2}\left[\frac{V}{r^2}-\cot\theta U(u,r_1,\theta)-U_2(u,r_1,\theta)\right]\left[\left[\frac{Ve^{2b}}{2r}\right]_1-(e^{2b})_0-U(e^{2b})_2\right]+\frac{e^{2b}V_2}{2r}[U(u,r_1,\theta)-U]\\&\quad+\frac{r}{2}\left\{e^{2b}\left[\frac{V}{r^2}-\cot\theta U(u,r_1,\theta)-U_2(u,r_1,\theta)\right]\right\}_0\\&\quad+\frac{Ur}{2}\left\{e^{2b}\left[\frac{V}{r^2}-\cot\theta U(u,r_1,\theta)-U_2(u,r_1,\theta)\right]\right\}\\&\quad-\frac{V}{4r}\left\{re^{2b}\left[\frac{V}{r^2}\cot\theta U(u,r_1,\theta)-U_2(u,r_1,\theta)\right]\right\}_1=0 \ ,\end{aligned} \tag{64}$$

$$-e^{-2b}\left[\left[\frac{Ve^{2b}}{2r}\right]_1-(e^{2b})_0-U(e^{2b})_2\right]+2b_2[U(u,r_1,\theta)-U]+\frac{e^{-2b}}{2}\left\{re^{2b}\left[\frac{V}{r^2}-\cot\theta U(u,r_1,\theta)-U_2(u,r_1,\theta)\right]\right\}_1=0\;, \tag{65}$$

$$\begin{aligned}&\frac{e^{2b}V_2e^{-g}}{2\sqrt{2}r^2}-\frac{r}{2\sqrt{2}}\left[\frac{V}{r^2}-\cot\theta U(u,r_1,\theta)-U_2(u,r_1,\theta)\right]\left[re^{g}U_1-\frac{2e^{2b}e^{-g}}{r}b_2\right]\\&\quad+\frac{re^{g}}{2}[U(u,r_1,\theta)-U]\left[\frac{V}{2r}\left[\frac{1}{r}+g_1\right]-Ug_2-g_0-U_2\right]-\frac{e^{-g}}{2\sqrt{2}}\left\{e^{2b}\left[\frac{V}{r^2}-\cot\theta U(u,r_1,\theta)-U_2(u,r_1,\theta)\right]\right\}_2\\&\quad+\frac{r}{\sqrt{2}}\{e^{g}[U(u,r_1,\theta)-U]\}\Big|_0-\frac{V}{2r}\left\{\frac{r}{\sqrt{2}}e^{g}[U(u,r_1,\theta)-U]\right\}_1+\frac{Ur}{\sqrt{2}}\{e^{g}[U(u,r_1,\theta)-U]\}_2=0\;,\end{aligned} \tag{66}$$

$$\begin{aligned}&-\frac{r}{2}g_1\left[\frac{V}{r^2}-\cot\theta U(u,r_1,\theta)-U_2(u,r_1,\theta)\right]-\left[U\left[g_2-\frac{\cot\theta}{2}\right]+g_0+\tfrac{1}{2}\left[U_2-\frac{V}{r}g_1\right]\right]\\&\qquad-\frac{e^{-g}}{2}\{e^{g}[U(u,r_1,\theta)-U]\}_2-\frac{(g_2-\cot\theta)}{2}[U(u,r_1,\theta)-U]=0\;.\end{aligned} \tag{67}$$

