Helicity formalism for transition amplitudes

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Transition amplitudes between states with spin ≤ 1 are considered and directly evaluated in terms of momenta and polarization vectors. A special algorithm is derived to reduce expressions where γ matrices of different lines are saturated. The application of the method is illustrated for radiative and nonradiative processes, including mass effects.

I. INTRODUCTION

In recent years the theory of strong, weak, and electromagnetic interactions has developed to a point that we are more in a position to make very accurate comparisons between theoretical predictions and experimental results. In this investigation we need to compute higher-order Feynman diagrams. The standard procedures, where we square the amplitude for a given process and use a covariant sum over polarizations, have become almost intractable. However, alternative techniques have been recently developed for analyzing bremsstrahlung cross sections[1] and transition amplitudes between Dirac spinors.[2] Motivated by these ideas we show that a unified approach can be formulated in which the amplitude for an arbitrary process, radiative or not, is directly computable in terms of the invariants which specify the process and for any set of spin indices. The formalism is discussed in Sec. II.

II. EVALUATION OF TRANSITION AMPLITUDES

What we need is a convenient procedure which eliminates spinors, spin-1 external wave functions, and γ matrices in terms of momenta and polarization vectors. In order to deal with spinors we first develop a method which makes use of an explicit representation of the γ matrices. Using the conventions of Ref. 2 we find

$$u_\lambda(p_i)\bar{u}_\rho(p_j) = -N(p_i)N(p_j)(\not{p}_i + im_i)\Gamma_{\lambda\rho}(\not{p}_j + im_j) ,$$

$$N^{-2}(p) = 2p_0(p_0+m) , \tag{1}$$

$$\Gamma_{\lambda\rho} = \frac{1+\gamma^4}{4}(\delta_{\lambda\rho} + i\gamma^5\vec{\gamma}\cdot\vec{\sigma}_{\rho\lambda}) ,$$

where $u(p)$ denotes a Dirac spinor and $\lambda = \pm 1$ gives the spin assignment in the p rest frame. Having in mind applications to high-energy physics we restrict our analysis to massless particles. v spinors are then converted into u spinors by $v_\lambda = -\lambda\gamma^5 u_{-\lambda}$. Next we proceed by reducing the numerator structure of an arbitrary diagram. Vector and axial-vector couplings of internal particles are replaced with a combination of scalar and pseudoscalar couplings. The basic reduction formula reads

$$\gamma^\mu S_n u_\lambda(p)\bar{u}_\rho(q) S_m \gamma^\mu = -\lambda\rho[S_m^R u_{-\rho}(q)\bar{u}_{-\lambda}(p)S_n^R - \gamma^5 S_m^R u_{-\rho}(q)\bar{u}_{-\lambda}(p)S_n^R\gamma^5 - \bar{u}_{-\lambda}(p)S_n^R S_m^R u_{-\rho}(q) + \gamma^5\bar{u}_{-\lambda}(p)S_n^R\gamma^5 S_m^R u_{-\rho}(q)] , \tag{2}$$

where S_n stands for an arbitrary string of n γ matrices and S_n^R is the same string in the reversed order. The formula can be easily proved by using the Chisholm identities.[3] In this way an arbitrary diagram consisting of n fermion lines connected by internal vector bosons is replaced by a collection of diagrams each formed by n disconnected fermion lines. However each application of the Chisholm identities doubles the number of terms. When many internal photons are present we could in principle avoid this problem by using the Kahane algorithm[3] which appears to minimize the number of terms in the final expression. The resulting amplitude is computed by introducing

$$\begin{aligned} S_{\lambda\rho}(p,q) &= -2p_0q_0\bar{u}_\lambda(p)u_\rho(q) , \\ P_{\lambda\rho}(p,q) &= -2p_0q_0\bar{u}_\lambda(p)\gamma^5 u(q) . \end{aligned} \tag{3}$$

(Matrix elements of this type have been tabulated for light-cone perturbation theory in Ref. 4.) For S and P we get

$$\begin{aligned} S_{\lambda\rho}(p,q) &= p\cdot q\,\delta_{\lambda\rho} + i(\vec{p}\times\vec{q})\cdot\vec{\sigma}_{\lambda\rho} , \\ P_{\lambda\rho}(p,q) &= (p_0\vec{q} - q_0\vec{p})\cdot\vec{\sigma}_{\lambda\rho} . \end{aligned} \tag{4}$$

