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Boundary conditions for the photon structure function in the leading and subleading order

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The importance of boundary conditions (at $Q^2=Q_0{}^2$) inherent in any general solution of the inhomogeneous photon evolution equations is stressed. Their inclusion is crucial at present energies, especially in connection with investigations concerning the relative importance of subleading corrections at small values of the scaling variable x. The framework for a direct x-variable analysis is formulated explicitly.

I. INTRODUCTION

Quantum-chromodynamic (QCD) corrections to the naive pointlike structure of the photon[1] were first studied by Witten[2] who, however, utilized the formal language of Wilson's operator-product expansion.[3] This language was adopted by Bardeen and Buras[4] in their study of subleading QCD corrections. An equivalent way of presentation is the one[5–7] utilizing the more explicit and transparent parton language which will be henceforth adopted by us. In this formulation the quarks and gluons in the photon satisfy an evolution equation similar to the Altarelli-Parisi equation with, however, additional inhomogeneous terms due to the practically constant (to the order α considered) photon component.

In contrast to the homogeneous Altarelli-Parisi equation this inhomogeneous term results in a nonvanishing quark distribution even in the Bjorken limit $Q^2\equiv -q^2\rightarrow\infty$ with $x\equiv Q^2/2p\cdot q$ fixed. The quark distribution in this limit is obviously independent of its initial value at some finite $Q_0{}^2$ (boundary condition) and corresponds to the so-called pointlike quark component of the photon. It has become customary to compare this unique and *predictable* limiting distribution with the data. However, at moderate values of Q^2 where present data are taken it is important to note that the quark distribution still depends on the boundary conditions and is therefore not uniquely predicted by the evolution equations alone. In particular for the second moment the pointlike asymptotic limiting solution in subleading order is meaningless and nonunique as $Q^2\rightarrow\infty$, making a specification of the boundary conditions mandatory. This has caused some confusion[8] in analyzing the significance and magnitude of the nonleading higher-order corrections in the small-x region.[9–11] As we shall see there exists, in fact, no special problem with the small-x region *if* the boundary conditions are treated appropriately, i.e., if the most general solution of the inhomogeneous evolution equations is taken into account and not just the particular (asymptotic) one.

To illustrate these points we introduce in Sec. II the appropriate definitions and notations necessary to discuss the inhomogeneous equation and its general boundary-condition-respecting solution in the Mellin-moment n space. The ensuing solution, though analytically transparent and manageable, entails very complicated n dependencies whose (Mellin) inversion into the experimentally more relevant Bjorken-x space is thus not recommendable for practical purposes. Instead we suggest pursuing the solution of the inhomogeneous evolution equations with their boundary conditions *directly* in x space. The necessary expressions and numerical procedure are introduced in Sec. III. In Sec. IV we finally discuss the merits of this direct approach as well as draw our conclusions.

II. SOLUTION OF THE EVOLUTION EQUATIONS

As in the deep-inelastic lepton-nucleon case it is advantageous to treat separately the nonsinglet (NS) quark sector $q^\gamma_{\rm NS}(x,Q^2)$ and the singlet (S) sector

$$q^\gamma_{\rm S}(x,Q^2)\equiv\begin{bmatrix}\Sigma^\gamma(x,Q^2)\\ G^\gamma(x,Q^2)\end{bmatrix}.$$

Here

$$\Sigma^\gamma(x,Q^2)=\sum_f[q_f^\gamma(x,Q^2)+\bar q_f^\gamma(x,Q^2)]\ ,$$

where f runs over all relevant quark flavors and $G^\gamma(x,Q^2)$ denotes the gluon distribution in the photon. The parton distributions $q_i^\gamma(x,Q^2)$ ($i=$NS,S) in the photon satisfy an inhomogeneous evolution equation[5–7]

$$\frac{dq_i^\gamma(x,Q^2)}{d\ln Q^2}=k_i(x,Q^2)+\int_x^1\frac{dy}{y}P_i\left(\frac{x}{y},Q^2\right)q_i^\gamma(y,Q^2)\ ,\quad (2.1)$$

