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Classical stability of round and squashed seven-spheres in eleven-dimensional supergravity

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Four-dimensional field equations are derived for perturbations of the seven-sphere size and squashing parameter in eleven-dimensional supergravity configurations in which the rank-four tensor gauge field strength is proportional to the four-dimensional volume element. Field equations are also derived for perturbations in the size of a round or squashed Einstein-metric S^7 and in the magnitude of gauge-field-potential components proportional to an antisymmetric torsion tensor on these seven-spheres. The known anti—de Sitter—invariant solutions with round or squashed seven-spheres are shown to be classically stable to these perturbations.

I. INTRODUCTION

There are now known to be a number of one-parameter families of classical solutions of the field equations of $N=1$ supergravity in $d=11$ dimensions[1] in which (i) the fermion field is set equal to zero, (ii) the geometry is the product of four-dimensional anti—de Sitter spacetime (AdS) and a compact seven-dimensional space that is topologically the seven-sphere (S^7), and (iii) the rank-four antisymmetric tensor gauge field strength is AdS invariant. (1) The S^7 may be metrically "round" [SO(8) invariant] and the gauge field strength may have the Freund-Rubin form[2] purely proportional to the AdS Levi-Civita tensor, giving $N=8$ supersymmetry in four-dimensional spacetime[3–6]; (2) the S^7 may be metrically "squashed" [maintaining only Sp(2)$\otimes$Sp(1) invariance] by the amount needed to give the nonstandard Einstein metric on it[7,8] while the gauge field strength maintains the Freund-Rubin form, leaving, in four dimensions, either $N=1$ or $N=0$ supersymmetry, depending on the orientation of the squashing[9,5,6,10]; (3) the S^7 may be left metrically round while the gauge field potential is given not only the Freund-Rubin spacetime components but also components proportional to an absolutely parallelizing torsion[11] for the S^7, breaking all supersymmetry[12–14]; or (4) the S^7 may be squashed and the gauge field potential given components proportional to an antisymmetric torsion which annihilates the Ricci tensor of the squashed S^7, again destroying all supersymmetry.[9,15,10]

In each of these solutions the AdS invariance imposed implies that various quantities are constants over the four-dimensional spacetime: (a) the size of the S^7, (b) the degree of squashing, (c) the proportionality between the Freund-Rubin spacetime components of the gauge field strength and the Levi-Civita tensor, and (d) the proportionality between the S^7 components of the gauge field potential and the appropriate torsion. The field equations give relationships between these constant quantities so that in each of the families listed above there is only one free parameter, which may be chosen to be the size of the S^7, for example.

In this paper various combinations of these four quantities are allowed to vary as Lorentz scalars over the spacetime, and the spacetime components of the metric are also allowed to vary. The other components of the metric, the fermion field, and the gauge field are taken to have the same *form* at each spacetime point as in the AdS-invariant solutions and, hence, are determined purely by the four Lorentz scalars. The 11-dimensional field equations for these configurations are reduced to four-dimensional field equations coupling the metric and scalar fields. The stability of the AdS-invariant solutions (candidate ground states) is analyzed against perturbations of the Lorentz scalar fields.

It is found that when the gauge field strength has the Freund-Rubin form so that the scalar (d) is zero, both (1) the round S^7 and (2) the squashed S^7 are classically stable to both (a) dilations and (b) squashing. The integral of the dual of the gauge field strength over the compact S^7 gives a conserved "charge" that is constant over the spacetime in this case and, hence, fixes (c) the coefficient of the Freund-Rubin components in terms of (a) and (b). This conserved charge prevents the S^7 from being unstable to changing size or shape if suitable asymptotic conditions are maintained in the four-dimensional spacetime. With a fixed value of this charge, the AdS-invariant (1) round and (2) squashed solutions have different asymptotic conditions (e.g., values of the effective cosmological constant in spacetime) and cannot tunnel into each other. The conserved charge and asymptotic conditions thus label different superselection sectors, each with its own vacuum or ground state, so it is meaningless to ask which is the true ground state of the system.

When the gauge potential is given nonzero components in the S^7 dimensions, the analysis is simple only when the S^7 is either round or squashed by the precise amount to give the nonstandard Einstein metric. Therefore, only these two values of the squashing parameter (b) were considered, and perturbations in the Lorentz scalars (a), (c), and (d) were analyzed. Again a conserved charge fixed (c) in terms of the other two scalars. Both (3) the round Englert solution[12–24] and (4) the squashed solution[9,15,10] with the gauge field components proportional to an S^7 torsion were found to be stable under small perturbations of these two scalars, (a) the dilations of the S^7 and (d) the magnitude of the gauge field proportional to the torsion. In each of these cases the AdS-invariant solution has no residual supersymmetry by which one might have been led to expect this stability.