Using the continuity of the spin coefficients and the metric components, Eqs. (64)–(67) may be rewritten as

$$\left[\frac{V}{2r}\right]\left[\left[\frac{Ve^{2b}}{2r}\right]_1-(e^{2b})_0-U(e^{2b})_2\right]-\frac{e^{2b}V_2U}{2r}+\left[\frac{e^{2b}}{2r}\right]_0-\frac{V}{2r}\left[\frac{e^{2b}V}{2r}\right]_1+U\left[\frac{e^{2b}V}{2r}\right]_2=0\;, \tag{68}$$

$$-e^{-2b}\left[\left[\frac{Ve^{b}}{2r}\right]_1-(e^{2b})_0-U(e^{2b})_2\right]-2b_2U+e^{-2b}\left[\frac{Ve^{2b}}{2r}\right]_1=0\;, \tag{69}$$

$$\begin{aligned}&\frac{e^{2b}V_2e^{-g}}{2\sqrt{2}r^2}-\frac{V}{2\sqrt{2}r}\left[re^{g}U_1-\frac{2e^{2b}e^{-g}b_2}{r}\right]\\&\quad-\frac{re^{g}U}{\sqrt{2}}\left[\frac{V}{2r}\left[\frac{1}{r}+g_1\right]-Ug_2-g_0-U_2\right]\\&\quad-\frac{e^{-g}}{r\sqrt{2}}\left[\frac{e^{2b}V}{2r}\right]_2-\frac{r}{\sqrt{2}}(e^{g}U)_0\\&\quad+\frac{V}{2\sqrt{2}r}(re^{g}U)_1-\frac{Ur}{\sqrt{2}}(e^{g}U)_2=0\;,\end{aligned} \tag{70}$$

$$-\frac{V}{2r}g_1-\left[U\left[g_2-\frac{\cot\theta}{2}\right]+g_0+\tfrac{1}{2}\left[U_2-\frac{V}{r}g_1\right]\right]+\frac{e^{-g}}{2}(e^{g}U)_2+\frac{U}{2}(g_2-\cot\theta)=0\;. \tag{71}$$

It follows at once from (68)–(71) that

$$V_0=b_0=U_0=g_0=0\;. \tag{72}$$

It is easy to check that the violation of any of the junction conditions automatically destroys the propagation of the Killing vector inside M.[14]

As a second example, let us consider one of the Killing vectors generating the spherical symmetry, namely,

$$\eta_S^{\mu}=\frac{-r}{\sqrt{2}}(\cos\phi+i\cos\theta\sin\phi)m_S^{\mu}-\frac{r}{\sqrt{2}}(\cos\phi-i\cos\theta\sin\phi)\bar{m}_S^{\mu}\;.$$

As before, we ask for the condition of existence of a Killing vector in M

$$\eta^{\mu}=Al^{\mu}+Bn^{\mu}+Cm^{\mu}+\bar{C}\bar{m}^{\mu}$$

such that

$$\begin{aligned}&A(u,r_1,\theta,\phi)=0\;,\\&B(u,r_1,\theta,\phi)=0\;,\\&C(u,r_1,\theta,\phi)=-\frac{r_1}{\sqrt{2}}(\cos\phi+i\cos\theta\sin\phi)\;.\end{aligned} \tag{73}$$

By a systematic inspection of the Killing equations, we obtain

$$\begin{aligned}&B(u,r,\theta,\phi)=0\;,\\&\mathrm{Re}C=\left[\frac{r}{r_1}\right]\mathrm{Re}C(u,r_1,\theta,\phi)\times\exp[g(u,r,\theta)-g(u,r_1,\theta)]\;,\end{aligned} \tag{74}$$

$$\mathrm{Im}C=\left[\frac{r}{r_1}\right]\mathrm{Im}C(u,r,\theta,\phi)\times\exp[g(u,r_1,\theta)-g(u,r,\theta)]\;,$$

and using the junction conditions together with (73),

$$\mathrm{Re}C=-\frac{r}{\sqrt{2}}\cos\phi e^{g}\,,$$
$$\mathrm{Im}C=-\frac{r}{\sqrt{2}}\cos\theta\sin\phi e^{-g}\,. \tag{75}$$

Feeding back (75) in (A13) gives

$$A(u,r,\theta,\phi)=0\,. \tag{76}$$

Finally, (74), (75), (76), together with the remaining Killing equations, lead to

$$V_2=b_2=g_2=U_2=0\,. \tag{77}$$

As in the first example, the propagation of the symmetry in the region M is closely related to the fulfillment of the junction conditions.