where for $i=$S, $P_{\rm S}(x,Q^2)$ is the standard two-by-two matrix of quark and gluon splitting functions and $k_s(x,Q^2)$ is obviously a two-component vector like $q^\gamma_{\rm S}(x,Q^2)$. The k_i's refer to the $\gamma\rightarrow$quark and $\gamma\rightarrow$gluon splitting functions to be specified below, i.e., represent the mixing between the photon and hadronic operators. Henceforth we shall suppress the index i unless its specification becomes necessary and our equations for the $i=$S case should be understood as matrix equations. For this reason care is taken everywhere on the *order* of terms written down. Furthermore, for the time being, we shall pass to the Mellin moments

$$q_i^\gamma(n,Q^2)\equiv\int_0^1dx\,x^{n-1}q_i^\gamma(x,Q^2)$$

which simplify Eq. (2.1) to

$$\frac{dq^\gamma(n,Q^2)}{d\ln Q^2}=k(n,Q^2)+P(n,Q^2)q^\gamma(n,Q^2)\ . \tag{2.2}$$

Henceforth we shall also omit for simplicity the obvious n dependence. Expanding in powers of the strong and electromagnetic coupling constants we have

$$\begin{aligned} k(Q^2)&=\frac{\alpha}{2\pi}k^{(0)}+\frac{\alpha\alpha_s(Q^2)}{(2\pi)^2}k^{(1)}+\cdots\ ,\\ P(Q^2)&=\frac{\alpha_s(Q^2)}{2\pi}P^{(0)}+\left[\frac{\alpha_s(Q^2)}{2\pi}\right]^2P^{(1)}+\cdots\ , \end{aligned} \tag{2.3}$$

where $\alpha_s(Q^2)$ satisfies[12]

$$\frac{d\alpha_s(Q^2)}{d\ln Q^2}=-\beta_0\frac{{\alpha_s}^2}{4\pi}-\beta_1\frac{{\alpha_s}^3}{(4\pi)^2}+\cdots \tag{2.4}$$

with $\beta_0=11-2f/3$ and $\beta_1=102-38f/3$. For later purposes it should be noted that the two-component singlet vectors $k^{(0,1)}$ have the following structure:

$$k_{\rm S}^{(0)}=\begin{bmatrix}k_q^{(0)}\\0\end{bmatrix},\quad k_{\rm S}^{(1)}=\begin{bmatrix}k_q^{(1)}\\k_G^{(1)}\end{bmatrix}\ . \tag{2.5}$$

It is now straightforward to solve Eq. (2.2) in leading order:[13]

$$q^\gamma(Q^2)=\frac{4\pi}{\alpha_s(Q^2)}\left[1-\left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{1-2P^{(0)}/\beta_0}\right]a+\left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{-2P^{(0)}/\beta_0}q^\gamma(Q_0^2)\ , \tag{2.6}$$

where[14]

$$a=\frac{\alpha}{2\pi\beta_0}\frac{1}{1-2P^{(0)}/\beta_0}k^{(0)} \tag{2.7}$$

and with $\alpha_s(Q^2)$ evolving according to

$$\frac{d\alpha_s}{d\ln Q^2}=-\beta_0\frac{{\alpha_s}^2}{4\pi}\ , \tag{2.4$'$}$$

i.e., $\alpha_s(Q^2)=4\pi/(\beta_0\ln Q^2/\Lambda^2)$. In the *next-to-leading* order one obtains

$$\begin{aligned} q^\gamma(Q^2)=&\frac{4\pi}{\alpha_s(Q^2)}\left[1-\left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{1-2P^{(0)}/\beta_0}\right]a+\left[1-\left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{-2P^{(0)}/\beta_0}\right]b\\ &+\alpha_s(Q^2)^{-2P^{(0)}/\beta_0}\{1+[\alpha_s(Q^2)-\alpha_s(Q_0^2)]R\}\alpha_s(Q_0^2)^{2P^{(0)}/\beta_0}q^\gamma(Q_0^2) \end{aligned} \tag{2.8}$$

