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Dynamical origin of black-hole radiance

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An explanation of the thermal quantum radiance of black holes discovered by Hawking is offered in terms of a black-hole metric undergoing quantum zero-point fluctuations of zero mean in its gravitational quasinormal modes. It is shown that such zero-temperature fluctuations, governed by the uncertainty principle, lead to the formation of a quantum ergosphere that enables matter of all species to tunnel out of the hole. The results confirm that a black hole cannot be in equilibrium at zero temperature. A dynamical temperature is calculated by equating the mean irreducible mass associated with the quantum ergosphere to the mean thermal energy of a quantum oscillator with the lowest quasinormal frequency. The result agrees with the Hawking temperature to within two per cent. The nature of the dynamical equilibrium and the higher modes are discussed, and it is calculated that the thermal excitations of the resonant modes have the canonical distribution to within several per cent. A calculation of the black-hole entropy using the statistical mechanics of the quasinormal modes yields a value $(0.27654)\hbar^{-1}(16\pi M^2)$, which is near the value usually assumed, $(0.25)\hbar^{-1}(16\pi M^2)$. Characteristic fluctuation scales are derived. The rms energy fluctuation of the physical ("dressed") event horizon is about 10^{17} GeV, independent of M. The physical metric fluctuations near the hole are of order unity when the hole has mass$\approx(0.15)$ (Planck mass)$\approx 1.8\times10^{18}$ GeV.

I. INTRODUCTION AND OVERVIEW

The thermal quantum radiance of black holes discovered by Hawking[1] demonstrated remarkable connections among gravity, quantum theory, and a thermodynamic interpretation of black-hole mechanics. Such connections had previously been conjectured and discussed by Bekenstein.[2] Hawking's demonstration of emission of energy by black holes put these ideas on a firm basis. Yet in the treatments of this effect that have appeared to date, the spacetime of the black hole plays no dynamical role; it serves only as an arena in which the spontaneous creation of energy quanta can occur. However, because there results an observable transfer of energy from the black hole to large distances, one concludes that there must exist a dynamical spacetime description of the Hawking effect. In this paper I give such a description by showing that the zero-point fluctuations of the metric of the black hole near its horizon necessarily create a "quantum ergosphere" that enables matter of all species to tunnel out of the hole by a process closely akin to the phenomenon of evanescence in physical optics. By using a semiclassical treatment of the quantum-mechanical uncertainty associated with a simple model spherical black-hole metric oscillating in its quasinormal modes, I am able to calculate a dynamical temperature that differs from the Hawking temperature by less than 2%. One can also establish from this model that the thermal excitations of a black hole in equilibrium are distributed over its quasinormal modes in accordance with the canonical Boltzmann probability to an accuracy of better than 4%. As I shall argue, and as indeed is necessary for the existence of thermal equilibrium, the dynamical effects are dominated by the low-frequency ringing modes. It is possible that a more sophisticated metric is needed for more precise results, though the qualitative nature of the various phenomena exhibited here is firmly established.

From this treatment it is also possible to calculate by simple statistical methods, rather than by mechanical-thermodynamical analogies, the entropy of a black hole. One finds a value close to the one usually assumed.

In the second section of this paper I summarize some recent findings concerning the backreaction on the horizon caused by the decay of a black hole "in vacuum."[3] This work ignores the zero-point metric fluctuations but serves to motivate consideration of them. The basic result is that the emission of energy by an uncharged spherical hole is necessarily associated with the formation of a "quantum ergosphere," which means that the quasistatic or timelike limit surface (TLS) must lie a bit outside the event horizon (EH). In the case of strictly spherical radiance, the TLS coincides with the apparent horizon (AH).[4] If A denotes the surface area, then the existence of a quantum ergosphere (QE) is characterized by the fact that $\delta A_{\rm QE} \equiv A_{\rm TLS} - A_{\rm EH} > 0$. Of course, $\delta A_{\rm QE} > 0$ is just the opposite of what occurs classically when a black hole accretes energy, so in itself $\delta A_{\rm QE} > 0$ is not surprising when a black hole is losing energy. What is of interest, however, is that when the Hawking temperature $T_H = \hbar(8\pi M)^{-1}$ and luminosity $L = B\hbar M^{-2}$ go to zero as M becomes arbitrarily large, the quantity $\delta A_{\rm QE}$ does not vanish, but takes on a value determined by Planck's constant and by the number of species of massless quanta that exist in Nature. $\delta A_{\rm QE}$ vanishes in the classical limit $\hbar \rightarrow 0$.

The nonvanishing of $\delta A_{\rm QE}$ as T_H approaches zero suggests that zero-temperature vacuum fluctuations of the spacetime metric of the horizon may play a role in the phenomenon of quantum radiance. This is the central issue considered in this paper. It addresses the previously unanswered question as to whether quantum gravity, through the phenomenon of vacuum or zero-point fluctuations, which are well established for other quantum fields, is related to and is consistent with quantum radiance as calculated using external quantum fields on a background whose own quantum-mechanical degrees of freedom are ignored. (Here I take the attitude that including massless propagating spin-2 external fields in the calculation of total luminosity is irrelevant to the issue of the effect of nonradiative zero-point fluctuations of the black-hole metric.)

I find that zero-point metric fluctuations, governed essentially by the uncertainty principle, do play an important role. In this introduction, I set out the basic ideas I have used. Remarkably, one can show that even if one ignores all the quantum fields except gravity, and disregards all models as to how it came into existence, a black hole that undergoes vacuum metric fluctuations cannot be in equilibrium at zero temperature because of a vacuum polarization effect ("tidal quadrupolarization") that creates a quantum ergosphere. One can, in a natural way, calculate an effective temperature that *results* from *zero*-temperature fluctuations. This is possible because $\hbar\omega/k_B T$ is small for all the relevant frequencies. Note that both the resonant frequencies ω and the temperature T scale with the mass M of the hole as M^{-1}. This scaling property is important because it will enable us to assert unambiguously that the black hole is "cold." That is, the "energy level" structure of the hole, as characterized by its ringing modes, is only slightly excited above its "ground state" at zero temperature.

There is in these calculations, ostensibly, a problem with the high frequencies. The temperature T at first sight appears to vary with the resonant ω's. However, consideration of the nature of the resonances shows how a black hole is able to come to an equilibrium temperature, the same for all modes, because the high-frequency resonances are associated with "gravitons" propagating effectively in a "storage ring" near the unstable circular photon orbit at $r \approx 3M$.[5] Because of this effect, the modes of angular momentum index l greater than 2 (the lowest) are not able to any large extent to raise tides at $r \cong 2M$. On the other hand, the $l = 2$ modes propagate nearly "freely" in a radial manner: the quadrupole tides will therefore dominate in equilibrium. These tides are the central dynamical mechanism because they necessarily have slightly different effects on the timelike limit surface, the apparent horizon, and the event horizon. This creates the quantum ergosphere and shows that dynamically the radiance is driven by *curvature* fluctuations, in accordance with the principle of equivalence. The equilibration of the tidal effects for the resonant frequencies associated with $l = 2,3,4,\ldots$ turns out to be equivalent to a Boltzmann probability distribution for the thermal excitations of the black hole that are produced by zero-point fluctuations in these modes.

The physical and geometrical features of a black-hole spacetime strongly suggest the use of null surfaces to calculate the quantum effects: ingoing null surfaces (there is no past event horizon) and outgoing null surfaces, among which there is "one" that is the future event horizon. Using null surfaces is analogous to the so-called "infinite-momentum-frame" quantization in flat spactime.[6] However, here the metric variables that define these surfaces are fluctuating, unlike the situation with a fixed "background" spacetime. This cannot be ignored in the present problem. For example, the "bare" surface $r = 2M$ is no longer null. One must also face the delicate issue that the physical event horizon defined in the usual manner is not locally determined by the incoming characteristic data, that are here prescribed in such a way as to model the zero-point fluctuations. Nevertheless, these difficulties can be overcome and the "background" Schwarzschild metric plays no role in the results; one uses only the "dressed" physical surfaces.

In this work the null surface commutation relations (which are obtained from the gravitational action) are applied only to the extent of obtaining a meaningful "uncertainty relation" to relate classically averaged metric variances to Planck's constant. One deals with the physical components of "transverse" metric fluctuations to obtain dimensionless amplitude values ϵ (proportional to $\hbar^{1/2}M^{-1}$). Hence, the method is semiclassical and represents only an initial step in a fully quantum-theoretic treatment of black holes. I avoid the problematical issue of pseudotensor expressions for gravitational zero-point energy. The notion of energy enters instead through the "irreducible mass"[7] associated with geometrically well-defined areas.

This way of dealing with metric fluctuations is somewhat like Welton's instructive semiclassical treatment of the Lamb shift.[8] Indeed, the phenomena resulting from metric zero-point fluctuations of a black hole are physically analogous to the Lamb shift for an electron in hydrogen in several ways. In both cases, it is the root-mean-square (rms) value of a dynamical variable (electron position; horizon location), rather than its mean fluctuation (which is zero), that determines the physical effect. In both cases, there is a small splitting of otherwise degenerate energy levels: In the hydrogen atom, the $2S_{1/2}$ and $2P_{1/2}$ electron states; in the black hole, the "irreducible" masses that can be associated with the timelike limit surface, apparent horizon, and event horizon. (These surfaces coincide—are degenerate—for a classical Schwarzschild black hole.) In the Lamb shift there are apparent ultraviolet problems that must be regulated; similarly for the black hole. However, unlike the Lamb shift, there are no infrared problems for the black hole: the lowest gravity frequencies for a given angular mode have very small transmission probabilities through the effective potential barrier and do not affect the horizon. A more important difference is that the atom can reach an equilibrium at zero temperature in a "Hohlraum" while a black hole cannot do so in a zero-temperature *Hohlraum*: the hole must acquire a nonzero temperature. This verifies a conjectured interpretation by Candelas and Sciama[9] of Hawking's original treatment of black-hole radiance,[1] in which they contrasted the atom and the black hole, and asserted that a black hole formed from gravitational collapse would not achieve an equilibrium state at temperature zero.

Another physical analogy supports and motivates this work and its interpretation: the phenomenon of evanescence in physical optics. Suppose an optical electromagnetic wave in a dense medium undergoes total internal reflection at an optically flat surface. If that surface is then perturbed by sprinkling upon in granules of appropriate size, the evanescent waves lying along the surface become real and propagate away. Or, if another similar optical medium is brought sufficiently near the first one, the waves will tunnel through the gap and propagate in the second medium as photons.

For a vacuum black hole, the "flat" surface can be imagined to be simply the event horizon $r=2M$ in a standard Eddington-Finkelstein diagram. When this surface is perturbed, energy radiates out. Now there *are,* in fact, purely imaginary resonant modes for every massless spin for spherical holes. It is known that if one adds some angular momentum to the hole, creating thereby a *classical* ergosphere, these modes acquire a real part and propagate.[10] It is just as if the time direction were rotated a bit in the complex plane.

When spherical holes undergo vacuum fluctuations, we will see that this "complex rotation" of the time direction is essentially $\pi/2$ in the correct sense during part of every cycle of an oscillation. The imaginary modes can become purely real part of the time without changing their moduli. Those whose frequency values are sufficiently large for a given total angular momentum (orbital plus spin) can escape from the hole to infinity. This is a tunneling phenomenon. Another way to picture it is as a blurring or oscillation of the physical light cone at the unperturbed horizon $r=2M$. When the cone opens, energy of every kind escapes. When it closes, nothing happens if there is no "matter" nearby. The net result is then leakage of energy. This verifies Hawking's remark:[1] "It should not be thought unreasonable that... because of quantum fluctuations of the metric, energy should be able to tunnel out... of a black hole." A black hole is thus a quantum mechanically unstable excited state of spacetime. Its "slow" instability—leakage of energy—might even be termed "secular," though of course there are no dissipative terms in the usual sense in its Lagrangian. In essence, there are boundary effects resulting from the boundary conditions that condition its existence.

I should emphasize that the present "geometrodynamical" treatment of quantum radiance is to be thought of as *complementary* to the usual ways of viewing the phenomenon. Consistency of the "fixed background plus dynamical external fields" approach and this one, which has "dynamical background plus no external fields," would seem to be significant. Moreover, the present approach, while being semiclassical and therefore approximate, has several bonuses besides giving some indications about the relations of quantum gravity and black holes. For example, I calculate by a simple statistical method the entropy and thermal fluctuations. I also show that one can find, independently of the mass of the black hole, a characteristic rms value for the irreducible mass fluctuation of the physical event horizon ($\sim 10^{17}$ GeV). Within the present approximation, one can discuss the breakdown of microcausality: the physical metric fluctuation is of order unity when the dimensionless parameter ϵ is still less than one ($\sim 10^{-1}$). This occurs, in the present approximation, when the black hole has mass $\approx(0.15)\times$(Planck mass).