II. GAUGE FIELD STRENGTH OF THE FREUND-RUBIN FORM

I shall use conventions similar to Duff *et al.*[3–6,9,10] so that the $d=11$ coordinates X with indices $A,B,\ldots$ are decomposed into $d=4$ spacetime coordinates x with indices $\alpha,\beta,\ldots$ and $d=7$ S^7 coordinates y with indices $a,b,\ldots$ When the fermion field is set equal to zero, the boson field equations are[1]

$$\bar{R}^A_B=\tfrac{1}{3}\bar{F}^{ACDE}\bar{F}_{BCDE}-\tfrac{1}{36}\delta^A_B\bar{F}^{CDEF}\bar{F}_{CDEF}\ , \tag{1}$$

$$\bar{F}_{ABCD}=4\bar{\nabla}_{[A}\bar{A}_{BCD]}\ , \tag{2}$$

$$\bar{\nabla}_A\bar{F}^{ABCD}=-\tfrac{1}{576}\bar{\epsilon}^{M_1\cdots M_8BCD}\bar{F}_{M_1\cdots M_4}\bar{F}_{M_5\cdots M_8}\ . \tag{3}$$

A general metric that is locally the direct sum of an arbitrary spacetime metric and a squashed S^7 metric may be written as

$$\begin{aligned}\bar{g}_{AB}dX^AdX^B=e^{-7u}g_{\alpha\beta}dx^\alpha dx^\beta &+e^{2u+3v}\left[\tfrac{1}{4}d\mu^2+\tfrac{1}{16}\sin^2\mu\sum_{i=1}^{3}\omega_i{}^2\right]\\ &+e^{2u-4v}\sum_{i=1}^{3}\tfrac{1}{16}(\nu_i+\cos\mu\omega_i)^2\ .\end{aligned} \tag{4}$$

The last two terms give the squashed S^7 metric[9] when $\nu_i=\sigma_i+\Sigma_i$, $\omega_i=\sigma_i-\Sigma_i$, with these one-forms satisfying the SU(2) algebra

$$d\sigma_i=-\tfrac{1}{2}\epsilon_{ijk}\sigma_j\wedge\sigma_k$$

and

$$d\Sigma_i=-\tfrac{1}{2}\epsilon_{ijk}\Sigma_j\wedge\Sigma_k\ .$$

The S^7 (a) size parameter $u=u(x)$ and (b) squashing parameter $v=v(x)$ are now taken to be scalar functions over the spacetime. The metric volume of the S^7 is $3^{-1}\pi^4e^{7u}$, and the squashing parameter[9] is $\lambda^2=e^{-7v}$. A Weyl rescaling of the actual spacetime metric $\bar{g}_{\alpha\beta}=e^{-7u}g_{\alpha\beta}$ has been performed to simplify the resulting equations for $g_{\alpha\beta}$. That is, $g_{\alpha\beta}(x)$, $u(x)$, and $v(x)$ will be taken as the basic gravitational field quantities, and unbarred spacetime curvature tensors and covariant differentiation will be with respect to $g_{\alpha\beta}$. Barred quantities will be with respect to the full metric $\bar{g}_{AB}$.

The Freund-Rubin[2] form for the gauge field strength is to take all components of $\bar{F}_{ABCD}$ zero except for the spacetime components

$$\bar{F}_{\alpha\beta\gamma\delta}=f(x)\bar{\epsilon}_{\alpha\beta\gamma\delta}=fe^{-14u}\epsilon_{\alpha\beta\gamma\delta}\ . \tag{5}$$

Taking f to be independent of the S^7 coordinates y implies that the gauge field strength automatically satisfies the integrability condition $\bar{\nabla}_{[A}\bar{F}_{BCDE]}=0$ for (2), so in any local region there is a gauge field potential $\bar{A}_{BCD}$ giving rise to this field strength. In a globally hyperbolic spacetime, which may be foliated by spatial hypersurfaces, the potential may be chosen locally proportional to the hypersurface volume three-form with the scalar function of proportionality integrated along the foliation by (2) to give a globally defined potential. If, on the other hand, the spacetime is compact without boundary, no global potential will exist and the field strength (5) will be closed but not exact.

Now the gauge field equation (3) applied to (5) in the metric (4) implies that $Q\equiv fe^{7u}$ is a constant over the spacetime, so

$$\bar{F}_{\alpha\beta\gamma\delta}=Qe^{-21u}\epsilon_{\alpha\beta\gamma\delta}\ . \tag{6}$$

This conserved "charge" Q is, up to a constant of proportionality, the integral of the Hodge dual of the field strength over the compact seven-dimensional space and thus is conserved (independent of moving the seven-dimensional hypersurface, i.e., changing the spacetime point at which it is chosen) so long as the right-hand side of (3) vanishes, as it does for the Freund-Rubin field-strength form.