As a bonus for working with an explicit representation of γ matrices we can avoid arbitrary phases. For simple processes the amplitude can be immediately computed. Consider $\mu^- e^-$ scattering in massless QED:

$$\frac{d\sigma}{dt}(\lambda_e\lambda_\mu \to \rho_e\rho_\mu) = 2\frac{\alpha^2}{s^3t^2}\,|M(\lambda_e\lambda_\mu \to \rho_e\rho_\mu)|^2 . \tag{5}$$

It follows

$$M(\lambda_e\lambda_p\to\rho_e\rho_\mu)$$
$$=-\lambda_e\rho_\mu(S_{\rho_e,-\rho_\mu}S_{-\lambda_e,\lambda_\mu}-P_{\rho_e,-\rho_\mu}P_{-\lambda_e,\lambda_\mu}$$
$$-S_{\rho_e\lambda_\mu}S_{-\lambda_e,-\rho_\mu}+P_{\rho_e\lambda_\mu}P_{-\lambda_e,-\rho_\mu})\ . \quad (6)$$

Thus in the c.m. system of the scattering particles

$$M(++\to++)=M(--\to--)=u^2\ ,$$
$$M(++\to-+)=M(--\to+-)=M(++\to+-)$$
$$=M(--\to-+)=-i(tu^3)^{1/2}\ ,$$
$$M(++\to--)=M(--\to++)=-tu\ , \quad (7)$$
$$M(+-\to++)=M(-+\to--)=M(+-\to--)$$
$$=M(-+\to++)=is(tu)^{1/2}\ ,$$
$$M(+-\to-+)=M(-+\to+-)=st\ ,$$
$$M(+-\to+-)=M(-+\to-+)=-su\ ,$$

in agreement with the well-known result

$$\sum_{\text{spin}}|M|^2=2s^4\left[1+\frac{t}{s}+\frac{1}{2}\frac{t^2}{s^2}\right]\ .$$

Calculations of QED processes with polarized particles, including higher-order corrections can be found in Ref. 5. In general we have expressions like

$$\bar{u}_\lambda(p_i)\prod_{l=1}^{n}\not{Q}_l u_\rho(p_j)\ , \quad (8)$$

where Q_l are linear combinations of external momenta (even when internal loops are present). This can be reduced to a product of S,P functions since

$$\not{Q}_l=2i\sum_m a_{ml}p_{0m}\sum_\tau u_\tau(p_m)\bar{u}_\tau(p_m)$$
$$\text{for } Q_l=\sum_m a_{ml}p_m\ . \quad (9)$$

Finally we must take into account a possible multiphoton radiation. Once the reduction formula is applied the formalism of Ref. 1, namely, the use of circularly polarized photon states, becomes particularly simple since each fermion line with its emitted photons can be analyzed independently from the rest of the diagram. Even when strong cancellations between different diagrams are not expected we use the fact that the polarization vector ϵ_μ^λ of Ref. 1 is explicitly given in terms of the external momenta and the previous formulas suffice in evaluating the amplitude. We also need an explicit representation for massive vector boson wave functions. A convenient way is to write

$$\epsilon_i^\lambda(k)=\delta_i^\lambda+\frac{1}{M(M+k_0)}k^\lambda k_i,\epsilon_0^\lambda(k)=\frac{1}{M}k^\lambda\ ,$$
$$\lambda,i=1,2,3\ ,\quad k^2=-M^2\ .$$

From Ref. 2 we learn that a specific reference to spinor components can be avoided (see also Ref. 6) if we allow for arbitrary phases. Here we derive the formalism for the massless limit with a new version of the reduction formula and also consider the general case; i.e., massive particles with arbitrary polarization vectors.

First however we want to compare the general features of this method with alternative works, in particular the one of Farrar and Neri.

To start with, we do not need in what follows a specific representation of the γ matrices and the method is completely general. As far as we consider only the massless spinor, the Farrar-Neri method can perhaps be faster for evaluating certain transition amplitudes, but the present method allows the extraction of helicity amplitudes in the general case. For instance, massive particles can easily be accommodated and no restriction is made on the polarization. Thus transverse polarization can be studied, and we have in mind polarization driving mechanisms in circular colliders which lead to transverse polarization of both e^+e^- beams. Also, external photons are easy to include in the scheme, which is simple and amounts to transform spinor amplitudes into a trace of γ matrices with no repeated indices.