A known physical feature of black holes that facilitates these calculations is the "quasinormal mode" spectrum of gravitational resonances ("ringing modes") that has been found in classical perturbation theory.[11] It has been found that, for each angular momentum index l, a spherical hole has a sequence of complex resonant frequencies. For each l there is one frequency whose Q is much greater than that of the others; this is called the "fundamental" resonance for that l. The real part of the fundamental is the greatest of any of those in the sequence for fixed l. On the other hand, the modulus of the fundamental is the least in the sequence; in other words, the nonfundamental modes are highly damped. In perturbations of spherical collapse, it has been found that at "late" times the gravitation radiation occurs at the least-damped quasinormal mode frequencies.[12] Hence, in this paper attention will be directed to these modes only, as they are expected to be characteristic of spacetime fluctuations. This, in essence, is "phenomenological" input based on results of the classical theory of small perturbations of black holes. In this paper the term "resonance" will be reserved for the least-damped (fundamental) resonance for a given l.

Each resonant mode of the hole is capable of being "spontaneously" activated. Classically, they all damp out to arbitrarily small amplitudes. However, quantum mechanically one does not expect the damping to be complete; one would expect a residual zero-point uncertainty in each such mode. These can be thought of as modeling the zero-point fluctuations of the black hole. However, there is an important point that must be kept in mind: the zero-point oscillations cannot be thought of as sharply localized at the horizon; this would force the amplitude of metric fluctuations of the horizon to be very large, which is physically incorrect. What will happen is that the uncertainty associated with an oscillation of frequency ω will be spread over a region comparable to the wavelength $2\pi/\omega$. Thus, the quantum ergosphere is a small region imprinted on the spacetime metric that results from fluctuations that cannot be regarded physically as being localized to within a region less than the longest relevant wavelength as determined by a distant observer.

An outline of the contents of this paper follows.

In Sec. II I review the results of a simple treatment of the backreaction on the horizon of a neutral spherical hole radiating spherically at a small rate. This work leads, in Sec. III, to the introduction of a very simple oscillating black-hole metric. (The Christoffel symbols, curvature tensors, and related geometric objects derived from this metric are displayed in the Appendx.) Section IV contains, for purely illustrative purposes, a calculation of the location of horizons and other quantities for purely spherical oscillations. The corresponding results for the physically relevant nonspherical oscillations characterized by angular indices $l \geq 2$ and frequencies ω_l are worked out in Sec. V. The results through this point may be regarded as essentially geometrical.

In Sec. VI, I outline the derivation of null-hypersurface commutation relations. Using these, I adopt a corresponding form of the uncertainty relation and calculate approximate dimensionless amplitudes for the relevant physical components of the metric. Section VII contains a key physical hypothesis that the mean irreducible mass associated with the splitting of horizons can be identified with the mean Planckian thermal energy of the field oscillator whose zero-point motion gave rise to the splitting effect. Effective temperatures are then calculated, and it is argued from the physical nature of the quasinormal resonant modes how thermal equilibrium can come about. This leads to a picture in which Boltzmann probability factors determined by the dominant (lowest) mode control the tidal (curvature) effects of the higher modes. I show in Sec. VIII that the black hole in equilibrium is only slightly excited from its putative zero-temperature "classical" state and therefore has, at any "time," only a very small statistical entropy, that is independent of its mass M, from the viewpoint of a distant observer. However, upon adding up this black-hole entropy as it decays very slowly into a heat bath with a temperature infinitesimally less than its own temperature, one finds a result quite close ($\approx 11\%$) in magnitude to the large thermodynamic entropy (0.25) $A\hbar^{-1}$ that is usually attributed to black holes on the basis of mechanical-thermodynamical analogies.

In Sec. IX, I show that from the equilibrium (Hawking) temperature and the frequency of the lowest quasinormal mode, one can calculate the characteristic rms irreducible mass fluctuation of the physical ("dressed") event horizon. Its value is about 10^{17} GeV, independent of the black-hole mass M. I also show that with this model one can obtain estimates of the spacetime scale for which a breakdown in microcausality occurs. In Sec. X, I attempt to assess the indications of these results for future work.

II. BACKREACTION TO BLACK-HOLE RADIANCE

I shall review briefly the response of an uncharged spherical black hole to a small spherical emission of energy.[3] In this paper, by "black hole" I shall always mean a "hole" with a future event horizon (EH) but *no* past event horizon. I shall not consider models as to how black holes form.

The spacetime of a spherical hole can be described by a metric of the type considered by Bardeen,[13]

$$ds^2 = -e^{2\psi}(1-2mr^{-1})dv^2 + 2e^{\psi}dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) , \qquad (2.1)$$

where ψ and m are functions of v and r. The advanced-time surfaces $v = \text{constant}$ are ingoing null surfaces with null tangent (normal) $\partial/\partial r$. All areas A that arise will be calculated on $v = \text{constant}$ surfaces so that r has the usual invariant geometrical meaning. If $\psi = 0$ and $m = \text{constant} > 0$, then (2.1) is the Schwarzschild metric in (advanced) Eddington-Finkelstein coordinates. If $\psi = 0$ and $m = m(v)$, (2.1) is a Vaidya metric.[4] The Einstein equations for (2.1) are $\partial_r m = -4\pi r^2 T^v_v$, $\partial_r \psi = 4\pi r T_{rr}$, and

$$\frac{\partial m}{\partial v} = 4\pi r^2 T^r_v , \qquad (2.2)$$

where $T^a{}_b$ is the effective stress-energy tensor. Near the horizon of an evaporating hole, in the "Unruh vacuum," it is consistent with calculations of the regularized stress-energy tensor to assume that the $T^a{}_b$'s are regular and of order LA^{-1}, where L is the small dimensionless luminosity.[13] If one defines a "Schwarzschild mass" $M(v) > 0$ as the value of $m(v,r)$ such that $g_{vv} = 0$, one can show that an observer at rest at $r \gg 2M$ sees a quasistatic geometry with $L \cong -dM/dv \cong -\partial m/\partial v$, where $\cong$ denotes equality through the first order in $L \ll 1$. The present coordinates are advantageous in that the components $T^a{}_b$ near the horizon are regular and approximately constant linear combinations of the physical components $T^{\hat{a}}{}_{\hat{b}}$ that correspond to an observer in an orthonormal frame freely falling at a speed $\cong c = 1$. I shall not assume, initially, that L has the Hawking form $L_H = B\hbar M^{-2}$.

There are actually three horizonlike loci of importance for black holes: the timelike (or quasistatic) limit surface (TLS), the apparent horizon (AH), and the event horizon (EH). For classical Schwarzschild holes these three coincide at $r = 2M$, which can be regarded as a degeneracy. In general, these surfaces, of which all will be regarded as three-dimensional histories of topologically spherical two-surfaces, do not coincide. Note that we have already "found" the TLS ($g_{vv} = 0$) by means of the definition of $M(v)$: $r_{\text{TLS}} = 2M(v)$. The vector $\partial/\partial v$ is spacelike for $r < r_{\text{TLS}}$.

The AH and EH are most easily found by defining ingoing null vectors β^a and outgoing null vectors l^a:

$$l^a = [l^v, l^r, l^\theta, l^\phi] = [1, \tfrac{1}{2}e^{\psi}(1-2mr^{-1}), 0, 0] , \qquad (2.3)$$

$$\beta^a = [0, -e^{-\psi}, 0, 0] , \qquad (2.4)$$

where I use the normalization $\beta_a l^a = -1$.[14] From (2.1) it now follows that the two-metric γ_{ab} to which β^a and l^a are orthogonal is given by

$$\gamma_{ab} = g_{ab} + l_a\beta_b + l_b\beta_a , \qquad (2.5)$$

$$\gamma_{ab}dx^a dx^b = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) . \qquad (2.6)$$

This is just ds^2 restricted to $v = \text{constant}$. Note that the outgoing null rays are parametrized by the time v of the distant observer. This is not an affine parameter for those rays except as $r \to \infty$.

For an outgoing null ray near $r=2M$, one finds in $O(L)$ $(d/dv \equiv l^a \nabla_a)$

$$\dot{r} \equiv \frac{dr}{dv} \cong \tfrac{1}{2}(1-2mr^{-1}) , \quad (2.7)$$

$$\ddot{r} \cong \dot{r}(4M)^{-1} + L(2M)^{-1} . \quad (2.8)$$

The contributions of ψ and $\partial_r m$ are negligible in $O(L)$, which greatly facilitates the calculations. Hence, one can take $\psi=0$ and $\partial_r m=0$, as there is no effect in the first order. This means that a Vaidya metric is sufficiently accurate for present purposes. Also, in this case r becomes an affine parameter for the ingoing null rays.

At $r=2M$, the "velocity" $\dot{r}$ for photons is zero, but $\ddot{r}>0$ as follows from (2.8). Hence, photons are only momentarily at rest on $r=2M$; they subsequently escape on a dynamical time scale $\kappa^{-1} \cong 4M$, κ=surface gravity. The $r=2M$ surface is timelike as one can see by substituting $\dot{r} \cong -2L$ into (2.1). We now show that $r=2M=r_{\rm TLS}$ is also the AH in the case of spherical symmetry.

The rate of change of area is

$$\frac{dA}{dv} = \oint \theta \, d^2S , \quad (2.9)$$

where

$$\theta = \gamma_{ab} \nabla^a l^b = \nabla_a l^a - \kappa \quad (2.10)$$

is the expansion of the outgoing rays and

$$\kappa = -\beta^a l^b \nabla_b l_a \quad (2.11)$$

is the surface gravity when evaluated on a horizon. In the case of spherical symmetry, from $A=4\pi r^2$ and $\dot{A}=\theta A$ one finds $\theta \cong r^{-1}(1-2mr^{-1})$. Therefore, $\theta \cong 0$ at $r=2M$. Hence, in this case the TLS coincides with the outermost "trapped" surface or apparent horizon.

The event horizon is necessarily a null surface and is defined by the outermost locus traced by outgoing photons that can "never" reach arbitrarily large distances r.[15] Inasmuch as the final state of an evaporating hole is somewhat problematical, however, one is forced to resort to a more practical approximate condition to locate the EH. We look for photons that could only reach large r in a time comparable to the evaporation time $\sim ML^{-1} \gg M$. This implies that the EH is accurately located by the "unaccelerated" or "stuck" photons with $\ddot{r} \cong 0$. It follows from the above, then, that[3]

$$\dot{r}_{\rm EH} \cong -2L, \theta_{\rm EH} \cong -2LM^{-1} , \quad (2.12)$$

$$r_{\rm EH}(v) \cong 2M(1-4L), \quad A_{\rm EH} \cong 16\pi M^2(1-8L) . \quad (2.13)$$

The value of $r_{\rm EH}$ agrees with Bardeen's result.[13] (This value is a bit large, by a negligible amount $\sim ML^2$.) One can show that these values satisfy the Raychadhuri curvature equation for null geodesics.[3]

Because the EH is inside the TLS, in contrast to the case of accretion of energy [replace L by $-L$ in (2.12) and (2.13)] the region between the horizons can be called a "quantum ergosphere" (QE), in analogy to the classical ergosphere of a rotating (Kerr) black hole.[16] Photons or ultrarelativistic particles (with $\gamma^2 \sim L^{-1}$) that originate in the QE can escape. However, this effect would only enhance the assumed luminosity in $O(L^2)$, a negligible amount. The main idea is that the QE is the indelible mark on the metric of a black hole that is losing energy at a mean rate determined by L.

An invariant measure of QE is provided by the difference of areas

$$\delta A_{\rm QE} \equiv A_{\rm TLS} - A_{\rm EH} = 16\pi M^2 (8L) . \quad (2.14)$$

Now, for the first time in this discussion, let us assume that L has the Hawking form $L_H = B\hbar M^{-2}$, where B is a dimensionless barrier factor.[17] We can substitute L_H into (2.14) and consider the limit of arbitrarily small Hawking temperature $T_H = \hbar(8\pi M)^{-1}$, that is, let M become very large. (In this context, "large" means $M \gg 10^{17}$g $\sim 10^{21}\hbar^{1/2}$, that is, only *massless* quanta can be produced at a significant rate.[17]) One sees that in this limit $\delta A_{\rm QE}$ is strictly independent of the mass M and *cannot go to zero:*

$$\delta A_{\rm QE} = 128\pi B\hbar , \quad (2.15)$$

where B now depends only on the massless quanta that exist in Nature.