More generally, if one defines the gauge-potential three-form

$$\bar{A}=\frac{1}{3!}\bar{A}_{BCD}dX^B\wedge dX^C\wedge dX^D$$

and the gauge-field-strength four-form

$$\bar{F}=\frac{1}{4!}\bar{F}_{ABCD}dX^A\wedge dX^B\wedge dX^C\wedge dX^D\ ,$$

then the gauge field equations (2) and (3) may be compactly written as $\bar{F}=d\bar{A}$ and

$$d*\bar{F}+\bar{F}\wedge\bar{F}=d(*\bar{F}+\bar{A}\wedge\bar{F})=0\ . \tag{7}$$

By Stokes's theorem, the integral of this throughout an eight-dimensional region may be replaced by an integral of $*\bar{F}+\bar{A}\wedge\bar{F}$ over the seven-dimensional boundary, which is thus also zero. If this boundary has two pieces, the integral is the same on each piece (up to a sign change which may be eliminated by reversing the orientation of one of the pieces from that induced from the eight-dimensional region). In particular, if the 11-dimensional space has the topology of the product of a Lorentzian four-dimensional spacetime with a compact seven-dimensional space, the integral over the latter at fixed x gives a conserved charge that is independent of the point x of the spacetime:

$$Q=-3\pi^{-4}\int_{x\ \mathrm{fixed}}(*\bar{F}+\bar{A}\wedge\bar{F})\ . \tag{8}$$

All this applies in general under the assumption that the fermion field vanishes, for otherwise it may provide a nonzero source term on the right-hand side of (7).

The next task is to solve the 11-dimensional Einstein equations (1), which, with the gauge field strength given by (6), become

$$\bar{R}^{\alpha}_{\beta}=-\tfrac{4}{3}Q^2e^{-14u}\delta^{\alpha}_{\beta}\,, \tag{9}$$

$$\bar{R}^{a}_{b}=+\tfrac{2}{3}Q^2e^{-14u}\delta^{a}_{b}\,, \tag{10}$$

$$\bar{R}^{\alpha}_{b}=\bar{R}^{a}_{\beta}=0\,. \tag{11}$$

An evaluation of the Ricci tensor of the metric (4) in terms of $g_{\alpha\beta}$, u, and v yields

$$\bar{R}^{\alpha}_{\beta}=e^{7u}(R^{\alpha}_{\beta}+\tfrac{7}{2}\delta^{\alpha}_{\beta}u^{;\gamma}_{;\gamma}-\tfrac{63}{2}u^{;\alpha}u_{;\beta}-21v^{;\alpha}v_{;\beta})\,, \tag{12}$$

$$\bar{R}^{5}_{5}=\bar{R}^{6}_{6}=\bar{R}^{7}_{7}=\bar{R}^{8}_{8}=12e^{-2u-3v}-6e^{-2u-10v}-e^{7u}(u^{;\alpha}_{;\alpha}+\tfrac{3}{2}v^{;\alpha}_{;\alpha})\,, \tag{13}$$

$$\bar{R}^{9}_{9}=\bar{R}^{10}_{10}=\bar{R}^{11}_{11}=2e^{-2u+4v}+4e^{-2u-10v}-e^{7u}(u^{;\alpha}_{;\alpha}-2v^{;\alpha}_{;\alpha})\,, \tag{14}$$

all other terms zero. Inserting (13) and (14) into (10) yields the field equations for u and v:

$$u^{;\alpha}_{;\alpha}=\tfrac{6}{7}e^{-9u+4v}+\tfrac{48}{7}e^{-9u-3v}-\tfrac{12}{7}e^{-9u-10v}-\tfrac{2}{3}Q^2e^{-21u}\,, \tag{15}$$

$$v^{;\alpha}_{;\alpha}=-\tfrac{4}{7}e^{-9u+4v}+\tfrac{24}{7}e^{-9u-3v}-\tfrac{20}{7}e^{-9u-10v}\,. \tag{16}$$

Putting (15) into (12) and equating this to the right-hand side of (9) gives

$$R^{\alpha}_{\beta}=\tfrac{63}{2}u^{;\alpha}u_{;\beta}+21v^{;\alpha}v_{;\beta}+\delta^{\alpha}_{\beta}(-3e^{-9u+4v}-24e^{-9u-3v}+6e^{-9u-10v}+Q^2e^{-21u})\,. \tag{17}$$

The field equations (15)—(17) are the Euler-Lagrange equations for the effective four-dimensional Lagrangian

$$\mathscr{L}=\sqrt{-g}\,(R-\tfrac{63}{2}u^{;\alpha}u_{;\alpha}-21v^{;\alpha}v_{;\alpha}+6e^{-9u+4v}+48e^{-9u-3v}-12e^{-9u-10v}-2Q^2e^{-21u})\,. \tag{18}$$

(The Weyl rescaling $\bar{g}_{\alpha\beta}=e^{-7u}g_{\alpha\beta}$ is thus effective in eliminating what would otherwise be a u-dependent coefficient of the scalar curvature of $g_{\alpha\beta}$ in the four-dimensional Lagrangian.)

The AdS-invariant ground-state solutions correspond to setting $u=$const, $v=$const, and having the spacetime curvature maximally symmetric:

$$R_{\alpha\beta\gamma\delta}=\tfrac{1}{3}\Lambda(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})\,. \tag{19}$$

Equation (16) then implies that either $v=v_1=0$ (the round S^7) or $v=v_2=\frac{1}{7}\ln 5$ (the squashed S^7 with the nonstandard Einstein metric).[7–9]