Let u_λ and v_λ be eigenstates of $\frac{1}{2}(1+\lambda\gamma^5)$ and $\frac{1}{2}(1-\lambda\gamma^5)$, respectively. The only property we need is

$$u_\lambda(p)\bar{u}_\lambda(p)=-i\frac{\not{p}}{2p_0}\pi_{+\lambda}\ ,$$
$$v_\lambda(p)\bar{v}_\lambda(p)=-i\frac{\not{p}}{2p_0}\pi_{-\lambda}\ , \quad (10)$$

with $\pi_\lambda=\frac{1}{2}(1+\lambda\gamma^5)$. Hence we may use $v_\lambda=-\lambda\gamma^5u_{-\lambda}$. Next we derive

$$u_\lambda(p)\bar{u}_{-\lambda}(q)=\frac{e^{-i\psi_-}}{2\sqrt{2}}(p_0q_0p\cdot q)^{-1/2}\not{p}\not{q}\pi_{-\lambda}\ , \quad (11)$$

where $e^{-i\psi_-}$ is an unspecified phase. When the helicity is the same $\bar{u}_\lambda u_\lambda=0$ and a different procedure must be used

$$u_\lambda(p)\bar{u}_\lambda(q)$$
$$=\frac{e^{-i\psi_+}}{2\sqrt{2}}(p_0q_0)^{-1/2}(2p\cdot nq\cdot n-p\cdot q)^{-1/2}\not{p}\not{n}\not{q}\pi_\lambda\ , \quad (12)$$

where n_μ is an arbitrary vector normalized to $n^2=1$. If R is the operator which reverses the order of a string of γ matrices, $RS=S^R$, we get

$$Ru_\lambda(p)\bar{u}_\rho(q)=u_{-\rho}(q)\bar{u}_{-\lambda}(p)\ .$$

Application of the Chisholm identities gives now[3]

$$\gamma^\mu S_n u_\lambda(p)\bar{u}_\lambda(q)S_m\gamma^\mu=-2S_m^R u_{-\lambda}(q)\bar{u}_{-\lambda}(p)S_n^R\quad(n+m\ \text{even})\ ,$$
$$\gamma^\mu S_n u_\lambda(p)\bar{u}_\lambda(q)S_m\gamma^\mu=\bar{u}_{-\lambda}(p)S_n^RS_m^Ru_{-\lambda}(q)-\gamma^5\bar{u}_{-\lambda}(p)S_n^R\gamma^5S_m^Ru_{-\lambda}(q)\quad(n+m\ \text{odd})\ ,$$
$$\gamma^\mu S_n u_\lambda(p)\bar{u}_{-\lambda}(q)S_m\gamma^\mu=\bar{u}_{-\lambda}(p)S_n^RS_m^Ru_\lambda(q)-\gamma^5\bar{u}_{-\lambda}(p)S_n^R\gamma^5S_m^Ru_\lambda(q)\quad(n+m\ \text{even})\ ,$$
$$\gamma^\mu S_n u_\lambda(p)\bar{u}_{-\lambda}(q)S_m\gamma^\mu=-2S_m^Ru_\lambda(q)\bar{u}_{-\lambda}(p)S_n^R\quad(n+m\ \text{odd})\ . \quad (13)$$

An alternative approach can be found by means of identity[3]

$$\mathrm{tr}(\gamma^\mu S)\,\mathrm{tr}(\gamma^\mu S') = 2\,\mathrm{tr}(S + S^R)S' .$$

As an example we consider

$$\bar u_4 \gamma^\mu u_1 \bar u_3 \gamma^\mu u_2 = \mathrm{tr}(\gamma^\mu u_1 \bar u_4)\,\mathrm{tr}(\gamma^\mu u_2 \bar u_3)\ ,\quad u_i = u_{\lambda_i}(p_i)$$
$$= 2\bar u_\rho(p_3) u_\lambda(p_1) \bar u_\lambda(p_4) u_\rho(p_2) + 2\bar u_\rho(p_3) u_{-\lambda}(p_4) \bar u_{-\lambda}(p_1) u_\rho(p_2)\ ,\quad \lambda=\lambda_1,\lambda_2\ ,\quad \rho=\lambda_2,\lambda_3\ . \tag{14}$$