V. CONCLUSIONS

We have seen so far that it is possible to state a set of appropriate junction conditions solely in terms of tetrads and spin coefficients. Furthermore, it appears, in some cases at least, that demanding the continuity of the spin coefficients implies the continuity of the metric components.

Finally, we would like to stress the close relationship between the propagation of the Killing vectors and the fulfillment of junction conditions. This should be taken into account, especially, when constructing material sources for gravitational fields.

APPENDIX

Let

$$\xi^{\alpha}=Al^{\alpha}+Bn^{\alpha}+Cm^{\alpha}+\bar{C}\bar{m}^{\alpha}$$

be a Killing vector. Then from the expressions

$$l_{\mu;\nu}=2\,\mathrm{Re}(\gamma)l_{\mu}l_{\nu}+2\,\mathrm{Re}(\epsilon)l_{\mu}n_{\nu}-(\alpha+\bar{\beta})l_{\mu}m_{\nu}-(\bar{\alpha}+\beta)l_{\mu}\bar{m}_{\nu}-\bar{\tau}m_{\mu}l_{\nu}-\bar{\kappa}m_{\mu}v_{\nu}$$
$$+\bar{\sigma}m_{\mu}m_{\nu}+\bar{\rho}m_{\mu}\bar{m}_{\nu}-\tau\bar{m}_{\mu}l_{\nu}-\kappa\bar{m}_{\mu}n_{\nu}+\rho\bar{m}_{\mu}m_{\nu}+\sigma\bar{m}_{\mu}\bar{m}_{\nu}\,,$$
$$n_{\mu;\nu}=-2\,\mathrm{Re}(\gamma)n_{\mu}l_{\nu}-2\,\mathrm{Re}(\epsilon)n_{\mu}n_{\nu}+(\alpha+\bar{\beta})n_{\mu}m_{\nu}+(\bar{\alpha}+\beta)n_{\mu}\bar{m}_{\mu}+\nu m_{\mu}l_{\nu}+\pi m_{\mu}n_{\nu}-\lambda m_{\mu}m_{\nu}-\mu m_{\mu}\bar{m}_{\nu}$$
$$+\bar{\nu}\bar{m}_{\mu}l_{\nu}+\bar{\pi}\bar{m}_{\mu}n_{\nu}-\bar{\mu}\bar{m}_{\mu}m_{\nu}-\bar{\lambda}\bar{m}_{\mu}\bar{m}_{\nu}\,,$$
$$m_{\mu;\nu}=\bar{\nu}l_{\mu}l_{\nu}+\bar{\pi}l_{\mu}n_{\nu}-\bar{\mu}l_{\mu}m_{\nu}-\bar{\lambda}l_{\mu}\bar{m}_{\nu}-\tau n_{\mu}l_{\nu}-\kappa n_{\mu}n_{\nu}+\rho n_{\mu}m_{\nu}+\sigma n_{\mu}\bar{m}_{\nu}+2i\,\mathrm{Im}(\gamma)m_{\mu}l_{\nu}+2i\,\mathrm{Im}(\epsilon)m_{\mu}n_{\nu}$$
$$-(\alpha-\bar{\beta})m_{\mu}m_{\nu}-(\beta-\bar{\alpha})m_{\mu}\bar{m}_{\nu}\,.$$

It can be shown that the Killing equations

$$\xi_{(\mu;\nu)}=0$$

may be written as

$$2A\,\mathrm{Re}(\gamma)+2\,\mathrm{Re}(C\bar{\nu})+\Delta A=0\,, \tag{A1}$$

$$2A\,\mathrm{Re}(\epsilon)-2B\,\mathrm{Re}(\gamma)+2\,\mathrm{Re}(C\bar{\pi})-2\,\mathrm{Re}(C\tau)+DA+\Delta B=0\,, \tag{A2}$$