First, examine the round S^7 and perturbations of it. For $v=v_1=0$, $u=u_1$, Eq. (15) implies that $e^{12u_1}=\frac{1}{9}Q^2$, and Eq. (17) gives $R^{\alpha}_{\beta}=\Lambda_1\delta^{\alpha}_{\beta}$ with

$$\Lambda_1=-12e^{-9u_1}=-12(\tfrac{1}{3}\,|\,Q\,|\,)^{-3/2}\,.$$

In the notation of Duff *et al.*,[3–6,9,10] where $\bar{F}_{\mu\nu\rho\sigma}=3m\bar{\epsilon}_{\mu\nu\rho\sigma}$, we have

$$m=m_1=\tfrac{1}{3}f_1=\tfrac{1}{3}Qe^{-7u_1}=e^{-u_1}\mathrm{sgn}Q\,,$$

and the cosmological constant of the full spacetime metric

$$\bar{g}_{\alpha\beta}=e^{-7u_1}g_{\alpha\beta}=|\,m_1\,|^{7}g_{\alpha\beta}$$

is

$$\bar{\Lambda}_1=e^{7u_1}\Lambda_1=-12m_1^2=-12(\tfrac{1}{3}\,|\,Q\,|\,)^{-1/3}\,. \tag{20}$$

This solution has one free parameter, which may be taken as the conserved charge Q, the linear size e^{u_1} of the S^7, the parameter m_1, or the cosmological constant of spacetime.

For small perturbations about the AdS-invariant solution, we linearize Eqs. (15) and (16) to obtain

$$(u-u_1)^{;\alpha}_{;\alpha}\approx 72e^{-9u_1}(u-u_1)=6(-\Lambda_1)(u-u_1)\,, \tag{21}$$

$$v^{;\alpha}_{;\alpha}\approx 16e^{-9u_1}v=\tfrac{4}{3}(-\Lambda_1)v\,. \tag{22}$$

Since the coefficients on the right-hand sides are positive, and since the spacetime metric signature is being taken as $(-+++)$, the round S^7 solution is stable to small perturbations of both dilations (changes in u) and squashings (changes in v). Alternately, we can write the four-dimensional Lagrangian

$$\mathscr{L}=\sqrt{-g}\,[R-2\Lambda_1-\tfrac{63}{2}u^{;\alpha}u_{;\alpha}-21v^{;\alpha}v_{;\alpha}-2V_1(u,v)]\,, \tag{23}$$

where the potential for the scalar fields is

$$V_1(u,v)=(-\Lambda_1)[1-\tfrac{1}{4}e^{-9(u-u_1)}\times(e^{4v}+8e^{-3v}-2e^{-10v})+\tfrac{3}{4}e^{-21(u-u_1)}]\,. \tag{24}$$

which has a local minimum at $u=u_1$, $v=0$.

Second, examine the squashed S^7, with $e^{7v_2}=5$. Equation (15) implies that a stationary solution has $u=u_2$ with

$$e^{12u_2}=\tfrac{1}{81}Q^2e^{10v_2}=3^{-4}5^{10/7}Q^2\,,$$

and Eq. (17) gives $R^{\alpha}_{\beta}=\Lambda_2\delta^{\alpha}_{\beta}$ with

$$\Lambda_2=-108e^{-9u_2-10v_2}=-2^2 3^6 5^{-5/2}\,|\,Q\,|^{-3/2}\,.$$

Now Duff's

$$m=m_2=\tfrac{1}{3}Qe^{-7u_2}=3^{4/3}5^{-5/6}\,|\,Q\,|^{-1/6}\mathrm{sgn}Q\,,$$

so

$$Q=3^8 5^{-5}m_2{}^{-6}\mathrm{sgn}m_2=2.099\,52m_2{}^{-6}\mathrm{sgn}m_2\,,$$

$$e^{u_2}=3\times 5^{-5/7}\,|\,m_2\,|^{-1}\,,$$

$$\Lambda_2=-2^2 3^{-6}5^5\,|\,m_2\,|^{9}\,,$$

and

$$\bar{\Lambda}_2=-12m_2{}^2=-4(3^{-11}5^5\,|\,Q\,|\,)^{-1/3}=3^{7/3}5^{-5/3}\bar{\Lambda}_1\simeq 0.887\,836\,35\bar{\Lambda}_1\,, \tag{25}$$

about 11% smaller in magnitude than the full cosmological constant $\bar{\Lambda}_1$ for the round S^7 with the same conserved charge Q. (The different values of $\bar{\Lambda}$ for the same Q mean the round and squashed S^7 solutions are in different superselection sectors with different asymptotic behavior of the corresponding AdS metrics, so their energies cannot be compared, nor can one tunnel into the other if the asymptotic conditions are kept fixed.)

To check the stability of the squashed AdS-invariant solutions, we linearize Eqs. (15) and (16) about $u=u_2$, $v=v_2$:

$$(u-u_2)_{;\alpha}^{;\alpha} \approx 648 e^{-9u_2-10v_2}(u-u_2)=6(-\Lambda_2)(u-u_2)\ , \tag{26}$$

$$(v-v_2)_{;\alpha}^{;\alpha} \approx -80 e^{-9u_2-10v_2}(v-v_2)=-\tfrac{20}{27}(-\Lambda_2)(v-v_2)\ . \tag{27}$$

The squashed S^7 is thus directly seen to be stable to dilations, but it naively appears to be unstable to the exponential growth of a change in the squashing parameter v. However, Breitenlohner and Freedman[16,17] have shown that in AdS spacetime, small perturbations in a test field ϕ with suitable asymptotic boundary conditions are actually stable if

$$\phi^{-1}\phi_{;\alpha}^{;\alpha} \geq -\tfrac{3}{4}(-\Lambda)\ . \tag{28}$$

Since $\frac{20}{27}<\frac{3}{4}$, small perturbations in the squashing parameter are stable, though only by a small margin of safety.

