PHYSICAL REVIEW D VOLUME 28, NUMBER 11 1 DECEMBER 1983

Phase transition and density of states in the quantum-chromodynamic bag model

R. Gagnon
Département de Physique-Mathématiques, Université de Moncton, Centre Universitaire de Moncton, Moncton, Nouveau-Brunswick, Canada E1A 3E9

(Received 6 June 1983; revised manuscript received 29 August 1983)

In the context of a two-phase model of hadronic matter suggested by Hagedorn and Rafelski, an analysis of a quark-gluon plasma described by perturbative quantum chromodynamics supplemented by a bag constant to account for nonperturbative effects is presented. A running coupling constant is explicitly implemented. The critical temperature and energy density are determined and the conditions under which $T_c < B^{1/4}$ are discussed. Next, the asymptotic density of hadronic states is derived by Laplace-inverting the partition function and the same form as in the statistical bootstrap model is obtained.

I. INTRODUCTION

It is widely accepted by now that a phase transition from hadronic matter to a quark-gluon plasma should occur at high enough temperature and/or density.[1] Lattice calculations of Yang-Mills theory suggest that deconfinement will take place at a critical temperature of approximately 150—225 MeV.[2–4] This is so far the only method to treat all phases within the same theory and a fully satisfactory lattice description of QCD with fermions has yet to come. Alternatively, two-phase approaches rely on a different description for each phase and usually critical quantities are determined by matching the pressures. For instance, in this manner, the quark-gluon plasma can be described by first-order perturbative QCD at finite temperature supplemented by a bag constant to account for nonperturbative effects. Results of lattice calculations[2] agree very well with such a description above the critical temperature, suggesting large cancellations among higher-order terms. To describe the hadron phase, the statistical bootstrap of Hagedorn[5] and Frautschi[6] is a natural candidate. Indeed, based on this idea, several models of hadronic reactions have been developed which account for many features of the data.[7–9] It is well known that in this case, the hadronic density of states is given by[10]

$$\rho(M) \propto \frac{e^{M/T_0}}{M^3} . \tag{1}$$

In this expression, T_0 (~ 160 MeV) should not be interpreted as the limiting temperature of the Universe, as first suggested,[5] but more likely as the critical temperature of a phase transition.[11]

Along these lines, a model of strongly interacting matter was developed by Hagedorn and Rafelski[12,13] in which the matching of the two phases occurs when the pressure vanishes in each phase. In particular, in the quark-gluon plasma the critical temperature is determined by the highest value of T on the $P(\mu,T)=0$ curve (μ being the chemical potential). We note that according to the bag model,[14] the plasma will then be at equilibrium, i.e., the pressure exerted by the constituents in the bag will be balanced by the pressure of the vacuum associated with the bag parameter. An estimate of the critical temperature was done[13] ($T_c \sim 160$ MeV assuming $B^{1/4}=190$ MeV) using a fixed value of the coupling constant ($\alpha_s = \frac{1}{2}$).

In this paper, we analyze in Sec. II the quark-gluon phase using explicitly a running coupling constant and determine the critical temperature and the resulting energy density for different flavor numbers and different bag parameters. We discuss the conditions of applicability of the model and the conditions under which the critical temperature will be smaller than the bag parameter. In Sec. III, we derive the asymptotic density of hadronic states using a saddle-point method and obtain an expression of the form found in the statistical bootstrap model [Eq. (1)]. This was already known in the case of an Abelian bag model[15,16] and of interacting fields in a bag with a fixed coupling constant.[17] We show that it also holds for perturbative QCD in a bag with a running coupling constant. In addition, we find that since the matching of the two phases is done at a vanishing pressure, the temperature T_0 in the density of states is the same as the critical temperature T_c of the phase transition. Section IV contains a summary and our conclusions.