One does not recover fully the classical Schwarzschild structure for a black hole in the limit considered unless $\hbar \to 0$. The nonvanishing of $\delta A_{\rm QE}$ as $T_H \to 0$ suggests consideration of vacuum or zero-temperature fluctuations, the treatment of which is begun in the next section. Focus of attention on area is suggested not only because it is a geometrically well-defined measure, but also because of its role in the theory of black holes in Bekenstein's conjecture that the black-hole entropy is proportional to its area,[2] in Hawking's area theorems,[18] and in the concept of "irreducible" mass introduced by Christodoulou and Christodoulou and Ruffini.[7]

The irreducible mass associated with an event horizon of area A is defined by[7]

$$M_{\rm irred} = \left[\frac{A_{\rm EH}}{16\pi} \right]^{1/2} . \quad (2.16)$$

Here, in order to provide a useful quasilocal measure of mass energy, following York and Piran,[19] I use this definition for any of the horizonlike loci:

$$M_{\rm irred}(H) = \left[\frac{A(H)}{16\pi} \right]^{1/2} , \quad (2.17)$$

where H=TLS, AH, or EH. Hence we have

$$M_{\rm irred}({\rm AH}) = M_{\rm irred}({\rm TLS}) = M \quad (2.18)$$

and

$$M_{\rm irred}({\rm EH}) = M(1-4L) . \quad (2.19)$$

Then we are able to associate a mass with the quantum ergosphere by means of

$$M_{\rm QE} \equiv M_{\rm irred}({\rm TLS}) - M_{\rm irred}({\rm EH}) . \quad (2.20)$$

Thus $M_{\rm QE} = 4LM$ from (2.19). The reasonableness of the definition is confirmed by the observation that $L = \kappa M_{\rm QE}$, where $\kappa^{-1} \cong 4M$ is the dynamical time scale of the hole.

III. FLUCTUATING METRIC

I shall model a neutral nonrotating black hole possessing residual zero-point fluctuations with the following simple metric:

$$ds^2 = -(1-2mr^{-1})dv^2 + 2dv\,dr + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 , \tag{3.1}$$

where

$$m = m(v,\theta) = M + \sum_l (2l+1)\epsilon_l M h_l(v) q_l(\theta) . \tag{3.2}$$

Here, M is a positive constant, l is the angular index, $h_l(v) = \sin\omega_l v$, and $q_l(\theta)$ is a spherical harmonic with azimuthal index zero. Because the hole is nearly spherical, one can use the axisymmetric modes without loss of generality. All q_l's will be normalized by requiring that the integral of $(q_l)^2$ over the unit sphere is 4π. The ω_l's are the resonant or ringing mode frequencies (real parts) for the given l.

Each oscillator is labeled by l and, for a given l, there are $(2l+1)$ of them. All oscillators are taken to be uncorrelated and independent. The dimensionless amplitude parameter ϵ_l will be taken in the form

$$\epsilon_l = \alpha_l \hbar^{1/2} M^{-1} , \tag{3.3}$$

where α_l is a pure number. The α_l's will later be evaluated to give (approximately) the correct uncertainty product for each mode. In the calculations, each oscillator will be treated separately and the subscript l will be dropped where no confusion can result.

Although it is unnecessary, one can regard (3.1) as a (not necessarily small) perturbation of the Schwarzschild metric in advanced-time Eddington-Finkelstein coordinates. Then, if we set $g_{ab} = g_{ab}(\text{Schwarzschild}) + \phi_{ab}$, we have

$$\phi_{vv} = \epsilon \frac{2M}{r} h(v) q(\theta) = \hbar^{1/2} \left[\frac{2\alpha}{r} h(v) q(\theta) \right] \tag{3.4}$$

and all other ϕ_{ab}'s are zero. This is a metric perturbation with zero trace with respect to either $g_{ab}(\text{Schwarzschild})$ or g_{ab}. It has the expected form and dimensions of a bosonic tensor field. The implicit gauge is such that r retains its usual geometric meaning; that is, on v-constant null surfaces, one has the usual two-sphere metric. Also, r is an affine parameter for incoming null rays. ϕ_{ab} satisfies a "transversality" condition $\phi_{ab}\beta^b = 0$.

The metric (3.1) has a number of remarkable simplifying features that follow from two facts: (1) It can be written in the form $g_{ab} = \eta_{ab} + k_a k_b$, where η_{ab} is flat and k^a is null (with respect to either g_{ab} or η_{ab}). (2) The simple form of $m(v,\theta)$ in (3.2). The latter property is assumed for simplicity; one could have $\partial m/\partial r \neq 0$, but this does not materially change the results of this paper while making the calculations messier. We shall see that the use of scalar spherical harmonics in g_{vv} will lead automatically to the appropriate tensor spherical harmonics when we compute the splitting of the three "horizons."

The remarkable features alluded to above are that the Einstein tensor G^a_b and the Weyl tensor $C^{ab}{}_{cd}$ (and hence the Riemann tensor $R^{ab}{}_{cd}$) are *strictly linear* in ϵ, as displayed in the Appendix. In fact, if we adopt the form (3.2), one finds further that G_{ab}, G^a_b, and G^{ab} are all strictly linear in ϵ. These properties make the statement "the time average of (3.1) is the classical Schwarzschild metric" meaningful. I define the time average of a quantity as being over the time v with period $2\pi/\omega$ at fixed (r,θ,ϕ) and denote it by an overbar. Hence one has, exactly, that

$$\bar{g}_{ab} = g_{ab}(\text{Schwarzschild}) , \tag{3.5a}$$

$$\bar{G}^a_b = \bar{G}_{ab} = \bar{G}^{ab} = 0 , \tag{3.5b}$$

$$\bar{C}^{ab}{}_{cd} = C^{ab}{}_{cd}(\text{Schwarzschild}) , \tag{3.5c}$$

$$\bar{R}^{ab}{}_{cd} = R^{ab}{}_{cd}(\text{Schwarzschild}) . \tag{3.5d}$$

We see that from the viewpoint of the classical Einstein equations $G^a_b = 8\pi T^a_b$, the "mean effective stress-energy tensor of matter" $\bar{T}^a_b$ is zero. Hence, on average, there is no "classical" stress energy passing into or out of the hole. This is an important point for, later, I shall have to compute certain quantities (horizon loci, areas) to second order in ϵ (with $\epsilon^2 \ll 1$) while discarding (for calculational purposes) higher-order terms. We shall find the quantum-radiance-tunneling phenomenon in order $\epsilon^2 = O(\hbar)$, and we see that the results will be determined strictly by the properties of horizons and will *not* be artifacts of having improperly truncated the curvature terms, which, in effect, would amount to having introduced by hand some sort of "classical dissipative matter." Let me emphasize that the idea behind this model is to suppose initially that the black hole *can* be in equilibrium in a zero-temperature environment (oscillations of zero mean; no net shrinkage or growth of the hole). We shall see, however, that it cannot be because the hole would necessarily be leaking away its mass were it in empty space. Thus, the results will imply that equilibrium would occur only in a "heat bath" of nonzero temperature. However, the effective ($\cong$Hawking) temperature that results leaves the modes so unexcited that feeding the resulting slightly larger mode amplitudes back into the calculations has negligible consequences.

It is obvious, of course, that $T^a{}_b$ is not strictly zero at all times. That is, during part of each cycle, the usual classical energy conditions are violated at $r = 2M$. However, such violations occur in the Hawking effect and must necessarily occur in the case of intrinsic *quantum* fluctuations of the metric.

The oscillating metric (3.1) is seen to be a simple generalized "Vaidya" type in which there is an oscillating "null fluid." This is a good approximation, at least as far as the computation of horizon loci is concerned, for the black-hole metric in a region from the singularity at $r = 0$ out to the wavelength $\Delta r = \lambda \approx 16.8M$ of the lowest resonant mode. (Because r is affine for incoming modes and ω is defined in terms of the time v of a distant observer, I take the period $2\pi/\omega$ as being synonymous with a radial wavelength measured by r; I do not employ the "tortoise" radial coordinate $r_* = r + 2M \ln|r/2M - 1|$

that is appropriate for the usual *spacelike* slices $t=\text{constant}$.)

The mass ("mass aspect") m oscillates around the constant mean value M. As m increases, one thinks heuristically of an absorption of positive "energy" from the past along $v=\text{constant}$; as m decreases one can think of the absorption of negative energy from the past. I assume that such oscillations appropriately normalized at the ringing frequencies give a good account of residual nonradiative "quantum noise" or uncertainty in the black-hole metric.

For each l, I shall find the loci and areas of the TLS, AH, and EH. We note that the TLS is already defined in the metric (3.1) by the form chosen for the "input perturbation" ϕ_{ab}. Setting $g_{vv}=0$, we find that

$$r_{\rm TLS}(v,\theta)=2M[1+\epsilon h(v)q(\theta)] . \qquad (3.6)$$

This surface has a normal that is sometimes spacelike, sometimes null, and sometimes timelike. The time average is of course $\bar{r}=2M$. On the other hand, we can intersect (3.6) with an incoming null slice $v=\text{constant}$ and compute its area from

$$A_{\rm TLS}(v)=\int_0^{2\pi}\int_0^{\pi}[r_{\rm TLS}(v,\theta)]^2\sin\theta\, d\theta\, d\phi . \qquad (3.7)$$

We are interested only in the mean area and the rms value of $r_{\rm TLS}$. The mean area is

$$\bar{A}_{\rm TLS}=16\pi M^2(1+\tfrac{1}{2}\epsilon^2) . \qquad (3.8)$$

As expected, there is no linear term in ϵ. The corresponding *mean irreducible mass* is *defined* by

$$\bar{M}_{\rm irred}({\rm TLS})\equiv\left[\frac{\bar{A}_{\rm TLS}}{16\pi}\right]^{1/2}\cong M(1+\tfrac{1}{4}\epsilon^2)$$
$$\cong M+\tfrac{1}{4}\alpha^2\frac{\hbar}{M} , \qquad (3.9)$$

where, anticipating that $\epsilon^2\ll 1$, I have retained only the second-order correction. From here on, I shall drop the subscript "irred" and denote mean irreducible masses by $\bar{M}$ (surface). It is important that this quantity is *not* defined by the mean value of r, which will always be $2M$.

That $\bar{A}_{\rm TLS}>16\pi M^2$ and $\bar{M}_{\rm TLS}>M$ is expected because the bare black hole is dressed by the "kinetic energy" of the oscillation. Hence, M and $A=16\pi M^2$ have no direct physical significance. The significant quantities will involve the *differences* among $\bar{A}_{\rm TLS},\bar{A}_{\rm AH},\bar{A}_{\rm EH}$, and among $\bar{M}_{\rm TLS},\bar{M}_{\rm AH},\bar{M}_{\rm EH}$. One sees that the gauge freedom through $O(\epsilon^2)$ in choosing the vector field $\partial/\partial v$ that defines the TLS will, therefore, not affect the results, which depend only on differences of areas.

IV. SPHERICAL OSCILLATIONS

It is helpful first to illustrate some of the calculations in the case of spherical oscillations: $l=0, q_0(\theta)=1$. Of course, in general relativity a black hole has no oscillatory spherical gravitational modes. Also, because we are considering electrically neutral holes, we shall not consider any $l=1$ modes of amplitude $O(\epsilon)=O(\hbar^{1/2})$ in the metric. Nevertheless, the $l=0$ case will be considered as an illustrative exercise. Some of the results carry over to the physically interesting cases $l\geq 2$.

With $q(\theta)=1$ we have $m=M[1+\epsilon h(v)]$ and (3.1) is a spherically fluctuating Vaidya metric. From a "classical" viewpoint, the only piece of the stress-energy tensor that is not precisely zero is

$$T_{ab}l^al^b=T_v^r=\frac{\epsilon\omega M\cos\omega v}{4\pi r^2} \qquad (4.1)$$

for which $\bar{T}_v^r=0$. Just as in Sec. II, we find for the outgoing null geodesics

$$\dot{r}=\tfrac{1}{2}(1-2mr^{-1}),\quad \theta=\frac{1}{r}(1-2mr^{-1}) . \qquad (4.2)$$

Hence the TLS and AH coincide at $r=2m(v)$. Clearly $\bar{r}_{\rm TLS}=\bar{r}_{\rm AH}=2M$. The mean areas are both given by (3.8).