One may alternately examine the stability of the squashed AdS-invariant solution by rewriting (18) as

$$\mathscr{L}=\sqrt{-g}\,[R-2\Lambda_2-\tfrac{63}{2}u^{;\alpha}u_{;\alpha}-21v^{;\alpha}v_{;\alpha}-2V_2(u,v)]\ , \tag{29}$$

where now with Λ_2 rather than Λ_1 extracted, the potential $V_2=V_1+\Lambda_1-\Lambda_2$ for the scalar fields has the form

$$V_2(u,v)=(-\Lambda_2)\{1-\tfrac{1}{36}e^{-9(u-u_2)}[25e^{4(v-v_2)}+40e^{-3(v-v_2)}-2e^{-10(v-v_2)}]+\tfrac{3}{4}e^{-21(u-u_2)}\}\ . \tag{30}$$

At fixed v, V_2 has no maximum as u is varied, so there can be no unstable equilibrium for the S^7 size parameter. However, at fixed u, V_2 has a minimum at $v=v_1=0$ (the round S^7) but a maximum at $v=v_2$ (the squashed S^7). One might think the squashed AdS-invariant solution would be unstable to a perturbation in which the S^7 squashing parameter v is different from v_2 in a region of AdS space, thereby lowering the potential energy in that region. However, the point of the Breitenlohner-Freedman argument is that in AdS space the ratio of the surface area to the volume of a region is always greater than some numerical constant times $(-\Lambda)^{1/2}$, so the spatial-gradient terms in (29), needed in order that the perturbation go to zero at spatial infinity, dominate the reduction in the potential energy and make the total energy of a small perturbation positive if the condition (28) is satisfied.

Although we have seen that when the gauge field strength has the Freund-Rubin form, both (1) the round S^7 and (2) the squashed S^7 AdS-invariant solutions are stable to small perturbations in (a) the S^7 size and (b) the degree of squashing, we might still ask whether there could be an instability to tunneling to a configuration with a large perturbation. If we hold the asymptotic conditions fixed in the AdS space, this tunneling would conserve the Abbott-Deser energy[18] and so would be possible only if a perturbed configuration existed with the same energy as the AdS-invariant configuration. One would expect that an extension of positive-mass theorems in four-dimensional supergravity theories[19–21,16,17,22] to this 11-dimensional theory might exclude such perturbed configurations of the same energy in the cases that the AdS-invariant solution has a residual four-dimensional supersymmetry.[23] If so, this would prove that the round S^7 solution with $N=8$ supersymmetry[3–6] and the left-squashed S^7 with $N=1$ supersymmetry[9,5,6,10] would be absolutely stable to all perturbations. The Lagrangian (29) is the same for both left- and right-handed squashings, so the right-squashed S^7 AdS-invariant solution with no supersymmetry[10] would also be stable to the perturbations considered in this paper.

To get a feel for whether a large perturbation exists with the same energy as the unperturbed AdS-invariant solution, one may consider a momentarily static spherical configuration. For simplicity, consider a perturbation in a single canonically normalized field ϕ with the Lagrangian

$$\mathscr{L}=\sqrt{-g}\,[R-2\Lambda-\phi^{;\alpha}\phi_{;\alpha}-2V(\phi)]\ . \tag{31}$$

For example, we could take $u=u_2$ and $\phi=\sqrt{21}(v-v_2)$. If the momentarily static spherical three-metric is written as

$$\left[1+\frac{r^2}{b^2}-\frac{2m(r)}{r}\right]^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\ , \tag{32}$$

where $b^2\equiv -3\Lambda^{-1}>0$ for the asymptotically AdS configuration being considered, then the Einstein equations give, with $\phi=\phi(r)$,

$$\frac{dm}{dr}=\tfrac{1}{4}r^2\left[1+\frac{r^2}{b^2}-\frac{2m}{r}\right]\left[\frac{d\phi}{dr}\right]^2+\tfrac{1}{2}r^2V(\phi(r))\ . \tag{33}$$

For a given configuration $\phi(r)$, the solution gives a total energy

$$M\equiv m(\infty)=\int_0^\infty dr\left[\frac{r^2}{4}\left[1+\frac{r^2}{b^2}\right]\left[\frac{d\phi}{dr}\right]^2+\frac{r^2}{2}V\right]\times\exp\left[-\int_r^\infty\frac{r'}{2}\left[\frac{d\phi}{dr'}\right]^2dr'\right]\ . \tag{34}$$