II. THE PHASE TRANSITION

For simplicity, we assume zero-mass quarks and start from the thermodynamical potential Ω to first order in α_s in the QCD bag model.[1,18]:

$$\frac{\Omega}{V} = -\left[\frac{8\pi^2}{45}+\frac{7\pi^2}{60}n_f\right]\frac{1}{\beta^4} - \sum_f \left[\frac{1}{4\pi^2}\mu^4 + \frac{1}{2}\frac{\mu_f^{\,2}}{\beta^2}\right] + g^2\left[\left(\tfrac{1}{6}+\tfrac{5}{72}n_f\right)\frac{1}{\beta^4} + \frac{1}{8}\sum_f\left[\frac{1}{\pi^4}\mu_f^{\,4} + \frac{2}{\pi^2}\frac{\mu_f^{\,2}}{\beta^2}\right]\right] + B , \tag{2}$$

where V is the volume of the system, n_f the number of quark flavors, μ_f the chemical potential of quarks of flavor f, B the bag constant, and β the inverse temperature. The running coupling constant as obtained from the renormalization group takes the form[2,19]

$$g^2 \to g^2(\beta) = \frac{-24\pi^2}{(33-2n_f)\ln(\beta\Lambda/4)} , \quad (3)$$

where Λ fixes the scale. Defining

$$A \equiv (\tfrac{8}{3} + \tfrac{7}{4} n_f)\frac{\pi^2}{15} \quad (4)$$

and

$$k \equiv \frac{4\pi^2}{(33-2n_f)}(1+\tfrac{5}{12}n_f) , \quad (5)$$

and since the highest critical temperature is obtained in the absence of valence quarks, setting all chemical potentials to zero, Eq. (2) reduces to

$$\frac{\Omega}{V} = -\left[A + \frac{k}{\ln(\beta\Lambda/4)}\right]\frac{1}{\beta^4} + B . \quad (6)$$

The equation of state is then obtained from

$$P = -\frac{\Omega}{V} , \quad (7)$$

and from

$$E = \Omega + \beta\frac{\partial\Omega}{\partial\beta}$$
$$= \frac{V}{\beta^4}\left\{3\left[A + \frac{k}{\ln(\beta\Lambda/4)}\right] + \frac{k}{\ln^2(\beta\Lambda/4)}\right\} + BV , \quad (8)$$

where P and E are, respectively, the pressure and energy of the system. We have

$$P = \frac{1}{3}\left[\frac{E}{V} - 4B - \frac{k}{\beta^4 \ln^2(\beta\Lambda/4)}\right] . \quad (9)$$

The result thus depends (though only slightly as we shall see below) on the temperature in contrast to the fixed coupling case. At the transition point, the pressure vanishes and therefore the critical temperature ($T_c \equiv 1/\beta_c$) must satisfy

$$T_c{}^4\left[A - \frac{k}{\ln(4T_c/\Lambda)}\right] = B . \quad (10)$$

It is clear, however, that for such a description to make sense, the quantity in large brackets must be positive. This puts a lower bound on the critical temperature and we must have

$$T_c > \tfrac{1}{4} e^{k/A}\Lambda . \quad (11)$$

At the same time, in order for perturbation theory to be applicable, one also requires $\alpha_s < 1$ (obviously this condition could be made more stringent) which also puts a lower limit on T_c:

$$T_c > \tfrac{1}{4}\left\{\exp\left[\frac{6\pi}{(33-2n_f)}\right]\right\}\Lambda . \quad (12)$$

Which condition is the most severe depends on the number of flavors as can be seen from Table I, but clearly in all cases, $T_c \gtrsim 0.5\Lambda$. Therefore, if the critical temperature does not turn out to be reasonably greater than half the scale parameter, the whole scheme should be abandoned.

It is also of interest to know whether or not the critical temperature is smaller than the bag parameter $B^{1/4}$. It will in fact be the case if, as given by Eq. (10),

$$\Lambda < 4e^{-k/(A-1)}B^{1/4} . \quad (13)$$

Numerical values of $(\Lambda/B^{1/4})_{\min}$ are listed also in Table I. Taking into account both constraints, i.e., (11) or (12) together with (13), one can conclude that T_c must be smaller than $B^{1/4}$ in the four-flavor case (more precisely provided $\alpha_s < 0.96$) and very likely also in the two- and three-flavor cases. Indeed, one only has to require $\alpha_s < 0.9$ if $n_f = 3$ and $\alpha_s < 0.8$ if $n_f = 2$. However, constraints on the pure gluon plasma are not strong enough to permit a definite conclusion. Therefore, assuming Λ to be flavor independent, one obtains $\Lambda \lesssim 2B^{1/4}$.