with

$$b=-\frac{1}{P^{(0)}}\left[2\left[P^{(1)}-\frac{\beta_1}{2\beta_0}P^{(0)}\right]a+\frac{\alpha}{2\pi}\left[k^{(1)}-\frac{\beta_1}{2\beta_0}k^{(0)}\right]\right]\ , \tag{2.9}$$

$$R=-\frac{1}{\pi\beta_0}\left[P^{(1)}-\frac{\beta_1}{2\beta_0}P^{(0)}\right]\ , \tag{2.10}$$

and where $\alpha_s(Q^2)$ evolves according to Eq. (2.4), keeping β_0 and β_1, with the approximate solution

$$\frac{4\pi}{\alpha_s(Q^2)}=\beta_0\ln\frac{Q^2}{\Lambda^2}+\frac{\beta_1}{\beta_0}\ln\ln\frac{Q^2}{\Lambda^2}\ . \tag{2.11}$$

Throughout this paper all higher-order results and expressions refer to the modified minimal-subtraction ($\overline{\rm MS}$) scheme and therefore the QCD scale in (2.11) has to be interpreted as $\Lambda_{\overline{\rm MS}}$.

The moments of the measured photon structure function $F_2^\gamma(x,Q^2)$,

$$F_2^\gamma(n,Q^2)\equiv\int_0^1dx\,x^{n-1}\frac{1}{x}F_2^\gamma(x,Q^2)\ ,$$

are given in terms of the distributions obtained in Eqs. (2.6) and (2.8) by

$$F_2^\gamma(n,Q^2)=\delta_{\rm NS}q_{\rm NS}^\gamma(n,Q^2)+\delta_{\rm S}\Sigma^\gamma(n,Q^2) \tag{2.12}$$

in the leading order, and by

$$\begin{aligned} F_2^\gamma(n,Q^2)=&\delta_{\rm NS}\left[1+\frac{\alpha_s(Q^2)}{4\pi}B_q(n)\right]q_{\rm NS}^\gamma(n,Q^2)+\delta_{\rm S}\left\{\left[1+\frac{\alpha_s(Q^2)}{4\pi}B_q(n)\right]\Sigma^\gamma(n,Q^2)+\frac{\alpha_s(Q^2)}{4\pi}B_G(n)G^\gamma(n,Q^2)\right\}\\ &+\delta_\gamma\frac{\alpha}{4\pi}B_\gamma(n) \end{aligned} \tag{2.13}$$

in the next-to-leading order, where[4]

$$\delta_{NS}=1,\ \ \delta_S=\langle e^2\rangle,\ \ \delta_\gamma=3f\langle e^4\rangle\ , \tag{2.14}$$

with

$$\langle e^k\rangle \equiv \frac{1}{f}\sum_f e_q{}^k\ .$$

The (Wilson-) expansion terms $B_q(n)$, $B_G(n)$, and $B_\gamma(n)$ are given explicitly in Ref. 4 where in addition the explicit forms for[13] $k^{(0)}(n)$ are presented and the numerical values for the more complicated quantities $k^{(1)}(n)$ are tabulated. Strictly speaking the $O(\alpha_s)$ terms arising through the combination of Eqs. (2.8) and (2.13) should be omitted here since their complete inclusion affords a treatment of Eq. (2.2) beyond the order here considered.

In the usual discussion and application of Eqs. (2.6) and (2.8) one assumes Q^2 to be so large as to justify the omission of the logarithmically decreasing terms $[\alpha_s(Q^2)/\alpha_s(Q_0{}^2)]^{-2P^{(0)}/\beta_0}$. However, for not too large x (or n) and Q^2 this is obviously unjustified. Furthermore, for $n=2$ this is absolutely *wrong* due to the vanishing of $\det P^{(0)}(n=2)$. As is clear from Eq. (2.9) $b(n)$ has a singularity at $n=2$ in the singlet case due to the last term proportional to $(1/P^{(0)})k^{(0)}$, whereas the term proportional to $k^{(1)}$ is well behaved and uniform due to the fact that $k_q^{(1)}=-k_G^{(1)}$ for $n=2$ in Eq. (2.5). However, as it appears in Eq. (2.8) this singularity of b is canceled by the corresponding zero of the quantity in square brackets multiplying b.[15] A simple and correct treatment of the boundary conditions, inherent in the most general solution of Eq. (2.1), thus automatically avoids the appearance of an apparent problem[8] and obviates the need for its solution[9–11] by some *ad hoc* regularization depending in addition on a nonperturbative and unknown parameter[11] λ.