The event horizon is found by simply requiring that it be a *null* surface near $r=2M$. This is justified by the assumption of equilibrium and of periodic oscillations with $r=2M$ as the mean value. We can simply solve the first-order equation for $\dot{r}=dr/dv$ in (4.2); the second-order forms of the equations we solve yield the same results. Throughout this work, I shall need only the first-order null radial equations because I am interested only in the average properties of the outgoing null trajectories. We set

$$r_{\rm EH}=2M[1+\epsilon f(v)+\epsilon^2\lambda(v)] . \qquad (4.3)$$

In computing mean areas, it will never be necessary to know explicitly the second-order term $\lambda(v)$ because it will be periodic and make no contribution to the mean area in $O(\epsilon^2)$. However, here I shall find both $f(v)$ and $\lambda(v)$ to illustrate the procedure, which is elementary.

We have

$$\left[1-\frac{2m(v)}{r_{\rm EH}(v)}\right]=\epsilon(f-h)+\epsilon^2(\lambda-f^2+\lambda f) . \qquad (4.4)$$

This gives the $O(\epsilon)$ and $O(\epsilon^2)$ equations

$$\dot{f}-\kappa f=-\kappa h , \qquad (4.5)$$

$$\dot{\lambda}-\kappa\lambda=-\kappa(f^2-hf) , \qquad (4.6)$$

where $\kappa=(4M)^{-1}$.

First solve (4.5). The solution has a spurious (homogeneous) part $\exp(\kappa v)$ that I discard. It is present even for a static metric and has nothing to do with the EH. We find

$$f(v)=(1+16\sigma^2)^{-1}(\sin\omega v+4\sigma\cos\omega v) , \qquad (4.7)$$

where $\sigma=\omega M$ is the dimensionless frequency. Note that

$$\overline{f^2}=\frac{1}{2}\frac{1}{1+16\sigma^2} \qquad (4.8)$$

so that the amplitude ($\sqrt{2}\times$rms value) associated with f is $(1+16\sigma^2)^{-1/2}$; whereas, the amplitude associated with h is unity. Hence, the amplitude of the fluctuation of the EH is *less* than those of the TLS and AH for all $\sigma>0$. This is an important property and will be seen to persist in all cases.

Substituting (4.7) into (4.6) enables us to find the second-order solution

$$\lambda(v)=B_1\cos 2\omega v+B_2\sin 2\omega v\ ,$$
$$B_1=\frac{32\sigma^2-256\sigma^4}{(1+16\sigma^2)^2(1+64\sigma^2)}\ , \tag{4.9}$$
$$B_2=\frac{2\sigma-160\sigma^3}{(1+16\sigma^2)^2(1+64\sigma^2)}\ .$$

The dependence on 2ω is expected and averages to zero.

The mean radius is $\bar{r}_{\rm EH}=2M$ and the area is [through $O(\epsilon^2)$]

$$A_{\rm EH}(v)=16\pi M^2[1+\epsilon(2f)+\epsilon^2(f^2+2\lambda)]\ , \tag{4.10}$$

which yields

$$\bar{A}_{\rm EH}=16\pi M^2[1+\tfrac{1}{2}\epsilon^2(1+16\sigma^2)^{-1}]\ . \tag{4.11}$$

It is evident that $\bar{A}_{\rm EH}<\bar{A}_{\rm TLS}=\bar{A}_{\rm AH}$. Hence, we define the difference of the mean areas as

$$\delta A_{\rm QE}\equiv\bar{A}_{\rm TLS}-\bar{A}_{\rm EH}=16\pi M^2\left[\tfrac{1}{2}\epsilon^2\frac{16\sigma^2}{1+16\sigma^2}\right]$$
$$=\frac{128\pi\alpha^2\sigma^2}{1+16\sigma^2}\hbar\ . \tag{4.12}$$

The property that the amplitude of the motion of the EH is less than that of the TLS has led to $\delta A_{\rm QE}>0$ and will be seen to persist in the realistic cases $l\geq 2$. Hence, for any mode, during part of each period there is a quantum ergosphere. This enables us to see that tunneling of energy out of the hole can occur spontaneously, as I have described earlier. This would lead, in the absence of a nonempty environment, to a *net* loss of mass by the hole. As all the modes are independent and uncorrelated, I therefore conclude from (4.12) that an equilibrium at zero temperature would be impossible, in accordance with the existence of the Hawking effect.

Using the definition of the mean irreducible mass, we can *define* a quasilocal measure of energy associated with the quantum ergosphere in terms of the *difference* of the mean irreducible masses of the TLS and the EH:

$$M_{\rm QE}\equiv\bar{M}_{\rm TLS}-\bar{M}_{\rm EH}=\frac{\bar{A}_{\rm TLS}-\bar{A}_{\rm EH}}{32\pi M}=\frac{4\epsilon^2\sigma^2 M}{1+16\sigma^2}$$
$$=\frac{4\alpha^2\sigma^2}{1+16\sigma^2}\frac{\hbar}{M}\ . \tag{4.13}$$

We observe that $\delta A_{\rm QE}$ and $M_{\rm QE}$ are zero only in the static ($\sigma\rightarrow 0$) or classical ($\hbar\rightarrow 0$) limits.

V. NONSPHERICAL OSCILLATIONS

We return to the metric (3.1) and, treating each mode labeled by l as independent, we have

$$m=M[1+\epsilon_l(\sin\omega_l v)q_l(\theta)]\ , \tag{5.1}$$

where $q_l=\sqrt{4\pi}\ Y_{l0}=(2l+1)^{1/2}P_l(\cos\theta)$ and P_l is a Legendre polynomial that defines the shape of the distorted oscillating black hole. Actually, of course, each such mode is $(2l+1)$fold degenerate, and this will be accounted for at the appropriate places. As I have mentioned, the mean spherical symmetry and absence of rotation enables us to work with the axisymmetric modes (azimuthal index $\mu=0$) without loss of generality because we shall deal only with mean properties. Thus, for $l=2$ there are five independent uncorrelated oscillations of the sphere $r\cong 2M$ that constitutes the black hole from the viewpoint of an outside observer. Note that I am not regarding these modes as being "gravitons" *directly,* in which case one would have (2 helicities)$\times$(5)=10 oscillators for $l=2$. The degrees of freedom to be "quantized" are nonradiative oscillatory modes of the "sphere" or "background" spacetime; each such mode can be thought of as being excited by two zero-point "gravitons," each with the given value of l and azimuthal index μ, each created or absorbed, in effect, at the singularity $r=0$. (From the viewpoint of formal scattering theory,[20] it is natural to regard absorption, and therefore emission, as occurring at $r=0$.) The two alternative points of view described here actually yield in the final analysis the same estimates for the ϵ_l's in the application of the uncertainty principle in Sec. VI; the point of view adopted here is, in my opinion, the natural way to view the problem. The distortions or "ripples" on the surface can be thought of as somewhat like the perturbations on an otherwise optically flat surface of a "dense" optical medium, that give rise to the propagation of evanescent waves.[21]

One recalls that in the definition of quasinormal modes,[11] although the effect on the horizon is ordinarily ignored in first-order perturbation theory, the boundary condition at $r=2M$ ($r_*=-\infty$) shows that the metric at $r=2M$ *is* actually undergoing transverse oscillations in the first order: the perturbation of the geometry goes as e^{ikr_*} as $r_*\rightarrow-\infty$ ($r\rightarrow 2M$). The situation is formally the same in considering quasinormal resonances for other massless fields, e.g., spins $0,\frac{1}{2},1$. However, the behavior e^{ikr_*} as $r_*\rightarrow-\infty$ for these fields do not directly involve the *metric* of a neutral spherical hole in $O(\epsilon)$ [for zero-point oscillations, $O(\epsilon)=O(\hbar^{1/2})$ for all fields]. This is an important reason to focus attention on metric fluctuations of $O(\hbar^{1/2})$ at the frequencies of the *gravitational* (spin-2) resonances. After all, the temperature, entropy, etc., of a black hole are intrinsic to the hole, which, having no "hair," is a purely gravitational object. The idea is that the uncertainty in these nonradiative or "inductive" modes creates, in a "gate" effect, a quantum ergosphere that allows radiative modes of *all* fields to tunnel out of the hole.

Another point should be made before performing the calculations for all $l\geq 2$. We are implicitly assuming, in the "equilibrium" situation with oscillations of zero mean value about $r=2M$, that all modes are equally efficient, modulo normalization of amplitude, in raising tides at $r=2M$. By "tides," I mean the distortions in the shape of the horizon, indexed by l and μ. Hence one is assuming that all the quasinormal "waves" are moving in and out radially in an effectively free manner. (One notes that ringing frequencies are defined by radially outgoing wave forms.) However, this assumption of effectively free radial oscillations can be seen, in fact, not to be a correct description of the nature of quasinormal resonances in an equilibrium situation when one considers how they become excited and what their observable effects are to an outside observer. There are very important *orbiting* effects for $l \geq 3$. Moreover, we will see that there cannot be a dynamically induced thermal equilibrium, i.e., one effective temperature pertaining to the quantum ergosphere, unless the actual "orbiting" nature of the modes with $l \geq 3$ is taken into account. I shall return to this point in Sec. VII.

I now proceed to the calculations for $l \geq 2$.

In the following, I shall drop the subscript "l" except where it is necessary. Unlike the model case $l=0$ considered in Sec. IV, the outgoing rays now have shear, which turns out to remove completely the degeneracy of the TLS, EH, and AH. (Recall that for $l=0$, TLS=AH. This will no longer hold.) Thus, one writes for the outgoing null rays a generator l^a parametrized by v

$$l^a=(1,l^r,l^\theta,0)\ , \tag{5.2}$$

$$l^r=\frac{1}{2}\left[1-\frac{2m}{r}\right]-\tfrac{1}{2}(rl^\theta)^2\ , \tag{5.3}$$

the latter guaranteeing that $l_a l^a=0$. We choose β^a the same as in (2.4) with $\psi=0$:

$$\beta^a=(0,-1,0,0)\ . \tag{5.4}$$

Thus, in the fluctuating geometry, the $v=$constant surfaces remain rigorously null and the ingoing rays are affinely parametrized by r.

The equations of motion for l^r and l^θ are now coupled. However, one has that $l^\theta=O(\epsilon)$ enters (5.3) quadratically. Therefore, we can solve the null geodesic equations

$$l^a\nabla_a l^\theta=\kappa l^\theta\ , \tag{5.5}$$

$$\kappa=-\beta^a l^b\nabla_b l_a\ , \tag{5.6}$$

in $O(\epsilon)$ near $r=2M$, which is all that is required. Setting $l^\theta=\epsilon\mu(v,\theta)$ and referring to the Appendix for the Christoffel symbols, we find the $O(\epsilon)$ equation

$$\frac{\partial\mu}{\partial v}-\kappa\mu=\frac{\kappa}{2M}(\sin\omega v)\frac{dq}{d\theta}\ , \tag{5.7}$$

where $\kappa=(4M)^{-1}+O(\epsilon)$. The solution is

$$l^\theta=\epsilon\mu=\frac{-\epsilon}{2M}f(v)\frac{dq}{d\theta}\ , \tag{5.8}$$

where $f(v)$ is the same function given in (4.7), here repeated for convenience:

$$f(v)=(1+16\sigma^2)^{-1}[\sin\omega v+4\sigma\cos\omega v]\ . \tag{5.9}$$

To find the *apparent horizon*, we must compute the expansion θ of the outgoing null rays and find where it vanishes. By definition,

$$\theta=\gamma^a_b\nabla_a l^b\ , \tag{5.10}$$

where the projection operator γ^a_b is found by substituting (5.2), (5.3), and (5.4) into (2.5). (The components of γ^a_b are displayed in the Appendix.) One finds in $O(\epsilon)$

$$\theta=\frac{2}{r}l^r+\partial_\theta l^\theta+l^\theta\cot\theta\ . \tag{5.11}$$

To find $r_{\rm AH}(v,\theta)$ we set it equal to $2M[1+\epsilon\beta(v,\theta)]$, substitute in (5.11) and (5.3), and equate θ to zero, from which one finds

$$\beta(v,\theta)=(\sin\omega v)q+f(v)\left[\frac{d^2q}{d\theta^2}+\cot\theta\frac{dq}{d\theta}\right]. \tag{5.12}$$