The first term contributes only for $\rho = -\lambda$, the second only for $\rho=\lambda$. Helicity conservation prevents in general from a proliferation of diagrams in the repeated use of the reduction formula. The mechanism which eliminates saturated γ matrices has the advantage that the result can still be written as a Feynman diagram. Also it can be applied directly, no matter where the γ matrices are located inside the graph. Scalar and pseudoscalar bilinear forms can now be derived:

$$\begin{aligned}
\bar u_\lambda(p) u_\rho(q) &= e^{i\psi_s}\left[\frac{p\cdot q}{2p_0 q_0}\right]^{1/2}\Lambda^-(\lambda,\rho)\ ,\\
\bar u_\lambda(p)\gamma^5 u_\rho(q) &= -\rho e^{i\psi_s}\left[\frac{p\cdot q}{2p_0 q_0}\right]^{1/2}\Lambda^-(\lambda,\rho) = e^{i\psi_p}\left[\frac{p\cdot q}{2p_0 q_0}\right]^{1/2}\Lambda^-(\lambda,\rho)\ ,
\end{aligned} \tag{15}$$

where $\Lambda^\pm(\lambda,\rho) = \frac{1}{2}(1\pm\lambda\rho)$. Also using Eq. (12) with $n = Q$ we get

$$\bar u_\lambda(p) \not{Q} u_\rho(q) = e^{i\psi_v}\left[\frac{2p\cdot Q q\cdot Q - Q^2 p\cdot q}{2p_0 q_0}\right]^{1/2}\Lambda^+(\lambda,\rho),\ \text{etc.} \tag{16}$$

The arbitrariness of the phases becomes relevant whenever different diagrams interfere. In the following example we use $e^{i\psi_p} = -\rho e^{i\psi_s}$ to derive

$$g_s{}^2 \bar u_1 u_2 \bar u_3 u_4 + g_p{}^2 \bar u_1 \gamma^5 u_2 \bar u_3 \gamma^5 u_4 = e^{2i\psi_s}\frac{1}{2}\left[\frac{p_1\cdot p_2 p_3\cdot p_4}{E_1E_2E_3E_4}\right]^{1/2}(g_s{}^2 + \lambda\rho g_p{}^2)\ ,\quad -\lambda_1=\lambda_2=\lambda\ ,\quad -\lambda_3=\lambda_4=\rho\ .$$

As an application we consider the radiative Coulomb potential scattering of an electron. The amplitude is

$$M_\gamma = -\frac{Ze^2}{|\vec q\,|^2}\,\mathrm{tr}\left[\left[\frac{p\cdot\epsilon^\sigma}{p\cdot k} - \frac{p'\cdot\epsilon^\sigma}{p'\cdot k}\right]\not{n} - \frac{1}{2}\frac{\not{n}\not{k}\not{\epsilon}^\sigma}{p\cdot k} + \frac{1}{2}\frac{\not{k}\not{\epsilon}^\sigma\not{n}}{p'\cdot k}\right]u_\lambda(p)\bar u_\lambda(p')\ ,\quad n^2=1\ . \tag{17}$$

Using[1]

$$\not{\epsilon}^\sigma = \frac{1}{\sqrt{2}N}(\not{k}\not{p}'\not{p}\pi_{-\sigma} - \not{p}'\not{p}\not{k}\pi_\sigma)\ ,$$

we find

$$M_\gamma = -\frac{Ze^2}{|\vec q\,|^2}\frac{e^{i\psi}}{N}(EE')^{-1/2}(2p\cdot np'\cdot n - p\cdot p')^{-1/2}[M_+^\sigma\Lambda^+(\sigma,\lambda) + M_-^\sigma\Lambda^-(\sigma,\lambda)]\ , \tag{18}$$

$$\begin{aligned}
M_+^\sigma &= p\cdot p'(p\cdot p' - 2p\cdot np'\cdot n + p\cdot k - p\cdot nk\cdot n) - p\cdot np'\cdot np\cdot k + (p\cdot n)^2 p'\cdot k + 2p'\cdot n\epsilon(n,p,p',k)\sigma\ ,\\
M_-^\sigma &= p\cdot p'(p\cdot p' - 2p\cdot np'\cdot n + p'\cdot k - p'\cdot nk\cdot n) - p\cdot np'\cdot np'\cdot k + (p'\cdot n)^2 p\cdot k - 2p\cdot n\epsilon(n,p,p',k)\sigma\ ,
\end{aligned} \tag{19}$$

where $\Lambda^\pm(\sigma,\lambda) = \frac{1}{2}(1\pm\lambda\sigma)$, $\epsilon(n,p,p',k) = \epsilon^{\mu\nu\alpha\beta}n_\mu p_\nu p'_\alpha k_\beta$. Since $\Lambda^\pm$ are projectors $M_\pm^\sigma$ do not interfere.