We can now proceed to the solution of Eq. (10) to determine the critical temperature. Owing to the fact that the logarithm is a slowly varying function of its argument, a small number of iterations using

$$T_c^{(n+1)} = \frac{B^{1/4}}{[A - k/\ln(4T_c^{(n)}/\Lambda)]^{1/4}} , \quad (14)$$

together with $T_c^{(0)} = B^{1/4}$ produces an accurate answer. Here Λ is taken to be 100 MeV as this is the value which best fits the Monte Carlo lattice calculations.[2] The bag parameter is not well known and we give results for three different choices: the usual $B^{1/4} = 145$ MeV from the original MIT bag model,[20] $B^{1/4} = 190$ MeV, the seemingly most generally accepted value as suggested by Satz[2] from

TABLE I. Minimum value of T_c/Λ and of $\Lambda/B^{1/4}$ for different numbers of quark flavors.

Number of flavors	$T_c^{\min}/\Lambda$ From Eq. (11)	$T_c^{\min}/\Lambda$ From Eq. (12)	$(\Lambda/B^{1/4})_{\min}$
0	0.49	0.44	0.82
2	0.46	0.48	1.77
3	0.47	0.50	1.83
4	0.48	0.53	1.82

TABLE II. Critical temperature, energy density, and coupling constant for different numbers of quark flavors and bag parameters.

$B^{1/4}$ Number of flavors	145 MeV $T_c/B^{1/4}$ T_c (MeV)	$\epsilon_c/4B$ ϵ_c (GeV/fm^3)	α_s	190 MeV $T_c/B^{1/4}$ T_c (MeV)	$\epsilon_c/4B$ ϵ_c (GeV/fm^3)	α_s	235 MeV $T_c/B^{1/4}$ T_c (MeV)	$\epsilon_c/4B$ ϵ_c (GeV/fm^3)	α_s
0	0.99	1.09	0.33	0.96	1.06	0.29	0.95	1.07	0.26
	143	0.25		183	0.72		224	1.67	
2	0.80	1.13	0.42	0.78	1.07	0.36	0.77	1.08	0.33
	116	0.26		149	0.73		182	1.68	
3	0.76	1.13	0.47	0.74	1.09	0.40	0.73	1.08	0.36
	110	0.26		141	0.74		172	1.69	
4	0.73	1.13	0.52	0.71	1.09	0.45	0.70	1.09	0.40
	106	0.26		135	0.74		165	1.70	

lattice calculations, and $B^{1/4}=235$ MeV from spectroscopic studies.[21]

In Table II, we present the resulting values of the critical temperature, the critical energy density ϵ_c, and of the running coupling constant evaluated at the transition point. The following observations can be made.

(1) For each number of flavors, the dependence on the bag parameter of the critical temperature is almost linear. The slope is approximately 0.75 except in the pure gluon case where it is almost unity. This difference is explained by the somewhat lower value of α_s in the absence of quarks. The actual values of the critical temperature agree with lattice calculations[2–4] and other estimates[13,22] for $B^{1/4}=190$ or 235 MeV, but not for $B^{1/4}=145$ MeV which seems really to low. In the three-flavor case, the statistical-bootstrap value of 160 MeV would result with $B^{1/4}=218$ MeV.

(2) Within 10%, $\epsilon_c=4B$ as used by Hagedorn and Rafelski.[12] This is because, as can be seen from Eqs. (8) and (10), the second term in its expression is proportional to $\alpha_s{}^2$ which is small. For $B^{1/4}=190$ MeV, one obtains $\epsilon_c \simeq 0.73$ GeV/fm^3 or about five times the normal nuclear density. This is in reasonable agreement with the lattice estimate of Ref. 2 though a little higher and well within previous estimates.[12,13,18,22,23] Obviously, from an experimental point of view, a higher value of the bag parameter would have dramatic consequences.