$$B\bar{\nu}-A(\bar{\alpha}+\beta+\tau)-C\bar{\lambda}-\bar{C}\mu-2iC\,\mathrm{Im}(\gamma)-\delta A+\Delta\bar{C}=0\,, \tag{A3}$$

$$-2B\,\mathrm{Re}(\epsilon)-C\kappa-\bar{C}\bar{\kappa}+DB=0\,, \tag{A4}$$

$$-A\kappa+B(\bar{\alpha}+\beta+\bar{\pi})+C\sigma+\bar{C}[\bar{\rho}-2i\,\mathrm{Im}(\epsilon)]-\delta B+D\bar{C}=0\,, \tag{A5}$$

$$2A\,\mathrm{Re}(\rho)-2B\,\mathrm{Re}(\mu)+2Re[C(\bar{\alpha}-\beta)]-\delta C-\bar{\delta}\bar{C}=0\,, \tag{A6}$$

$$A\sigma-B\bar{\lambda}-\delta\bar{C}-\bar{C}(\bar{\alpha}-\beta)=0\,. \tag{A7}$$

For the metric (30), the equations (A1)–(A7) take the form

$$Ae^{-2b}\left[\left[\frac{Ve^{2b}}{2r}\right]_1-(e^{2b})_0-U(e^{2b})_2\right]+\frac{e^{2b}V_2}{\sqrt{2}r^2}e^{-g}\mathrm{Re}(C)+A_0-\frac{V}{2r}A_1+UA_2=0\,, \tag{A8}$$

$$-Be^{-2b}\left[\left[\frac{Ve^{2b}}{2r}\right]_1-(e^{2b})_0-U(e^{2b})_2\right]+\frac{2\sqrt{2}e^{-g}b_2\mathrm{Re}(C)}{r}+e^{-2b}A_1+B_0-\frac{V}{2r}B_1+UB_2=0\,, \tag{A9}$$

$$\frac{Be^{2b}Ve^{-g}}{2\sqrt{2}r^2}-\frac{Ae^{-2b}}{\sqrt{2}}\left[re^{g}U_1-\frac{2e^{-g}e^{2b}}{r}b_2\right]-C\left[U\left[g_2-\frac{\cot\theta}{2}\right]+g_0+\frac{1}{2}\left[U_2-\frac{V}{r}g_1\right]\right]-\bar{C}\left[\frac{U}{2}\cot\theta-\frac{V}{2r^2}+\frac{U_2}{2}\right]-\frac{1}{r\sqrt{2}}(e^{-g}A_2+ie^{g}\csc\theta A_3)+\bar{C}_0-\frac{V}{2r}\bar{C}_1+U\bar{C}_2=0\,, \tag{A10}$$

$$B_1=0\ , \tag{A11}$$

$$\frac{e^{-2b}re^{g}U_1B}{\sqrt{2}}-e^{-2b}g_1C-\frac{\bar{C}e^{-2b}}{r}+e^{-2b}\bar{C}_1-\frac{1}{r\sqrt{2}}(B_2e^{-g}+ie^{g}\csc\theta B_3)=0\ , \tag{A12}$$

$$-\frac{2Ae^{-2b}}{r}-B\left[U\cot\theta-\frac{V}{r^2}+U_2\right]+\frac{\sqrt{2}e^{-g}}{r}(g_2-\cot\theta)\mathrm{Re}(C)-\frac{\sqrt{2}}{r}\mathrm{Re}(e^{-g}C_2+ie^{g}\csc\theta C_3)=0\ , \tag{A13}$$

$$-Ae^{-2b}g_1-B\left[U\left[g_2-\frac{\cot\theta}{2}\right]+g_0+\tfrac{1}{2}\left[U_2-\frac{V}{r}g_1\right]\right]-\frac{1}{r\sqrt{2}}(e^{-g}\bar{C}_2+ie^{g}\csc\theta\bar{C}_3)-\frac{\bar{C}e^{-g}}{r\sqrt{2}}(g_2-\cot\theta)=0\ . \tag{A14}$$

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