Even in a situation where masses are not negligible there are many advantages in computing directly the amplitude. The relevant formalism has been developed in Ref. 2. The reduction formula is now more complicated and we derived it only for a simple example. Let

$$M = \bar u_4\gamma^\mu u_1 \bar u_3 \gamma^\mu u_2\ ,$$

where $u_i = u(p_i, n_i, \lambda_i)$ denotes a Dirac spinor with polarization $\lambda_i = \pm 1$ along n_i with $n_i{}^2 = 1, n_i\cdot p_i = 0$. Thus

$$\begin{aligned}
M &= \mathrm{tr}(\gamma^\mu u_1\bar u_4)\,\mathrm{tr}(\gamma^\mu u_2 \bar u_3) = \mathrm{tr}(\gamma^\mu M_{14}^{\mathrm{odd}})\,\mathrm{tr}(\gamma^\mu M_{23}^{\mathrm{odd}})\ ,\\
M_{ij} &= M_{ij}^{\mathrm{odd}} + M_{ij}^{\mathrm{even}} = N_{ij}(\not{p}_i + im_i)\Gamma_{ij}(\not{p}_j + im_j)\ ,\\
\Gamma_{ij} &= 1 + i\gamma^5(\lambda_i\not{n}_i + \lambda_j\not{n}_j) + \lambda_i\lambda_j\not{n}_i\not{n}_j\ .
\end{aligned} \tag{20}$$

N_{ij} is a normalization factor and even (odd) denotes the part of M_{ij} with an even (odd) number of γ matrices. We can easily prove that

$$M_{ij}^{\text{odd}}+M_{ij}^{\text{even}}=u(p_i,n_i,\lambda_i)\bar{u}(p_j,n_j,\lambda_j)\ ,$$
$$M_{ij}^{\text{odd}}-M_{ij}^{\text{even}}=-u(-p_i,-n_i,\lambda_i)\bar{u}(-p_j,-n_j,\lambda_j)\ , \tag{21}$$
$$M_{ij}^{R}=u(p_j,-n_j,\lambda_j)\bar{u}(p_i,-n_i,\lambda_i)\ .$$

Also

$$M=2\,\text{tr}(M_{14}^{\text{odd}}+M_{14}^{\text{odd},R})M_{23}^{\text{odd}}$$
$$=\tfrac{1}{2}(\bar{u}_3u_1\bar{u}_4u_2-u_3u_1'''\bar{u}_4'''u_2+\bar{u}_3u_4''\bar{u}_1''u_2-\bar{u}_3u_4'\bar{u}_1'u_2-\bar{u}_3'''u_1\bar{u}_4u_2'''+\bar{u}_3'''u_1'''\bar{u}_4'''u_2'''-\bar{u}_3'''u_4''\bar{u}_1''u_2'''+\bar{u}_3'''u_4'\bar{u}_1'u_2''')\ , \tag{22}$$

where $u_i'=u(-p_i,n_i,\lambda_i), u_i''=u(p_i,-n_i,\lambda_i), u_i'''=u(-p_i,-n_i,\lambda_i)$. Using the formalism we are able to derive the expression for $u_\lambda(p)\bar{u}_\lambda(q)$ previously given in the massless case. It turns out that n_μ can be chosen to satisfy $n\cdot p=n\cdot q=0$. Indeed we start by computing $u(p_1,n_1,\lambda_1)\bar{u}(p_2,n_2,\lambda_2)$ with

$$n_{i\mu}=\frac{\cos\psi}{m\beta_i}(\vec{p}_i,i\beta_i^{\,2}E_i)+\sin\psi\, n_\mu\ ,$$
$$n_\mu=(\vec{n},0), \vec{n}\cdot\vec{p}_i=0, \vec{n}^{\,2}=1\ , \tag{23}$$
$$0\le\psi\le\pi/2\ ,\ \ p_i^{\,2}=-m^2\ ,\ \ \beta_i^{\,2}=1-\frac{m^2}{E_i^{\,2}}\ ,$$

for $\psi=0$ this corresponds to longitudinal polarization. As usual

$$u_1\bar{u}_2=\tfrac{1}{2}(E_1E_2)^{-1/2}(n_+^{\ -1/2}\Lambda^++n_-^{\ -1/2}\Lambda^-)(-i\not{p}_1+m)\tfrac{1}{2}(1+i\lambda_1\gamma^5\not{n}_1)\tfrac{1}{2}(1+i\lambda_2\gamma^5\not{n}_2)(-i\not{p}_2+m)\ . \tag{24}$$