Using some recursion properties of the Legendre polynomials yields

$$\beta(v,\theta)=[\sin\omega v-l(l+1)f(v)]q(\theta)\ . \tag{5.13}$$

Hence we can compute the mean area of the apparent horizon by intersecting it with $v=$constant and averaging over a period:

$$\bar{A}_{\rm AH}=16\pi M^2\left[1+\tfrac{1}{2}\epsilon^2\frac{[l(l+1)-1]^2+16\sigma^2}{1+16\sigma^2}\right]. \tag{5.14}$$

The timelike limit surface is where $g_{vv}=0$ or $r_{\rm TLS}=2m=2M[1+\epsilon(\sin\omega v)q(\theta)]$. Its mean area is found as above as

$$\bar{A}_{\rm TLS}=16\pi M^2[1+\tfrac{1}{2}\epsilon^2]\ . \tag{5.15}$$

To find the event horizon, defined as a strictly null surface near $r=2M$, we return to (5.3) and write $r_{\rm EH}=F(v,\theta)$. Then

$$l^r=\frac{dF}{dv}=l^a\nabla_a F=\frac{\partial F}{\partial v}+l^\theta\frac{\partial F}{\partial\theta}\ . \tag{5.16}$$

Setting $F=2M[1+\epsilon\gamma(v,\theta)]$ and substituting into (5.3), one obtains the $O(\epsilon)$ equation

$$\frac{\partial\gamma}{\partial v}-\kappa\gamma=-\kappa(\sin\omega v)q(\theta)\ , \tag{5.17}$$

from which follows

$$r_{\rm EH}(v,\theta)=2M[1+\epsilon f(v)q(\theta)]\ . \tag{5.18}$$

The mean area is

$$\bar{A}_{\rm EH}=16\pi M^2\left[1+\tfrac{1}{2}\epsilon^2\left[\frac{1}{1+16\sigma^2}\right]\right]. \tag{5.19}$$

Note that we have obtained the important relations

$$16\pi M^2<\bar{A}_{\rm EH}<\bar{A}_{\rm TLS}<\bar{A}_{\rm AH}\ . \tag{5.20}$$

Hence, though all surfaces have a mean gravitational radius $\bar{r}=2M$ the amplitudes of the fluctuations are always different, being ordered by (5.20). Observe that the physical event horizon, being null, remains in effect nearest to the background event horizon $r=2M$, while the TLS and AH make larger excursions. This is because, as three-dimensional surfaces, they are sometimes timelike ("emission") and sometimes spacelike ("absorption") with respect to the background geometry or the physical geometry.

From (5.20) it follows that there exists a quantum ergosphere ($\bar{A}_{\rm EH}<\bar{A}_{\rm TLS}$) with which we can associate an irreducible mass (per mode)

$$M_{\rm QE}=\frac{\overline{A}_{\rm TLS}-\overline{A}_{\rm EH}}{32\pi M}=\epsilon^2\frac{4\sigma^2 M}{1+16\sigma^2} . \qquad (5.21)$$

This expression depends, as we shall see, explicitly on l because both ϵ^2 and the ringing frequencies σ do so.

That $\overline{A}_{\rm AH}>\overline{A}_{\rm TLS}$ results from the presence of shear, which, though its mean value is zero, has a nonvanishing rms value. I display here its angle-averaged variance per mode, though it is not needed explicitly in the sequel

$$\langle \underline{\sigma}^2\rangle=\frac{\epsilon^2}{16M^2}\frac{(l+2)(l+1)l(l-1)}{1+16\sigma^2} , \qquad (5.22)$$

where, by definition, $\underline{\sigma}^2=\sigma_{ab}\sigma^{ab}$ and

$$\sigma_{ab}\equiv\gamma_{ac}\gamma_{bd}\nabla^c l^d-\tfrac{1}{2}\gamma_{ab}\theta . \qquad (5.23)$$

The angular brackets denote a time average plus an average over the sphere. The components of the shear tensor are displayed in the Appendix. It is interesting to note that their angular dependence is determined by the Gegenbauer functions $C_{l+2}^{-3/2}(\theta)$ that are ordinarily found to be characteristic of separable massless spin-2 perturbation equations for spherical black holes in terms of methods that are not only entirely different from the present ones, but are much more complicated and not easily visualized.[22]

The explicit importance of $\overline{A}_{\rm AH}>\overline{A}_{\rm TLS}>\overline{A}_{\rm EH}$ in this work, as we shall see, is that the fluctuations of the apparent horizon determine the physically significant transverse metric variances to which the uncertainty relation is applied. This will determine the ϵ^2's in $M_{\rm QE}$. That the behavior of the apparent horizon is crucial is suggested by the fact that, of the mean area formulas, only $\overline{A}_{\rm AH}$ depends explicitly on l, for fixed ϵ^2 and σ.

VI. UNCERTAINTY RELATION AND FLUCTUATION AMPLITUDES

I have modeled the zero-point oscillations with characteristic data belonging to "retarded time" null surfaces, among which are past null infinity ("scri minus") and the (future) event horizon. The classical Poisson brackets and quantum commutators of such data are "equal time" brackets, where "equal time" means $v=$constant. This is analogous to "infinite momentum frame" quantization in flat spacetime.[6]

To find the $v=$constant brackets when one has a Lagrangian $\mathscr{L}=\mathscr{L}(\psi;\,\partial_a\psi)$, with indices on ψ suppressed, and $v=$constant is a null surface, one proceeds formally just the same as in the standard "space plus time" canonical formalism. One defines the canonical momentum as

$$\Pi=\frac{\partial\mathscr{L}}{\partial(\partial_v\psi)} , \qquad (6.1)$$

but, as is well known, one does not find that $\Pi\sim\partial_v\psi$; rather, one finds $\Pi\sim\partial_u\psi$, where $\partial/\partial u$ is the null normal=null tangent of the $v=$constant slice. (In flat spacetime, for example, $v\sim t+r, u\sim t-r$.) The classical Poisson brackets corresponding to (6.1) have the form (using tensors and proper Dirac δ functions)[6]

$$\{\Pi^A(u,\vec{x}),\psi_B(u',\vec{x}')\}_{v={\rm const}}=\tfrac{1}{2}\delta^A_B\delta(u-u')\delta^{(2)}(\vec{x}-\vec{x}') , \qquad (6.2)$$

where $\vec{x}$ denotes the two transverse (tangent to the sphere) directions and Π^A and ψ_B refer to the transverse physical components, i.e., $\Pi^A=(\Pi^{\hat\theta},\Pi^{\hat\phi})$ and $\psi_B=(\psi_{\hat\theta},\psi_{\hat\phi})$. The factor "$\frac{1}{2}$" on the right is understood[6] and can be obtained systematically by following Dirac's procedure for the construction of brackets.[23] The characteristic transverse data are typically unconstrained and the fixing of a suitable gauge is understood. The quantum commutator has an extra factor $(i\hbar)$ on the right.

In the case of metric (3.1), one has $\partial/\partial u\leftrightarrow-\partial/\partial r=\beta^a$, which is the null normal (tangent) of the $v=$constant null slices. It points out of the exterior region, across the future event horizon, and into the black hole. It is irrelevant that $\partial/\partial v$ is only null at $r=2m$ and that the slices $r=$constant are not in general null. One needs explicitly only one family of null surfaces to set up the brackets. However, physical consistency of the corresponding classical characteristic (as opposed to "Cauchy") problem would demand that there be no fields propagating along the null surfaces crossed by $v=$constant; this is true at the physical future event horizon, where the gravitational analog of (6.2) is to be applied.

To obtain the gravitational analog of (6.2), one uses the first-order gravitational Lagrangian $\mathscr{L}=(16\pi)^{-1}\sqrt{-g}\,[R-(\text{divergence terms})]$, where R is the scalar curvature. In the usual "3 + 1" formalism, this is the ADM or "$(\Gamma\Gamma-\Gamma\Gamma)$" Lagrangian. A suitable geometrical form for $\mathscr{L}$ can be obtained easily from the results of d'Inverno, Smallwood, and Stachel[24] or by the ADM method.[25] In any case, the "kinetic" part of $\mathscr{L}$ is separately quadratic in both the ingoing (β^a) null second fundamental tensor and in the outgoing (l^a) null second fundamental tensor. The latter is of course constructed from σ_{ab} and θ. However, choosing as a typical transverse variable $g_{\theta\theta}$, one finds that the corresponding momentum on $v=$constant involves the *ingoing* second fundamental tensor

$$\chi_{ab}=\gamma^c_a\gamma^d_b\nabla_c\beta_d . \qquad (6.3)$$

The *tensor* momentum in physical components corresponding to $g_{\theta\theta}$ is

$$p^{\hat\theta\hat\theta}=\frac{-1}{16\pi}(\chi^{\hat\theta\hat\theta}-\gamma^{\hat\theta\hat\theta}{\rm tr}\chi) \qquad (6.4)$$

which is formally identical to the tensor form of the Dirac-ADM momentum of the 3 + 1 formalism.[25,26] Evaluating (6.4) for metric (3.1) gives

$$p\equiv p^{\hat\theta\hat\theta}=p^{\theta}_{\theta}=\frac{1}{16\pi r} . \qquad (6.5)$$

The method I have used in this paper to make the calculations tractable with pen and ink has only allowed us to determine fluctuations near $r=2M$, so in an uncertainty relation based on (6.2) I must calculate variances such as $(\Delta p)^2\equiv\langle(\delta p)^2\rangle$ between pairs of surfaces $r=r(v,\theta)$ near $r=2M$. It is clear from the fact that incoming characteristic data determine the *apparent horizon,* that

$r = r_{\rm AH}$ should be one of the surfaces. From the requirement of consistency mentioned above, the other surface must be a null surface crossed by v = constant, along which no physical signals (at least classically) are propagating: This is the *physical event horizon* $r = r_{\rm EH}$. Note that there is here a slight "acausal feedback" because the data do not fix the physical event horizon "instantaneously." The scale over which this feedback occurs, as we shall see later, is very small, comparable within a few orders of magnitude to the Planck distance. I believe this feature is in principle essential and unavoidable. In reality, there can be no fixed "background light cone" at any nontrivial order of $\hbar^{1/2}$ in the metric fluctuations.

Likewise, in computing $(\Delta g)^2 = \langle (\delta g)^2 \rangle$, where angular brackets include averages over the sphere, I use the physical components of the difference of $g_{\theta\theta} = r^2$ for the surfaces $r = r_{\rm AH}(v,\theta)$ and $r = r_{\rm EH}(v,\theta)$. One has then

$$\delta g \equiv g^{\theta\theta}({\rm EH})[g_{\theta\theta}({\rm AH}) - g_{\theta\theta}({\rm EH})] \ . \tag{6.6}$$

One then finds a very simple relation between Δg and Δp, namely,

$$\Delta p = (16\pi)^{-1}\kappa\Delta g \ , \tag{6.7}$$

where $\kappa = (4M)^{-1} + O(\epsilon)$ is the surface gravity. Using the results for $r_{\rm AH}(v,\theta)$ and $r_{\rm EH}(v,\theta)$, we obtain

$$\Delta g = 2\epsilon \langle (\beta - fq)^2 \rangle^{1/2} \ . \tag{6.8}$$

The semiclassical "least variance" or minimal uncertainty relation that follows from (6.2) is constructed by the procedure described by DeWitt.[27] The left-hand side corresponding to (6.2) is $(\Delta\psi)(\Delta\pi)$ or $(\Delta g)(\Delta p)$, where, for example, (Δg) is obtained from the average $\langle (\delta g)^2 \rangle_\Omega$ of δg over a spacetime domain Ω. For Ω I choose the natural spherical "box," defined by a radius $r = \lambda = 2\pi\omega^{-1}$ extending from the singularity at $r = 0$ to $r = \lambda$, times the corresponding period $\tau = 2\pi\omega^{-1}$. Thus

$$\Omega = \int_{v_0}^{v_0+\tau} \int_0^\lambda \int_0^\pi \int_0^{2\pi} \sqrt{-g}\, dv\, dr\, d\theta\, d\phi \ , \tag{6.9}$$

where $\sqrt{-g} = r^2 \sin\theta$ for the metric (3.1). Hence, we have $\Omega = \tau({\rm vol})$, where $({\rm vol}) = \frac{4}{3}\pi\lambda^3$, just as in flat spacetime. This simple result follows from the facts that r is an affine parameter for ingoing rays and thus measures the wavelength λ as in flat space, and that the period τ refers to the time v that is the proper time of a distant observer.