(3) At the critical point, the value of the coupling constant is small enough to ensure a good convergence of the perturbation expansion above the phase transition.

(4) Finally, one notes that the value of Λ that we used fulfills the condition (13) in the pure gluon case, also resulting in a critical temperature smaller than $B^{1/4}$.

III. THE DENSITY OF STATES

We now address the problem of determining the asymptotic density of states in this model. These are actually *hadronic states* of a bag of volume V containing our quark-gluon plasma. We use the fact that it can be obtained by Laplace-inverting the partition function. In order to exclude the spurious excitations of the center-of-mass system, we write (following Ref. 24)

$$Z(\beta,\vec{Q})=\frac{V}{(2\pi)^3}\int dE\, d^3P\, \rho(E,\vec{P})e^{-\beta E-\vec{P}\cdot\vec{Q}}\ , \quad (15)$$

where $\rho(E,\vec{P})$ is the number of states per unit of energy at a given total momentum $\vec{P}$. Under Laplace inversion, one has

$$\rho(E,\vec{P})=\frac{(2\pi)^3}{V}\frac{1}{(2\pi i)^4}\int_{c-i\infty}^{c+i\infty} d\beta\, d^3Q\, e^{f(\beta,\vec{Q})}\ , \quad (16)$$

where we have defined

$$f(\beta,\vec{Q})=\beta E+\vec{P}\cdot\vec{Q}+\ln Z(\beta,\vec{Q})\ . \quad (17)$$

To evaluate Eq. (16), we use the saddle-point method which approximates the integral by its main contribution, i.e., in a region where $f(\beta,\vec{Q})$ takes its maximum value. The well-known procedure[24] then leads to the result

$$\rho(E,\vec{P})\cong\frac{2\pi}{V}\frac{e^{f(\beta_0,Q_0)}}{|\det|^{1/2}}\ , \quad (18)$$

where $(\beta_0,\vec{Q}_0)$ is the saddle point determined by

$$\frac{\partial f}{\partial\beta}=0 \text{ and } \vec{\nabla}_Q f=0\ , \quad (19)$$

and det is the four-by-four determinant of the matrix made of the second-order derivative of f with respect to β and Q_j and evaluated at the saddle point.

To proceed, one needs $\ln Z(\beta,\vec{Q})=-\beta\Omega$. Following Kapusta,[17] one has from Eq. (2)

$$\ln Z(\beta,\vec{Q})=\left[\left[\frac{8\pi^2}{45}-\frac{g^2}{6}\right]+\left[\frac{7\pi^2}{60}-\frac{5}{72}g^2\right]n_f\right]\frac{\beta V}{(\beta^2-Q^2)^2}-\beta BV=\left[A+\frac{k}{\ln(\beta\Lambda/4)}\right]\frac{\beta V}{(\beta^2-Q^2)^2}-\beta BV\ . \quad (20)$$

Using (17), (19), and (20) to determine the saddle point, we obtain

$$\vec{\nabla}_Q f=\vec{P}+\left[A+\frac{k}{\ln(\beta\Lambda/4)}\right]\frac{4\beta V\vec{Q}}{(\beta^2-Q^2)^2}=0\ , \tag{21}$$

and since $\vec{P}=0$, we also have $\vec{Q}=0$. Therefore, as one readily verifies, all thermodynamical quantities derived in Sec. II remain valid at the saddle point. Moreover,

$$\begin{aligned}\frac{\partial f}{\partial\beta}&=(E-BV)+\left\{\left[A+\frac{k}{\ln(\beta\Lambda/4)}\right]\left[1-\frac{4\beta^2}{\beta^2-Q^2}\right]-\frac{k}{\ln^2(\beta\Lambda/4)}\right\}\frac{V}{(\beta^2-Q^2)^2}\\&=(E-BV)-\frac{V}{\beta^4}\left\{3\left[A+\frac{k}{\ln(\beta\Lambda/4)}\right]-\frac{1}{\ln^2(\beta\Lambda/4)}\right\}=0\ ,\end{aligned} \tag{22}$$

which is the same as Eq. (8). One thus anticipates that the saddle point β_0 will have to satisfy condition (10). Indeed, at equilibrium the pressure vanishes and using (9), (22) becomes

$$\frac{1}{{\beta_0}^4}\left[A+\frac{k}{\ln(\beta_0\Lambda/4)}\right]=B\ . \tag{23}$$