The configuration must have ϕ tending toward an extremum of $V(\phi)$ [which we will here take to be at $\phi=0$, with $V(0)=0$ in order that M be defined] faster than $r^{-3/2}$ at radial infinity in order that the integral of the gradient term converge, and if

$$V(\phi) \geq -\frac{9\phi^2}{8b^2}, \tag{35}$$

which is equivalent to the Breitenlohner-Freedman condition (28) for a solution of the field equation $\phi_{;\alpha}^{;\alpha}=\partial V/\partial\phi$, then $M \geq 0$ with equality only if $\phi(r)=0$ for all r. However, the behavior of the potential (30) for $u=u_2$ and large ϕ is

$$V_2\left[u_2, v=v_2+\frac{\phi}{\sqrt{21}}\right] \sim -\frac{25}{12b^2}\exp\left[\frac{4\phi}{\sqrt{21}}\right], \tag{36}$$

which drops far below the criterion (35) for large ϕ. If the last factor were not present in Eq. (34) for M, one could easily construct configurations with $M<0$. Nevertheless, the exponential factor at the end of the expression for M, which acts to reduce the magnitude of the local energy density by a gravitational redshift factor, seems to prevent $M \leq 0$ from being possible unless $\phi=0$ everywhere. Although the integrand may be made negative at small r, it must become positive at large r as ϕ approaches zero and the condition (35) applies. The exponential factor in the integral for M reduces the effect of the negative contribution much more than that of the positive contribution and apparently always keeps $M>0$ (unless $\phi=0$), though I have no rigorous proof of this. It would be interesting to know the fastest decreasing $V(\phi)$ possible that does not allow $M<0$. It can be shown that

$$V(\phi) \sim -\frac{3}{5b^2}\exp\left[\frac{5\sqrt{6}}{4}\phi\right]$$

allows $M<0$, so the borderline case is presumably asymptotically exponential.

III. GAUGE FIELDS HAVING COMPONENTS PROPORTIONAL TO AN S^7 TORSION

In these configurations the gauge field strength not only has the Freund-Rubin components (5) but also has components obtained from differentiating a gauge field potential proportional to a suitable, totally antisymmetric torsion tensor S_{bcd} on the S^7. Such a suitable torsion is only known to exist for the round S^7 metric[12] and for the squashed S^7 with the nonstandard Einstein metric.[9,15,10] Hence, I shall restrict attention to the metric (4) with either $v=v_1=0$ or $v=v_2=\frac{1}{7}\ln 5$.

It is convenient to rewrite the 11-dimensional metric as

$$\begin{aligned}\bar{g}_{AB}dX^A dX^B &= \bar{g}_{\alpha\beta}dx^\alpha dx^\beta+\bar{g}_{ab}dy^a dy^b \\ &= e^{-7u}g_{\alpha\beta}dx^\alpha dx^\beta+e^{2u}g_{ab}dy^a dy^b,\end{aligned} \tag{37}$$

where now g_{ab} is normalized so $R_{ab}=6g_{ab}$. [This means that if $v=v_1$, the new u is the same as the old u in the metric (4), but if $v=v_2$, the factor of e^{2u} in the last term of (37) is the same as $3^{-2}5^{10/7}e^{2u}$ in (4).] Unbarred tensors will be defined and manipulated using the unbarred metrics $g_{\alpha\beta}$ and g_{ab}; barred tensors will be defined and manipulated using the barred metrics $\bar{g}_{\alpha\beta}$ and $\bar{g}_{ab}$. The torsion tensor in the normalized g_{ab} obeys the equations[12,15,10]

$$S^{acd}S_{bcd}=6\delta_b^a, \tag{38}$$

$$S_{ab}{}^{c}S_{cd}{}^{e}S_{ef}{}^{a}=3S_{bdf}, \tag{39}$$

$$S_{abc;d}=S_{[abc;d]}=S_{e[ab}S^{e}{}_{c]d}, \tag{40}$$

$$S^{abc}{}_{;c}=0, \tag{41}$$

$$S_{abc}=-\tfrac{1}{24}\epsilon_{abcdefg}S^{def;g}. \tag{42}$$

On the round S^7 there is also a suitable torsion of the opposite handedness,[12] but for simplicity I shall just consider this one, since the other is equivalent under a reversal of the coordinate orientation and the sign of f.

Now, in addition to the Freund-Rubin components (5), I shall take the field-strength components

$$\bar{F}_{Abcd}=-\bar{F}_{bAcd}=\bar{F}_{bcAd}=-\bar{F}_{bcdA}=4\bar{\nabla}_{[A}\bar{A}_{bcd]} \tag{43}$$

with the gauge-potential components

$$\bar{A}_{bcd}=\tfrac{1}{4}e^{3u+w}S_{bcd}, \tag{44}$$

where e^{3u} gives the appropriate scaling when the S^7 metric g_{ab} is scaled by e^{2u} to give $\bar{g}_{ab}$, and where $w=w(x)$ is a Lorentz scalar field giving the magnitude of these gauge-potential components:

$$\bar{A}_{bcd}\bar{A}^{bcd}=\tfrac{21}{8}e^{2w}. \tag{45}$$

Evaluating the conserved charge (8) yields

$$Q=qS, \tag{46}$$

$$q=fe^{7u}-\tfrac{7}{4}e^{6u+2w}, \tag{47}$$

$$\begin{aligned}S&=3\pi^{-4}\int\sqrt{g}\,d^7y \\ &=1 \text{ for } v=v_1,\ 3^7 5^{-5} \text{ for } v=v_2.\end{aligned} \tag{48}$$