The procedure used in this work has not allowed an explicit determination of δg for the canonical variables except near $r = 2M$. Hence, I make the simplest possible assumption, that the average near $2M$ is equal to the average throughout Ω: $\langle (\delta g)^2 \rangle_\Omega = \langle (\delta g)^2 \rangle = (\Delta g)^2$ as in (6.8). This should give a reasonably good approximation.

The right-hand side of the semiclassical variance relation corresponding to (6.2) is found by assuming the correspondence

$$\langle {\rm vac} | \{ \hat\psi, \hat\Pi \} | {\rm vac} \rangle^2 \rightarrow \left[\hbar \frac{1}{\Omega} \int \{ \psi, \Pi \} d^4\Omega \right]^2 . \tag{6.10}$$

The right-hand side of (6.10) is

$$\left[\hbar \frac{1}{\tau({\rm vol})} \int_{v_0}^{v_0+\tau} \int_0^\lambda \oint \sqrt{-g}\, \delta(r - r') \, \delta(\text{2-sphere}) dv' dr' d\theta' d\psi' \right]^2 , \tag{6.11}$$

where the δ functions are proper. Hence, (6.11) becomes $\frac{1}{4}\hbar^2({\rm vol})^{-2}$ as expected. For the minimal uncertainty relation I thus obtain

$$(\Delta g)(\Delta p) = \frac{1}{2} \frac{\hbar}{({\rm vol})} \ , \tag{6.12}$$

which could have been anticipated. Note that this relation holds for each of the $2l+1$ oscillators labeled by l with quasinormal frequencies ω_l, and for all $l \geq 2$.

We now collect the results (6.6), (6.7), and (6.12) to obtain

$$\epsilon_l^{\,2} = \alpha_l^{\,2} \frac{\hbar}{M^2} = \frac{\hbar}{M^2} \left[\frac{3}{2} \left[\frac{\sigma_l}{\pi} \right]^3 \frac{1 + 16\sigma_l^{\,2}}{l^2(l+1)^2 + 16\sigma_l^{\,2}} \right] . \tag{6.13}$$

The numerically calculated dimensionless frequencies $\sigma_l = M\omega_l$ of the first five ringing modes are given in the Appendix. A good approximation that can be deduced from the Regge-Wheeler and Zerilli effective potentials is also given in the Appendix. For instance, for $l = 2$ we have the numerically calculated value $\sigma_l = 0.373\,67$ ($\lambda_2 = 16.814\,80M$), and we calculate from (6.13)

$$(\epsilon_2)^2 = 2.135\,04 \times 10^{-4} \frac{\hbar}{M^2} \ . \tag{6.14}$$

Note that $\epsilon_l^{\,2}$ goes as l, as l becomes very large because as $l \rightarrow \infty$, $\sigma_l \rightarrow l(27)^{-1/2}$; this gives rise to a high-frequency problem to be dealt with later.

VII. TEMPERATURE

We can now use the value of ϵ_l determined by (6.13) in the expression (5.21) to quantify $M_{\rm QE}$ per mode. The existence of this quasilocal energy has resulted from the temperature-independent fluctuations of the black-hole metric and prompts the question as to whether the results obtained thus far can be given a thermal interpretation: Does the energy of the quantum ergosphere have a temperature? If so, the spacetime dynamical theory and the Hawking effect might be seen to have an even closer relationship.

To see if this is the case, I shall now conjecture that $M_{\rm QE}$ can be *identified* with the mean *thermal* energy U of a Planckian oscillator of frequency ω_l in a heat bath of temperature T. This thermal energy is regarded as having *resulted* from the zero-point fluctuations. If the T calculated from this assumption satisfies for a given mode $\hbar\omega/T \gg 1$, as it always does, then the mode is relatively unexcited ("cold"). This means that if we were to then go back to ϵ^2 and correct it for the fact that $T > 0$ [in the uncertainty principle, one would have on average slightly more than $(\frac{1}{2}\hbar)$ per mode], we would find little change. One would then iterate and find convergence to some new T. I have done this, and the new T does not differ significantly from the original one.

The mean energy of an oscillator at temperature T is given by the familiar expression

$$U_{\text{Total}}=\tfrac{1}{2}\hbar\omega+\hbar\omega(e^{\hbar\omega/T}-1)^{-1} . \quad (7.1)$$

The first term is the zero-point energy that gives rise to the quantum ergosphere. The second term is the mean thermal energy U. I equate U and M_{QE} to find

$$T_l=\frac{\hbar}{M}\sigma_l\left[\ln\left[1+\frac{1+16\sigma_l^2}{4\alpha_l^2\sigma_l}\right]\right]^{-1} . \quad (7.2)$$

Here I shall present a few results, later tabulated (Table I) more precisely (temperature in units $\hbar M^{-1}$):

$$\sigma_2=0.373\,67\rightarrow T_2=0.040\,51 ;$$

$$\sigma_3\approx 0.60\rightarrow T_3\approx 0.07 ;$$

$$\sigma_4\approx 0.8\rightarrow T_4\approx 0.10 ;$$

$$\sigma_5\approx 1.01\rightarrow T_5\approx 0.12 ;$$

$$\sigma_6\approx 1.21\rightarrow\approx T_6\approx 0.14 .$$

Note that T_2 differs from the Hawking temperature $T_H=(8\pi)^{-1}\cong 0.039\,79$ fractionally by about 1.8%. The $l\geq 3$ "temperatures" are larger. As $l\rightarrow\infty$, $\sigma_l\rightarrow l(27)^{-1/2}$ and

$$\sigma_l T_l^{-1}\rightarrow\ln[1+6^{-1}\pi^3(27)^2]\cong 8.234\,37 .$$

One has for all l that $\sigma_2 T_2^{-1}>\sigma_l T_l^{-1}>8.234\,37$ with $\sigma_2 T_2^{-1}\cong 9.224\,14$. Hence, all modes are relatively unexcited. If we define in the usual way a mean occupation number $\langle n_l\rangle$ by $U_l=M_{\text{QE}}(l)=\langle n_l\rangle\hbar\omega_l$, or

$$\langle n_l\rangle=(e^{h\omega_l/T_l}-1)^{-1} , \quad (7.3)$$

we find $10^{-4}\lesssim\langle n_l\rangle\lesssim 2.7\times 10^{-4}$.

Of course, if all the T_l's are not equal, there is actually no clear-cut temperature T associated with the quantum ergosphere, and it may be doubted whether the dynamical effect has a simple thermodynamic interpretation. However, heuristic arguments can be given to effect a better understanding. As I mentioned earlier, the description I have used assumes that all the quasinormal modes can be interpreted in terms of "waves" passing essentially freely in a radial manner in and out of the hole. However, studies by Press,[28] Goebel,[29] Ruffini,[5] and others[11,12] have given clear pictures of how these modes are excited and what appears to an outside observer. Ruffini,[5] in particular, gave a helpful description. One can excite a mode by injecting relativistic matter into the unstable circular photon orbit at $r\approx 3M$, near the top of the effective potential. Most of the observed gravitational radiation occurs at $l=2$; the $l=2$ modes pass in an essentially free radial manner to the distant observer. Thus, the *equilibration* of such modes would occur as in my description.

On the other hand, radiation to infinity in the modes $l\geq 3$ is highly suppressed because most of it orbits the hole in a "storage ring" effect and what "free" radiation there is spirals almost entirely down the hole.[5] One can see that to equilibrate such modes requires waves spiraling out of the hole to stabilize (in an unstable equilibrium) the storage ring at $r\approx 3M$. The modes thus circling the hole in all directions incoherently would not on average be efficient in raising tides on the black-hole surface. Hence the true amplitudes ϵ_l(EQ) in a dynamical equilibrium would not be as large as I have estimated, except for $l=2$. In this description, because the orbiting effects increase with l, one thinks of most of the zero-point energy of the higher modes as having a rapidly decreasing observable effect. (Ruffini also pointed out that for electromagnetic radiation and charged holes, the orbiting effects are again very important for all but the lowest mode, which is $l=1$ in the case of electromagnetism.)

There is another, I believe quite strong, indication that the tidal fluctuations $l\geq 3$ are strongly suppressed in equilibrium. Indeed, the suppression is *required* for thermal equilibrium ($T=T_{\text{EQ}}$ for all l) from the present perspective. Let us regard the observable energy levels of a black hole as being defined by the gravitational ringing mode frequencies. Let us treat each mode as being either "on" ($n=1$) or "off" ($n=0$), which should suffice for a cold system. Then assume that the relative probability of the excitation of the lth mode with respect to the quadrupole mode is given by the canonical ratio of Boltzmann probability factors:

$$\frac{\exp(-\hbar\omega_1/T_{\text{EQ}})}{\exp(-\hbar\omega_2/T_{\text{EQ}})}\equiv B(l,2,T_{\text{EQ}}) . \quad (7.4)$$

Taking $T_{\text{EQ}}=T_2$ of the "free" quadrupole mode, we have then

$$\epsilon_l^2\rightarrow\epsilon_l^2(\text{EQ})=\epsilon_l^2 B(l,2,T_2) \quad (7.5)$$

in the expression for $M_{\text{QE}}(l)$. This substitution modifies the temperature formula (7.2) via α_l^2. One then finds, remarkably, that the new T_l's, for all $l=2,3,\ldots,\infty$ (denoted T_{EQ}), are equal to T_2 with fractional differences of less than 3.8%. The calculations are fully summarized in Table I, with temperatures in units $\hbar M^{-1}$.

The near constancy of the derived T_{EQ}'s and their small differences from the Hawking temperature suggest that a dynamical equilibrium temperature exists and can be identified with $T_H=\hbar(8\pi M)^{-1}$. I shall henceforth assume $T_{\text{EQ}}=T_H$ for all modes in the sequel. If $T_{\text{EQ}}=T_2=T_H$, then the value of $(\epsilon^2)^2$ would be, from (7.2)

$$\epsilon_H^2=1.8056\times 10^{-4}\frac{\hbar}{M^2} , \quad (7.6)$$

TABLE I. Quasinormal frequencies and computed temperatures to be compared to $T_H=(8\pi)^{-1}\cong 0.039\,79$.

l	σ_{res}	T	σ_{res}/T	T_{EQ}
2	0.373 67	0.040 51	9.224 14	0.040 51
3	0.599 445	0.068 91	8.698 56	0.042 00
4	0.809 175	0.095 12	8.506 91	0.042 02
5	1.012 29	0.120 31	8.414 24	0.041 87
6	1.212 01	0.144 94	8.362 12	0.041 71
∞	$l(27)^{-1/2}$	l(0.023 37)	8.234 37	0.040 51

rather than $(\epsilon_2)^2=2.1350\times10^{-4}\hbar M^{-2}$. I shall adopt (7.6) for $l=2$.

The results above imply that the quadrupole modes are of dominant interest. This makes for an interesting analogy with the semiclassical theory of the Lamb shift. Welton[8] showed that one obtains good results with a high-frequency cutoff for the electromagnetic zero-point modes given by $\omega\lesssim \lambdabar_C{}^{-1}$ with $\lambdabar_C=\hbar m_e{}^{-1}$ being the Compton radius or reduced Compton wavelength of the electron. Kramers[30] gave the same cutoff in terms of the *dipole* approximation to semiclassical nonrelativistic quantum electrodynamics. The general physical argument is to the effect that shorter wavelengths would act incoherently over the quantum-mechanically defined volume "occupied" by the electron. A general description of the main effects of zero-point fluctuations for any kind of field is given in these terms, with analogous conclusions, in Misner, Thorne, and Wheeler.[16] (In particular, they discuss zero-point fluctuations of the spacetime metric.) Such arguments invite a comparison of the gravitational ringing mode frequencies with the reciprocal of the gravitational radius $\cong 2M$ of a hole of mass M. One sees from Table I that only the quadrupole mode satisfies $\omega\lesssim(2M)^{-1}$, which is very suggestive. The standard argument that a quantum gravitational high-frequency cutoff should be comparable to the Planck frequency is irrelevant to the tidal curvature fluctuations at the surface of the black hole unless the mass of the hole is comparable to the Planck mass. See Sec. IX.

Obviously, none of the heuristic arguments I have advanced about regulating the high-frequency modes can be regarded as final, although I regard the orbiting effects in the $l\geq 3$ modes as persuasive. The self-consistency of the hypothesis that the tidal effects (geometrical) are suppressed by a factor equal to the Boltzmann factor (thermal) is also highly nontrivial evidence. In the case of the Lamb shift, one has a consistent theory of renormalization that makes the semiclassical theory unnecessary, though still helpful. There should be possible a similar more nearly complete treatment for black holes, thus rendering the present semiclassical theory unnecessary.