Therefore, we obtain

$$f(\beta_0,\vec{Q}_0)=\beta_0E+\left[A+\frac{k}{\ln(\beta_0\Lambda/4)}\right]\frac{V}{{\beta_0}^3}-\beta_0BV=\beta_0E\ . \tag{24}$$

The determinant is easily calculated and is given by

$$\det=768\frac{\beta^3V^4}{{\beta_0}^8}R_0\ , \tag{25}$$

where, by definition,

$$\begin{aligned}R_0&\equiv A+\frac{k}{\ln(\beta_0\Lambda/4)}\left[1+\frac{7}{12\ln(\beta_0\Lambda/4)}+\frac{1}{6\ln^2(\beta_0\Lambda/4)}\right]\\&=\frac{8\pi^2}{45}+\frac{7\pi^2}{60}n_f-\tfrac{1}{6}g^2(\beta_0)(1+\tfrac{5}{12}n_f)\left[1+\frac{7}{12\ln(\beta_0\Lambda/4)}+\frac{1}{6\ln^2(\beta_0\Lambda/4)}\right]\ .\end{aligned} \tag{26}$$

From (9), the volume at equilibrium is

$$V=\frac{E}{[4B+k/{\beta_0}^4\ln^2(\beta_0\Lambda/4)]}\ . \tag{27}$$

Collecting terms, we have for the density of states

$$\rho(E)\cong a\frac{e^{\beta_0E}}{E^3}\ , \tag{28}$$

where the coefficient is given by

$$a\equiv\frac{\pi}{8\sqrt{3}}{\beta_0}^4\frac{\{4B+k/[{\beta_0}^4\ln^2(\beta_0\Lambda/4)]\}^3}{B^{3/2}{R_0}^{1/2}}\ . \tag{29}$$

Therefore, as in the case of the Abelian bag model[15,16] or of a bag model with a fixed coupling constant,[17] we get a density of states of the form obtained in the statistical bootstrap model [Eq. (1)]. We already noted that the equations determining β_0 and $1/T_c$ are exactly the same and therefore of course $\beta_0=1/T_c$. This is due to the fact that the condition of equilibrium of the bag, i.e., the pressure of the constituents equals the "external" pressure, is the same as the condition determining the matching of the two phases of the model, i.e., $P(\mu,T)=0$. Moreover, it is clear that in addition to changing the actual value of β_0, the only effect of the coupling constant being running is to modify slightly the coefficient. Indeed, using the values of Table II, we estimate the numerical error of neglecting terms of $O(1/\ln^2(\beta_0\Lambda/4))$ to 10–20%. It is easy to show that the coefficient then reduces to

$$a\cong\frac{8\pi}{\sqrt{3}}B{\beta_0}^2\ , \tag{30}$$

which is the exact result when the coupling constant is not running.[17]

IV. CONCLUSION

In this paper, a quark-gluon plasma described by perturbative QCD at finite temperature supplemented by a bag constant to account for nonperturbative effects has been studied. The critical temperature of a phase transition from hadronic matter to this quark-gluon plasma was determined for different numbers of flavors and different bag parameters. For simplicity, all quarks were assumed massless but a running coupling constant was implemented. After a short discussion on the applicability of the model, it was concluded that the critical temperature would be smaller than the bag parameter when the number of flavors is two or more and that the scale-fixing parameter Λ should be smaller than $\sim 2\ B^{1/4}$. Using $\Lambda = 100$ MeV, it was found that in the pure gluon case, $T_c/B^{1/4}$ was close to unity but otherwise was approximately 0.75. This is a little lower than the value of 0.83 obtained by Rafelski.[13] Also, the value of 145 MeV for the bag parameter appears to be too low as it leads to really low critical temperatures. A comparison with other estimates[2-4,13] favors a value of approximately 200 MeV. The resulting critical energy density was in all cases a little over the unperturbed result of $4B$. Indeed, the coupling constant α_s turned out to be relatively small.