When $\lambda_1=\lambda_2=\lambda$ we find $n_+=-2e^{+i\phi}\sin^2\psi\, p_1\cdot p_2+O(m)$. Thus

$$u(p_1,n_1,\lambda)\bar{u}(p_2,n_2,\lambda)=e^{i\Phi}\frac{1}{4\sqrt{2}}(E_1E_2\,p_1\cdot p_2)^{-1/2}\not{p}_1[\sin\psi+i(\cos\psi+\lambda\gamma^5)\not{n}]\not{p}_2+O(m)\ . \tag{25}$$

The previous formula and Eq. (13) form the base for computing high-energy cross sections for particles with an arbitrary degree of transverse polarization. In the limit $\psi=0$, $m=0$

$$u_\lambda(p_1)\bar{u}_\lambda(p_2)=ie^{i\Phi}\frac{1}{2\sqrt{2}}(E_1E_2\,p_1\cdot p_2)^{-1/2}\not{p}_1\not{n}\not{p}_2\pi_\lambda\ . \tag{26}$$

As a final application we consider the bremsstrahlung amplitude for $e^+e^-\to F^+F^-\gamma$ where the F-mass effects are explicitly taken into account. Moreover we only include final-states radiation, simulating in this way a QCD three-jet cross section for heavy quarks. The amplitude is

$$M=\bar{u}(p_3,n_3,\lambda_3)[\gamma^\mu\Delta(p_4+k)\not{\epsilon}^\sigma+\not{\epsilon}^\sigma\Delta(-p_3-k)\gamma^\mu]v(p_4,n_4,\lambda_4)\bar{v}(p_1,\lambda_1)\gamma^\mu u(p_2,\lambda_2)$$
$$\Delta^{-1}(p)=-i\not{p}+m\ , \tag{27}$$

where factors due to couplings and an overall s^{-1} from the photon propagator have been omitted. Using[1]

$$\epsilon_\mu^\sigma=\frac{1}{\sqrt{2}N}(p_4\cdot k\,p_{3\mu}-p_3\cdot k\,p_{4\mu}-\sigma\epsilon_{\mu\nu\alpha\beta}p_4^\nu p_3^\alpha k^\beta)\ ,$$
$$\not{\epsilon}^\sigma=\frac{1}{\sqrt{2}N}(\not{p}_3\not{p}_4\not{k}\pi_\sigma-\not{k}\not{p}_3\not{p}_4\pi_{-\sigma})\ ,\ \ \pi_\sigma=\tfrac{1}{2}(1+\sigma\gamma^5)\ , \tag{28}$$

we find

$$M=\frac{i}{\sqrt{2}N}\bar{u}_3\gamma^\mu\left[p_3\cdot p_4+m^2\frac{p_3\cdot k}{p_4\cdot k}+im\frac{p_3\cdot k}{p_4\cdot k}\not{k}\pi_{-\sigma}+\not{k}\not{p}_3\pi_\sigma\right]v_4\bar{v}_1\gamma^\mu u_2$$
$$+\frac{i}{\sqrt{2}N}\bar{u}_3\left[p_3\cdot p_4+m^2\frac{p_4\cdot k}{p_3\cdot k}-im\frac{p_4\cdot k}{p_3\cdot k}\pi_{-\sigma}\not{k}+\pi_\sigma\not{p}_4\not{k}\right]\gamma^\mu v_4\bar{v}_1\gamma^\mu u_2\ . \tag{29}$$

Next we use

$$v_4\bar{v}_1=\tfrac{1}{4}N_{14}(\not{p}_4-im)(1-i\lambda_1\lambda_4\not{n}_4)\not{p}_1\pi_{-\lambda_1}\,,$$
$$u_2\bar{u}_3=\tfrac{1}{4}N_{23}\pi_{-\lambda_2}\not{p}_2(1+i\lambda_2\lambda_3\not{n}_3)(\not{p}_3+im)\,, \tag{30}$$
$$N_{if}=(mp_i\cdot n_f+p_i\cdot p_f)^{-1/2}\Lambda^-(\lambda_i,\lambda_f)+(-mp_i\cdot n_f+p_i\cdot p_f)^{-1/2}\Lambda^+(\lambda_i,\lambda_f)$$
$$=\sum_{k=\pm}N_{ifk}\Lambda^k(\lambda_i,\lambda_f)\,.$$