We are left with the picture of a quantum black hole in equilibrium as an excited state of spacetime associated with a temperature. Dynamical equilibrium with its own zero-point fluctuation is impossible in the absence of an appropriate "heat bath." The zero-point fluctuations are much larger than the thermal fluctuations ΔU associated with $U=M_{\rm QE}$. In turn, $\Delta U\gg U$ (Sec. VIII). From the viewpoint of statistical thermodynamics, the degree of excitation of its energy levels is very slight, as $\hbar\omega_l/T_H=8\pi\sigma_l\geq 8\pi\sigma_2=9.391\,35$, independent of the black-hole mass.

VIII. ENTROPY

First we consider the entropy S corresponding to the mean thermal energy

$$U=\sum_{l=2}^{\infty}(2l+1)U_l \tag{8.1}$$

associated with the quantum ergosphere at temperature $T=T_H=\hbar(8\pi M)^{-1}$:

$$S_l=(\langle n_l\rangle+1)\ln(\langle n_l\rangle+1)-\langle n_l\rangle\ln\langle n_l\rangle$$
$$=\frac{1}{T}(U_l-F_l)\,, \tag{8.2}$$

$$S=\sum_{l=2}^{\infty}(2l+1)S_l\,. \tag{8.3}$$

Here

$$F_l=T\ln[1-\exp(-\hbar\omega_l/T)]$$

is the free energy. Clearly S_2 is the dominant term in (8.3). This entropy at any "moment" is small and is *independent of M* because of the scaling property $T\sim M^{-1}$ and $\omega_l\sim M^{-1}$. For example,

$$5S_2=4.335\,75\times10^{-3}\,. \tag{8.4}$$

However it may be that S is small and independent of M, a large statistical entropy $S_{\rm BH}\sim A\hbar^{-1}, A=16\pi M^2$, results when we add up all the excited states that disappear if we allow the hole to evaporate down to a final mass zero. We must do this by allowing the black hole to evaporate very slowly, *nearly in equilibrium*, into a heat bath of temperature θ, in the limit as $\theta\to T$ from below. We do not define $S_{\rm BH}$ using an evaporation into the vacuum. (This goes as $|dM/dv|\sim M^{-2}$ and is considered below.) Then, each time the hole surrenders thermal energy $U=\Delta M$ into the heat bath, the thermal states being continually repopulated by the zero-point oscillations, the corresponding entropy $S=\Delta S_{\rm BH}$ in (8.3) disappears. Note that in an "equilibrium" decay one does not, by the principle of detailed balancing, need to take account of the quasinormal mode opacities (transmission coefficients), which, in any event, are nearly equal (see below). In this argument, the types of matter into which the hole decays is not important; we are concerned only with the entropy of the black hole and not that of the radiated matter.

Recall that S is independent of M and that U goes as $(M^{-1})\times$(factors independent of M). The decay may be taken sufficiently slowly such that the factor M^{-1} in U changes arbitrarily little in any time interval of interest. Note that $UM^{-1}\lesssim\hbar M^{-2}(1.6\times10^{-4})$. Hence, in the limit of infinitely slow decay one has $|\Delta S_{\rm BH}/\Delta M_{\rm BH}|=S/U$ which we identify as $|dS/dM|_{\rm BH}$. Thus,

$$\left|\frac{dS}{dM}\right|_{\rm BH}=\frac{8\pi M}{\hbar}\left[1-\frac{\sum_{l=2}^{\infty}(8\pi)^{-1}(2l+1)\ln(1-e^{-8\pi\sigma_l})}{\sum_{l=2}^{\infty}\sigma_l(2l+1)(e^{8\pi\sigma_l}-1)^{-1}}\right]. \tag{8.5}$$

The only explicit dependence on M occurs in the first factor on the right of (8.5). The second factor is independent of M and has a value $\cong 1.106\,17$. (The "ln" terms are negative.) Integrating (8.5) from zero to M yields

$$S_{\rm BH} \cong (1.106\,17)\left[\frac{1}{4}\frac{A}{\hbar}\right] \cong (0.276\,54)\frac{A}{\hbar} \qquad (8.6)$$

if we assume $S_{\rm BH}=0$ when $M=0$.

This value of $S_{\rm BH}$ is close to the value usually assumed, $(0.25)A\hbar^{-1}$. That (8.6) is roughly 11% greater than this results here from the fact that the oscillator modes have a non-negligible free energy. Another way to look at it is that in the present treatment the black hole *cum* normal modes is not really localizable within $r=2M$; it has an effective radius about equal to the wavelength of the lowest mode, $\lambda_2 \approx 16.8M$. In fact, the high-frequency limit of (8.5) implies precisely $(0.25)A\hbar^{-1}$. To see this, we replace each of the dimensionless physical values of σ_l corresponding to the quasinormal modes in (8.5) by a large value "x" independent of l. Then the factor in brackets in (8.5) becomes approximately $[1+(8\pi x)^{-1}$ (quotient of identical sums)]. Hence, as $x\to\infty$, integration from zero to M of this limiting form of (8.5) yields $(0.25)A\hbar^{-1}[1+(8\pi x)^{-1}]\to(0.25)A\hbar^{-1}$.

It is worthwhile to recall that the value 0.25 was originally obtained from a mechanical-thermodynamical analogy based on

$$dM=\kappa(8\pi)^{-1}dA \ , \qquad (8.7)$$

where $\kappa=(4M)^{-1}$. This formula, which is usually obtained by differentiation of the relation $M=\kappa(4\pi)^{-1}A$ that holds for static vacuum holes, was verified by the perturbative dynamical treatment given in Sec. II.[31] Upon the identification by Hawking that $T=\hbar(8\pi M)^{-1}=\kappa\hbar(2\pi)^{-1}$, it would follow from the hypothesis that (8.7) can be written in thermodynamic form, with $dS_{\rm BH}=(0.25)\hbar^{-1}dA$, if and *only* if one assumes that the thermodynamic law for neutral nonrotating holes has the form $dM=T_H dS_{BH}$, that is, that there is present no term corresponding to free energy, e.g., "$-pdV$" for a simple mechanical system. Given $T=\kappa\hbar^{-1}(2\pi)^{-1}$, the formulas for M and dM do not by themselves imply uniquely a value for the entropy; $dM=T_H dS_{\rm BH}$ is the "simplest" possibility.

It was noted recently[32] that there existed no *statistical* calculation[33] independently confirming $S_{\rm BH}=(0.25)A\hbar^{-1}$. However, one sees that if the entropy is defined in terms of the excitations of the "energy levels" of the hole, identified here with the quasinormal modes, then a simple argument based on ordinary statistical thermodynamics gives a value close to the usual one. If we adopt the value given by (8.6), then the mechanical-thermodynamic relations previously written $M=2T_H S_{\rm BH}, dM=T_H dS_{\rm BH}$ become, respectively,

$$M=2(T_H S_{\rm BH}-\lambda\kappa A) \ , \qquad (8.8)$$

$$dM=T_H dS-\lambda\kappa dA \ , \qquad (8.9)$$

where $\lambda=\text{constant}=(8\pi)^{-1}(0.106\,17)$. Writing the "extra pair" of variables as κ and A is convenient and simple, but is not unique and is done here only because I do not at present know a more illuminating presentation. It seems to me quite plausible that if one slowly adds "heat" $T_H dS_{\rm BH}$ to a black hole in equilibrium, thereby causing it to expand, that it should do "work" in lifting itself in its own gravitational "potential well." I do wish to stress that (8.8) and (8.9) are mathematically fully equivalent to $M=\kappa(4\pi)^{-1}A$ and $dM=\kappa(8\pi)^{-1}dA$. The partition between "heat" and "work" is a physical question depending upon an independent calculation of $S_{\rm BH}$, even if we are given the Hawking temperature.

To give an independent confirmation of the ideas behind the above estimate of black-hole entropy, let us consider the decay of the hole into *vacuum*. We then expect to find a larger entropy since this process is irreversible.[3,32] Moreover, inasmuch as our thermal energies are defined by excitations of the black-hole geometry, it would not be surprising to find a value comparable to the entropy that would be radiated by the hole in a hypothetical decay purely into gravitons, which themselves are the excitations of the geometry that are "counted" by a distant observer in this situation. Page has recently given this value as $(1.3481)(4\pi M^2)\hbar^{-1}$ over the lifetime of the hole.[34] In performing this calculation, I shall assume for simplicity that the transmission coefficients Γ_l for the quasinormal modes are equal and thus, in the method given below, can be ignored. (In fact $\Gamma_2\approx 0.47,\ldots,$ $\Gamma_l\approx$ a bit less than 0.5 for large l.[35]) Moreover, the fact that each excitation of the sphere would correspond to two gravitons (two helicities) is easily seen to be irrelevant in the method of calculation below.

Now the rate of decay $|dM/dv|\sim M^{-2}$ cannot be ignored. One could suppose the loss of mass (thermal energy) to occur in M/U events, each yielding the entropy S, except for the fact the $M/U=(M^2\hbar^{-1})\times$(a pure number) and that M^2 changes in accordance with $|dM/dv|\sim M^{-2}$. This implies that, with an initial mass M, the average of M^2 over the decay from M to zero is $\frac{3}{5}M^2$. Hence, the total entropy calculated this way is

$$S\cong\frac{3}{5}\left[\frac{8\pi M^2}{\hbar}\right]\left[1-\frac{F}{U}\right] , \qquad (8.10)$$

where $(1-FU^{-1})$ was evaluated in (8.5) as 1.106 17. Hence we find $S\cong(1.3274)(4\pi M^2)\hbar^{-1}$, which differs fractionally from Page's value by about 1.5%. This result confirms the reasonableness of regarding black-hole entropy as being associated with excitations of its gravitational quasinormal modes.

IX. FLUCTUATIONS

We begin by computing the rms thermal fluctuation for the lth oscillator at the Hawking temperature and the corresponding ringing frequency. We take into account here that it is $(2l+1)$fold degenerate. The standard calculation for an oscillator gives

$$\Delta U_l=\frac{\hbar}{M}[(2l+1)^{1/2}\sigma_l(1-e^{-8\pi\sigma_l})^{-1}e^{-4\pi\sigma_l}] \ , \qquad (9.1)$$

whereas $U=(2l+1)U_l$. For $l=2$ and $l=3$ considered independently we find

$$\left[\frac{\Delta U}{U}\right]_{l=2}\approx 49, \quad \left[\frac{\Delta U}{U}\right]_{l=3}\approx 706 \ . \qquad (9.2)$$

The numbers are $\gg 1$ because the modes are relatively unexcited at $T=T_H$. For large l we find

$$\left[\frac{\Delta U}{U}\right]_{l\gg 1}\approx(2l+1)^{-1/2}\exp[4\pi l(27)^{-1/2}] \ . \tag{9.3}$$

These values can be compared to the corresponding larger ratios of zero-point energy ($U_0=\frac{1}{2}\hbar\omega_l$ per mode) and thermal energy, for example,

$$\left[\frac{U_0}{U}\right]_{l=2}\approx 6\times10^3;\ \left[\frac{U_0}{U}\right]_{l=3}\approx 2\times10^6 \ . \tag{9.4}$$

One notices that U_0, U, and ΔU all go as $\hbar M^{-1}$; they depend on the mass of the hole. However, there is a potentially important rms energy fluctuation that is independent of the mass of the hole, being determined only by Planck's constant, the ringing frequencies, and the Hawking temperature. This is the characteristic rms fluctuation of the irreducible mass of the physical event horizon. This quantity will be determined predominantly by the five $l=2$ modes and would be expected to be comparable to the Planck mass $\approx 1.22\times10^{19}$ GeV. We have then, with $M_{\rm EH}=(A_{\rm EH}/16\pi)^{1/2}$ corresponding to one of the five modes

$$\begin{aligned}\Delta M_{\rm EH}&=\sqrt{5}[\overline{(M_{\rm EH}{}^2)}-(\overline{M}_{\rm EH})^2]^{1/2}\\ &=\sqrt{5}\left[\frac{\overline{A}_{\rm EH}}{16\pi}-M^2\right]^{1/2}\\ &=\sqrt{5}[\tfrac{1}{2}M^2(\epsilon_H)^2(1+16\sigma_2{}^2)^{-1}]^{1/2}\\ &\approx 1.18\times10^{-2}\hbar^{1/2}\approx 1.44\times10^{17}\text{ GeV} \ ,\end{aligned} \tag{9.5}$$

about 2 orders of magnitude less than the Planck energy.