Next, the asymptotic density of states was derived by Laplace-inverting the partition function. The center-of-mass degrees of freedom were explicitly excluded through the constraint $\vec{P}_{\rm c.m.} = 0$. A form identical to that found in the statistical bootstrap model was obtained. As in the case of a fixed coupling,[17] the running coupling constant affects only the coefficient a and the actual value of β_0. However, one should realize that as in other treatments, the plasma states were not required to be color singlets. Therefore, additional states have been included in the density and this should affect the pre-exponential power and the coefficient but not the exponent.[17,25] In this connection, some work has already been done by Gorenstein and collaborators.[26] On another hand, it has been shown by Jennings and Bhaduri[16] that accounting for the finite size of the bag in a noninteracting gluon gas leads to a modification of the exponent itself. In fact, it is quite difficult to take into account all relevant aspects at once and since (it is hoped) different approaches complement each other, our purpose here was mainly to study the consequences of implementing a running coupling constant.

[1]See, for instance, E. V. Shuryak, Phys. Rep. 61, 71 (1980); *Quark Matter Formation and Heavy Ion Collisions,* proceedings of the XVIII Rencontre de Morrond, Les Arcs, France, 1982, M. Jacob and J. Trần Thanh Vân [Phys. Rep. 88, 321 (1982)].

[2]H. Satz, Phys. Lett. 113B, 245 (1982); Phys. Rep. 88, 349 (1982).

[3]K. Kajantie, C. Montonen, and E. Pietarinen, Z. Phys. C 9, 253 (1981).

[4]I. Montvay and E. Pietarinen, Phys. Lett. 110B, 148 (1982); 115B, 151 (1982).

[5]R. Hagedorn, Nuovo Cimento Suppl. 3, 147 (1965).

[6]S. C. Frautschi, Phys. Rev. D 3, 2821 (1971).

[7]R. Hagedorn and J. Ranft, Nuovo Cimento Suppl. 6, 169 (1968).

[8]F. Johns *et al.*, Phys. Rev. Lett. 29, 756 (1972).

[9]R. Gagnon, Phys. Rev. D 22, 616 (1980).

[10]C. J. Hamer and S. C. Frautschi, Phys. Rev. D 4, 2125 (1971).

[11]N. Cabibbo and G. Parisi, Phys. Lett. 59B, 67 (1975).

[12]R. Hagedorn and J. Rafelski, Phys. Lett. 97B, 180 (1980); in *Statistical Mechanics of Quarks and Hadrons,* proceedings of the International Symposium, University of Bielefeld, Federal Republic of Germany, 1980, edited by H. Satz (North-Holland, Amsterdam, 1981), p. 237.

[13]J. Rafelski, Phys. Rep. 88, 331 (1982).

[14]A. Chodos *et al.*, Phys. Rev. D 9, 3471 (1974).

[15]J. I. Kapusta, Phys. Rev. D 23, 2444 (1981).

[16]B. K. Jennings and R. K. Bhaduri, Phys. Rev. D 26, 1750 (1982).

[17]J. Kapusta, Nucl. Phys. B196, 1 (1982).

[18]S. A. Chin, Phys. Lett. 78B, 552 (1978).

[19]J. I. Kapusta, Nucl. Phys. B148, 461 (1979).

[20]T. DeGrand *et al.*, Phys. Rev. D 12, 2060 (1975).

[21]P. Hasenfratz *et al.*, Phys. Lett. 95B, 299 (1980).

[22]K. Kajantie and H. I. Miettinen, Z. Phys. C 9, 341 (1981).

[23]R. Gagnon, Can. J. Phys. 60, 797 (1982).

[24]B. K. Jennings, L. Satpathy, and S. Das Gupta, Phys. Rev. C 24, 440 (1981).

[25]M. I. Gorenstein *et al.*, Theor. Math. Phys. 52, 843 (1983).

[26]M. I. Gorenstein *et al.*, Z. Phys. C. 18, 13 (1983).