Thus $M=(i/16\sqrt{2}N)N_{14}N_{23}\,\mathrm{tr}T$. After applying Chisholm identities and rearranging the terms in the trace we find

$$T=\Lambda^-(\lambda_1,\lambda_2)t\,,$$
$$t=\sum_{ijk=\pm}\mathrm{tr}D_{ijk}\Lambda^i(\sigma,\lambda_1)\Lambda^j(\lambda_1,\lambda_4)\Lambda^k(\lambda_2,\lambda_3)\,. \tag{31}$$

The Λ are projectors and the different terms in the sum never interfere in the cross section. However the matrices D for arbitrary polarizations contain up to a maximum of eight γ matrices and therefore their trace is cumbersome but clearly not impossible (compare with the standard procedure)

$$D_{ijk}=A_i\pi(i\lambda_1)B_{ij}C_k+A_{-i}\pi(-i\lambda_1)B_{-ij}C_k+A_i'C_j'B_{ik}'\pi(i\lambda_1)+A_{-i}'C_j'B_{-ik}'\pi(-i\lambda_1)\,,\quad i,j,k=+,-\,, \tag{32}$$

$$A_+=A+\not{k}\not{p}_3\,,\quad A_-=A+im\frac{p_3\cdot k}{p_4\cdot k}\not{k}\,,\quad A=p_3\cdot p_4+m^2\frac{p_3\cdot k}{p_4\cdot k}\,,$$
$$A_+'=A'+\not{p}_4\not{k}\,,\quad A_-'=A'-im\frac{p_4\cdot k}{p_3\cdot k}\not{k}\,,\quad A'=p_3\cdot p_4+m^2\frac{p_4\cdot k}{p_3\cdot k}\,,$$
$$B_{+\pm}=\not{p}_4\not{p}_1\mp m\not{n}_4\not{p}_1\,,\quad B_{-\pm}=-im\not{p}_1\mp i\not{p}_4\not{n}_4\not{p}_1\,, \tag{33}$$
$$C_\pm=p_2\cdot p_3\mp mp_2\cdot n_3-2im\not{p}_2\mp 2i\not{p}_3\not{n}_3\not{p}_2\,,$$
$$C_\pm'=p_1\cdot p_4\mp mp_1\cdot n_4+2im\not{p}_1\pm 2i\not{p}_1\not{n}_4\not{p}_4\,,$$
$$B_{+\pm}'=im\not{p}_2\pm i\not{p}_2\not{n}_3\not{p}_3\,,\quad B_{-\pm}'=\not{p}_2\not{p}_3\mp m\not{p}_2\not{n}_3\,.$$

Finally

$$M=\frac{i}{16\sqrt{2}N}\Lambda^-(\lambda_1,\lambda_2)\sum_{ijk=\pm}N_{14j}N_{23k}\,\mathrm{tr}D_{ijk}\Lambda^i(\sigma,\lambda_1)\Lambda^j(\lambda_1,\lambda_4)\Lambda^k(\lambda_2,\lambda_3)\,. \tag{34}$$

In our opinion the present method has many advantages, some of which are, however, more conceptual than practical. The use of the projection operators, introduced in this context by Caffo and Remiddi in Ref. 2, allows us to extract automatically the helicity amplitudes, which can be written without internal vector and axial-vector couplings. The result is a highly compact expression even for massive particles and arbitrary polarization vectors.

The physics of polarized beams in e^+e^- colliders can be fully analyzed in a covariant way, while complicated processes such as π-π scattering where everything is massless and each quark is assumed to be collinear with the hadron are perhaps better computed by other methods, such as those of Farrar and Neri.

In this respect one should make a distinction between methods where the study of polarization effects is the main topic and methods to compute the cross section for particular processes by using the best strategy. What we have discussed in this paper belongs more to the first category but we feel that it is also very useful when viewed as a short way for performing complicated computations, especially in the massless case used in conjunction with an algebraic program as SCHOONSCHIP.

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