It is of interest to compare the above value to a mass scale determined by the dynamical metric fluctuations associated with black-hole radiance. We ask: For what value of the black-hole mass do the rms fluctuations of the physical components of the metric become of order unity? This mass is denoted by M_*. This scale may be thought of as corresponding to the onset of a modification of our usual ideas of the causal microstructure of spacetime. Of course, this is something of a speculative exercise, but does turn out to be just within the domain of mathematical validity of the approximations adopted in the previous work. Again, the five $l=2$ modes dominate. The physical metric fluctuation described in Sec. VI in this case yields an rms fluctuation

$$\Delta g=2\sqrt{5}\epsilon_H\left[\frac{1}{2}\frac{36+16\sigma_2{}^2}{1+16\sigma_2{}^2}\right]^{1/2} \ . \tag{9.6}$$

Equating Δg to unity implies that

$$\epsilon_H{}^2\cong 1.8056\times10^{-4}\frac{\hbar}{M_*{}^2}\approx 8.46\times10^{-3} \ . \tag{9.7}$$

Note that when $\Delta g=1$, $\epsilon_H\approx 0.09$. Solving for M_* we find

$$M_*\approx 0.15\hbar^{1/2}\approx 1.8\times10^{18}\text{ GeV} \ . \tag{9.8}$$

This is about an order of magnitude less than the Planck mass and is consistent with the widely held belief that such large metric fluctuations would occur at about the Planck scale. Observe that $\Delta M_{\rm EH}\sim 10^{-1}M_*$. The effective radius of the hole at this mass is about $\lambda_2=16.8M_*\approx 2.5$ times the Planck length.

Elsewhere,[3] based on an expanded version of the analysis in Sec. II, in which one computes the backreaction on the horizons to the creation of radiation, ignoring the metric zero-point fluctuations, I extrapolated the results in $O(\hbar)$ and defined a mass $M_{\rm dis}$ at which the disappearance of the hole might occur.[36] This heuristic definition was essentially that disappearance would occur when the mass of the black hole equals the irreducible mass $M_{\rm QE}$. The value obtained depended on the black-hole luminosity and hence on an assumed particle spectrum. Using a minimal GUT (Georgi-Glashow)× Einstein theory I obtained $M_{\rm dis}\approx 0.15\hbar^{1/2}$.[37] Using the $N=8$ supergravity particle content treated as external radiation fields, I obtained $M_{\rm dis}\approx 0.2\hbar^{1/2}$.[37] Both of these values are consistent with the value in the $M_*\approx 0.15\hbar^{1/2}$ where $\Delta g=1$. When $\Delta g=1$, it is not unreasonable to think of the disappearance of the hole. The exact numbers, of course, cannot be taken too seriously. However, they again point to the consistency of regarding metric zero-point fluctuations as being in essence the origin of black-hole radiance, which is the thesis of this work.

X. CONCLUSION

I have adopted a simple model of a black hole undergoing zero-point fluctuations of zero mean (with respect to a "background" Schwarzschild metric) in its least-damped quasinormal modes. Using just this discrete set of frequencies simplifies the calculations and, as I have described, is in accord with what has been learned classically from perturbation and scattering "experiments." This, in effect, is a partially phenomenological input. It would be desirable to consider in more detail all the frequencies that can have tidal effects on the surface of the hole. One would also like to generalize the radial dependence of the metric (3.1) to incorporate more nearly exactly the radial waveforms of the modes. The correct angular functions came out directly in the analysis of the apparent horizon and the outgoing shear.

Attention was directed to the quantum ergosphere. This small region near $r=2M$ is the imprint on the metric associated with the inability of a black hole to be in equilibrium at zero temperature. The method of computation suggests that the effective size of the region associated with energy production probably extends out to about $r=\lambda_2=16.8M$ or so. It proved natural to associate the thermal energy and entropy with the quantum ergosphere. The issue of black-hole entropy deserves a more detailed study, and this will appear elsewhere.

The characteristic energies and fluctuation scales associated with black-hole metric fluctuations and the possibility of relating them to the particle content of the radiated matter suggests that a yet-to-be-found complete and consistent theory of quantum gravity applicable to black holes might not be possible without some kind of "unification" with the other fields, such as is currently being studied from the "particle theory" point of view (supergravity, Kaluza-Klein theories, etc.). However, the "spacetime" point of view, which is complementary, is also important. This "complementarity" can be stated as follows: The Hawking temperature is the lowest temperature consistent with the quantum nature of the metric of spacetime. It seems that black holes constitute a problem for quantum gravity with a relative importance not unlike that of the hydrogen atom and the electromagnetic *Hohlraum* in the early history of quantum physics. The central ideas are the principle of equivalence, spacetime fluctuations, and the uncertainty principle. Little else in the way of physics was used in this work. In particular, the Einstein equations played only an interpretational role in defining the idea of fluctuations with a zero mean.

It should be possible to give a path-integral version of this work. Preliminary work[38] shows that the basic effect (quantum ergosphere) corresponds to a second variation of the action ("one loop") as might be expected.

One should consider as well charged and rotating holes to see if their quantum radiance can be characterized in the present terms. Although more difficult, this seems likely. One would also consider other types of event horizons.

The final goal involves, among other things, doing away with the semiclassical theory presented in this paper, although its heuristic "pictorial" power seems very helpful.

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APPENDIX

I use MTW (Ref. 16) conventions throughout, with $G=c=k_B=1$. The metric is given in (3.1). Its determinant is given by $\sqrt{-g}=r^2\sin\theta$. Define $F\equiv 1-2mr^{-1}=F(v,r,\theta)$. The nonvanishing contravariant components of the metric are

$$g^{vr}=1,\ g^{rr}=F,\ g^{\theta\theta}=r^{-2}, \quad g^{\phi\phi}=(r^2\sin^2\theta)^{-1}\ . \tag{A1}$$

The nonvanishing Christoffel symbols are

$$\begin{aligned}
&\Gamma^v_{vv}=\tfrac{1}{2}F_{,r},\ \Gamma^v_{\theta\theta}=-r,\ \Gamma^v_{\phi\phi}=-r\sin^2\theta\ ,\\
&\Gamma^r_{vv}=-\tfrac{1}{2}F_{,v}+\tfrac{1}{2}FF_{,r},\ \Gamma^r_{vr}=-\tfrac{1}{2}F_{,r}\ ,\\
&\Gamma^r_{v\theta}=-\tfrac{1}{2}F_{,\theta},\ \Gamma^r_{\theta\theta}=-rF,\ \Gamma^r_{\phi\phi}=-rF\sin^2\theta\ ,\\
&\Gamma_{vv}{}^{\theta}=\tfrac{1}{2}r^{-2}F_{,\theta},\ \Gamma^\theta_{r\theta}=r^{-1},\ \Gamma_{\phi\phi}{}^{\theta}=-\sin\theta\cos\theta\ ,\\
&\Gamma^\phi_{v\phi}=r^{-1},\ \Gamma^\phi_{\theta\phi}=\cot\theta\ .
\end{aligned} \tag{A2}$$

The Einstein tensor G^a_b has nonvanishing components

$$\begin{aligned}
&G^v_v=r^{-1}F_{,r}-r^{-2}(1-F)\ ,\\
&G^r_v=r^{-1}F_{,v}+(2r^2)^{-1}(F_{,\theta\theta}+F_{,\theta}\cot\theta)\\
&G^r_r=G^v_v;G^r_\theta=-F_{,r\theta}-(2r)^{-1}F_{,\theta}\ ,\\
&G^\theta_v=r^{-2}G^r_\theta,\ \ G^\theta_\theta=G^\phi_\phi=\tfrac{1}{2}F_{,rr}+r^{-1}F_{,r}\ .
\end{aligned} \tag{A3}$$

The scalar curvature is

$$R=-F_{,rr}-4r^{-1}F_{,r}+2r^{-2}(1-F)\ . \tag{A4}$$

The Weyl tensor $C^{ab}{}_{cd}$ has nonvanishing components

$$\begin{aligned}
&C^{vr}{}_{vr}=-\tfrac{1}{6}[F_{,rr}-2r^{-1}F_{,r}-2r^{-2}(1-F)]\ ,\\
&C^{vr}{}_{v\theta}=-\tfrac{1}{2}(F_{,r\theta}-2r^{-1}F_{,\theta})\ ,\\
&C^{v\theta}{}_{v\theta}=C^{v\phi}{}_{v\phi}=-\tfrac{1}{2}C^{vr}{}_{vr}\ ,\\
&C^{r\theta}{}_{vr}=-r^{-2}C^{vr}{}_{v\theta}\ ,\\
&C^{r\theta}{}_{v\theta}=(4r^2)^{-1}(F_{,\theta\theta}+F_{,\theta}\cot\theta)\ ,\\
&C^{r\theta}{}_{r\theta}=-\tfrac{1}{2}C^{vr}{}_{vr}=C^{r\phi}{}_{r\phi}\ ,\\
&C^{r\phi}{}_{v\phi}=(4r^2\sin^2\theta)^{-1}(2F\cot\theta-F_{,\theta\theta}-F_{,\theta}\cot\theta)\ ,\\
&C^{r\phi}{}_{\theta\phi}=r^2C^{r\theta}{}_{vr},\ \ C^{\theta\phi}{}_{v\phi}=C^{r\theta}{}_{vr}\ ,\\
&C^{\theta\phi}{}_{\theta\phi}=C^{vr}{}_{vr}\ .
\end{aligned} \tag{A5}$$

The Riemann tensor is given by

$$\begin{aligned}
R^{ab}{}_{cd}=C^{ab}{}_{cd}&+\tfrac{1}{2}\left[\delta^a_cG^b_d-\delta^a_dG^b_c+\delta^b_dG^a_c-\delta^b_cG^a_d\right]\\
&+\tfrac{1}{3}R\left[\delta^a_c\delta^b_d-\delta^a_d\delta^b_c\right]\ .
\end{aligned} \tag{A6}$$

The nonvanishing components of the projection operator $\gamma^a_b=\delta^a_b+l^a\beta_b+\beta^al_b$ are given by

$$\begin{aligned}
&\gamma^r_v=r^2(l^\theta)^2,\ \ \gamma^r_\theta=-r^2l^\theta,\ \ \gamma^\theta_v=-l^\theta\ ,\\
&\gamma^\theta_\theta=\gamma^\phi_\phi=1\ .
\end{aligned} \tag{A7}$$

The only nonvanishing components of the outgoing shear tensor σ^a_b are in $O(\epsilon)$

$$\sigma^\theta_\theta=-\sigma^\phi_\phi=\epsilon\kappa f(v)\left[\frac{dq_l}{d\theta}\cot\theta-\frac{d^2q_l}{d\theta^2}\right]\ , \tag{A8}$$

where

$$\begin{aligned}
\frac{d^2q_l}{d\theta^2}-\frac{dq_l}{d\theta}\cot\theta=&\tfrac{1}{3}(2l+1)^{1/2}(l+2)(l+1)\\
&\times l(l-1)C_{l+2}^{-3/2}(\sin^2\theta)^{-1}\ ,
\end{aligned} \tag{A9}$$

and C^α_n is a Gegenbauer function with standard normalization.[22]

The least-damped gravitational quasinormal frequencies for $l=2,3,\ldots,6$ are given in units M^{-1} by[39]

$$\begin{aligned}
&\sigma_2=(0.373\,67)+i(0.088\,96)\ ,\\
&\sigma_3=(0.599\,445)+i(0.092\,71)\ ,\\
&\sigma_4=(0.809\,175)+i(0.094\,16)\ ,\\
&\sigma_5=(1.012\,29)+i(0.094\,87)\ ,\\
&\sigma_6=(1.212\,01)+i(0.095\,27)\ .
\end{aligned} \tag{A10}$$

A good approximation for the real parts is[35]

$$\mathrm{Re}\sigma_l \approx \left[\frac{(l-1)(l+2)}{27} - \left[\frac{1}{2\sqrt{27}} \right]^2 \right]^{1/2} \underset{\text{large } l}{\rightarrow} \frac{l}{\sqrt{27}} , \quad \text{(A11)}$$

and for the imaginary parts[40]

$$\mathrm{Im}\sigma_l \approx \frac{1}{2\sqrt{27}} . \quad \text{(A12)}$$

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