Holding Q fixed is equivalent to solving the gauge field Eq. (3) with BCD being spacetime indices. The other nontrivial gauge field equations to be solved are when BCD are S^7 indices. This yields, using (42),

$$\begin{aligned}\bar{\nabla}_A\bar{F}^{Abcd}&=\{\tfrac{1}{4}e^{7u}[e^{-3u+w}(3u+w)^{;\alpha}]_{;\alpha}-4e^{-5u+w}\}S^{bcd} \\ &=-\tfrac{1}{576}\bar{\epsilon}^{M_1\cdots M_8bcd}\bar{F}_{M_1\cdots M_4}\bar{F}_{M_5\cdots M_8} \\ &=2e^{-4u+w}fS^{bcd},\end{aligned} \tag{49}$$

so if $\bar{A}^{bcd}=\frac{1}{4}e^{-3u+w}S^{bcd}\neq 0$ in order that it may be divided out,

$$f=-2e^{-u}+\tfrac{1}{8}e^{8u}(3u_{;\alpha}^{;\alpha}+w_{;\alpha}^{;\alpha}-9u^{;\alpha}u_{;\alpha}+w^{;\alpha}w_{;\alpha}). \tag{50}$$

The nontrivial Einstein equations (1) become

$$\begin{aligned}\bar{R}^{\alpha}_{\beta}&=e^{7u}(R^{\alpha}_{\beta}+\tfrac{7}{2}\delta^{\alpha}_{\beta}u_{;\gamma}^{;\gamma}-\tfrac{63}{2}u^{;\alpha}u_{;\beta}) \\ &=\tfrac{1}{3}\bar{F}^{\alpha BCD}\bar{F}_{\beta BCD}-\tfrac{1}{36}\delta^{\alpha}_{\beta}\bar{F}^{ABCD}\bar{F}_{ABCD} \\ &=-\tfrac{4}{3}\delta^{\alpha}_{\beta}(f^2+\tfrac{7}{2}e^{-2u+2w}) \\ &\quad+\tfrac{7}{24}e^{7u+2w}[3(3u+w)^{;\alpha}(3u+w)_{;\beta} \\ &\qquad-\delta^{\alpha}_{\beta}(3u+w)^{;\gamma}(3u+w)_{;\gamma}],\end{aligned} \tag{51}$$

$$\bar{R}^a_b=\delta^a_b(6e^{-2u}-e^{7u}u^{;\alpha}_{;\alpha})$$
$$=\tfrac{1}{3}\bar{F}^{aBCD}\bar{F}_{bBCD}-\tfrac{1}{36}\delta^a_b\bar{F}^{ABCD}\bar{F}_{ABCD}$$
$$=\delta^a_b[\tfrac{2}{3}f^2+\tfrac{10}{3}e^{-2u+2w}$$
$$+\tfrac{1}{12}e^{7u+2w}(3u+w)^{;\alpha}(3u+w)_{;\alpha}] . \qquad (52)$$

Equations (50) and (52) then give four-dimensional field equations for u and w:

$$u^{;\alpha}_{;\alpha}=6e^{-9u}-\tfrac{10}{3}e^{-9u+2w}-\tfrac{2}{3}e^{-7u}f^2$$
$$-\tfrac{1}{12}e^{2w}(3u+w)^{;\alpha}(3u+w)_{;\alpha} , \qquad (53)$$
$$w^{;\alpha}_{;\alpha}=10e^{-9u+2w}-2e^{-9u}+8e^{-8u}f$$
$$+2e^{-7u}f^2+9u^{;\alpha}u_{;\alpha}-w^{;\alpha}w_{;\alpha}$$
$$+\tfrac{1}{4}e^{2w}(3u+w)^{;\alpha}(3u+w)_{;\alpha} . \qquad (54)$$

Inserting (53) into (51) allows us to write the four-dimensional Einstein equations as

$$G^\alpha_\beta=\tfrac{63}{2}u^{;\alpha}u_{;\beta}+\tfrac{7}{8}e^{2w}(3u+w)^{;\alpha}(3u+w)_{;\beta}$$
$$+\delta^\alpha_\beta[-\tfrac{63}{4}u^{;\gamma}u_{;\gamma}-\tfrac{7}{16}e^{2w}(3u+w)^{;\gamma}(3u+w)_{;\gamma}$$
$$+21e^{-9u}-7e^{-9u+2w}-e^{-7u}f^2] . \qquad (55)$$

Equations (53)–(55) are the Euler-Lagrange equations for the four-dimensional Lagrangian

$$\mathcal{L}=\sqrt{-g}\,[R-\tfrac{63}{2}u^{;\alpha}u_{;\alpha}$$
$$-\tfrac{7}{8}e^{2w}(3u+w)^{;\alpha}(3u+w)_{;\alpha}-2V(u,w)] , \qquad (56)$$

$$V(u,w)=-21e^{-9u}+7e^{-9u+2w}$$
$$+e^{-7u}(qe^{-7u}+\tfrac{7}{4}e^{-u+2w})^2 , \qquad (57)$$

where the explicit expression for f from Eq. (47) has been inserted, since it is determined by the gauge field equations in terms of the constant q and the Lorentz scalars u and w and, hence, is not a dynamical variable if u and w are.

The potential (57) has no extremum at finite u, w if $Q=qS>0$, but if $Q<0$ it has a single extremum at

$$u=u_3=\tfrac{1}{6}\ln\left[-\frac{4q}{15}\right]$$

and $w=w_3=0$. This gives $f=f_3=-2e^{-u_3}$, in agreement with Duff, Nilsson, and Pope[10] since their $m=m_3=e^{-u_3}$ so that $\bar{R}_{ab}=6m_3^2\bar{g}_{ab}$ at the extremum. (Their unbarred tensors are equivalent to my barred tensors.) Thus, $q=-\frac{15}{4}m_3^{-6}$. The total conserved charge (8) is, by (48),

$$Q=q=-2^{-2}3\times5m_3^{-6}=-3.75m_3^{-6}$$

for the round S^7 and

$$Q=3^75^{-5}q=-2^{-2}3^85^{-4}m_4^{-6}=-2.6244m_4^{-6}$$

for the squashed S^7, where m_3 has now been used for the round S^7 and m_4 for the squashed S^7 AdS-invariant solutions in order that these may be compared at a fixed value of the conserved charge.

The value of the potential (57) at the extremum is

$$V_3=-10e^{-9u_3}=-10\left[\frac{-4q}{15}\right]^{-3/2}=-10m_3^9 ,$$

which is the effective cosmological constant Λ_3 of the metric $g_{\alpha\beta}$. The cosmological constant of the full spacetime AdS metric $\bar{g}_{\alpha\beta}=e^{-7u_3}g_{\alpha\beta}$ is then

$$\bar{\Lambda}=e^{7u_3}V_3=-10e^{-2u_3}=-10m_3^2 .$$

Expressed in terms of the conserved charge Q,

$$\bar{\Lambda}_3=-2^{1/3}3^{1/3}5^{4/3}(-Q)^{-1/3}$$
$$=(3750Q^{-1})^{1/3}$$
$$=2^{-5/3}3^{-1}5^{4/3}\bar{\Lambda}_1\approx0.897\,681\,12\bar{\Lambda}_1 , \qquad (58)$$
$$\bar{\Lambda}_4=-2^{1/3}3^{8/3}5^{-1/3}(-Q)^{-1/3}$$
$$=(2624.4Q^{-1})^{1/3}$$
$$=2^{-5/3}3^{4/3}5^{-1/3}\bar{\Lambda}_1\approx0.796\,993\,9\bar{\Lambda}_1 , \qquad (59)$$

where the values are compared at the same

$$Q=2^63^4\bar{\Lambda}_1^{-3}=5184\bar{\Lambda}_1^{-3} .$$

Since the extremum (existing for $Q<0$) of the potential (57) is an absolute minimum, and since the kinetic terms in the Lagrangian (56) have the usual signs, the AdS-invariant solutions with gauge-potential "torsion" components are absolutely stable to perturbations of the S^7 size and the magnitude of the "torsion" components. For small perturbations, one may linearize Eqs. (53) and (54) about $u=u_3$, $w=w_3=0$ and diagonalize the resulting mass matrix to obtain

$$(u-u_3+\tfrac{1}{15}w)^{;\alpha}_{;\alpha}\approx60e^{-9u_3}(u-u_3+\tfrac{1}{15}w)$$
$$=6(-\Lambda_3)(u-u_3+\tfrac{1}{15}w) , \qquad (60)$$

$$w^{;\alpha}_{;\alpha}\approx20e^{-9u_3}w=2(-\Lambda_3)w . \qquad (61)$$

Both coefficients are positive, so we see directly that these small perturbations are stable. Note that neither of these AdS-invariant solutions have any residual supersymmetry in four dimensions, so there was no formal argument that they ought to be stable. Of course, my analysis considers only two of the very simplest modes and, therefore, does not prove that the solutions are stable to all perturbations, though if an instability occurs one would rather expect it to show up in a fairly simple mode, since more complicated modes of the full mass matrix are likely to have higher values of the mass squared.[24]

The stability analysis used here for the bosonic degrees of freedom depended crucially on the constancy of the "charge" Q over the spacetime when the topology is that of a product manifold of four-dimensional spacetime with a compact seven-dimensional space and when the fermion field vanishes so that (7) is the gauge field equation. One might ask what effect fermion excitations have on Q. For example, could they tend to cause Q to get bigger or smaller as the spacetime evolved? If Q somehow got bigger than $\sim 10^{360}$, the cosmological constant would be reduced below the observational limits ($\overline{\Lambda} \lesssim 10^{-120}$ in Planck units). However, this would not seem to be a satisfactory solution, for then the compact space would be very large so that the coupling constants for the four-dimensional gauge fields resulting from the symmetries of the compact space would be unmeasurably small. Also, the infinite tower of massive states would become very light, so that presumably we would be able to excite them and see the effects of the extra dimensions (e.g., move around in them). But, whether the effects are good or bad, it would be interesting to know how Q might vary over spacetime.

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