From our discussion it is clear that the only unknown quantities are the input quark and gluon distributions of the photon at $Q^2=Q_0{}^2$. These can only be determined by *measuring* $F_2^\gamma(x,Q^2)$ at some value (or values) of Q^2 just as done in the analogous case of the hadronic quark and gluon distributions in deep-inelastic lepton-hadron scattering. The theory is then *tested* only by performing *further* measurements at some other values of Q^2 *differing* from the ones used for the determination of the photonic quark and gluon distributions. It is needless to say that the present *restricted* range of available Q^2 does not provide the required testing conditions unless one *assumes* $q_{NS}^\gamma(x,Q_0{}^2)$, $\Sigma^\gamma(x,Q_0{}^2)$, and $G^\gamma(x,Q_0{}^2)$ to be negligible there. For example, vector-meson-dominance (VMD) considerations[7] are usually put forward to argue the negligibility of $q^\gamma(x,Q_0{}^2)$ and $G^\gamma(x,Q_0{}^2)$ at $x\gtrsim 0.4$, $Q_0{}^2\simeq 1$ GeV^2. This, however, should be *tested* without further prejudice according to the above-mentioned procedure.

III. DIRECT EVALUATION OF $F_2^\gamma(x,Q^2)$

In the common approach, $(1/x)F_2^\gamma(x,Q^2)$ is evaluated by first calculating $F_2^\gamma(n,Q^2)$ according to the results in Sec. II for $2\le n\le 20$ and then fitting the parameters of some given form of $(1/x)F_2^\gamma(x,Q^2)$ to the calculated moments $F_2^\gamma(n,Q^2)$. Usually one neglects also all logarithmically decreasing terms. However, even for this simplified calculation one has to deal with rather involved expression in n space, e.g., Eq. (2.9).

For the full, boundary-condition-respecting solutions this calculation is even more involved and in particular results in a Q^2 dependence which is *not* as simply parametrizable as in the so-called pointlike solution.[4] Thus $F_2^\gamma(x,Q^2)$ must be fitted to the corresponding moments at *each* relevant value of Q^2 *separately*. Obviously in this case it is of great advantage to start from Eq. (2.1) directly and solve by iteration from some initial $Q_0{}^2$. This differs from the direct x-space (leading-order) calculation of Ref. 6 where an integral equation in x space has been written down for the function $a(n)$ in Eq. (2.7). In fact, multiplying Eq. (2.7) from the left on both sides by $1-2P^{(0)}/\beta_0$ and taking the inverse Mellin transform of this equation, one immediately obtains the integral equation

$$a(x)=\frac{\alpha}{2\pi\beta_0}k^{(0)}(x)+\frac{2}{\beta_0}\int_x^1\frac{dy}{y}P^{(0)}\left[\frac{x}{y}\right]a(y) \tag{3.1}$$

of Ref. 6. Similarly Eq. (2.9) is thus equivalent to

$$-\int_x^1\frac{dy}{y}P^{(0)}\left[\frac{x}{y}\right]b(y) =\frac{\alpha}{2\pi}\left[k^{(1)}(x)-\frac{\beta_1}{2\beta_0}k^{(0)}(x)\right] +2\int_x^1\frac{dy}{y}\left[P^{(1)}\left[\frac{x}{y}\right]-\frac{\beta_1}{2\beta_0}P^{(0)}\left[\frac{x}{y}\right]\right]a(y)\ . \tag{3.2}$$

With $a(y)$ as obtained from Eq. (3.1) one could similarly *directly* solve Eq. (3.2) for $b(y)$ thus avoiding the intermediate use of Eq. (2.9) in n space and its subsequent inversion either by fitting or by explicit Mellin integration in n space. However, the determination of $a(x)$ and $b(x)$ is useful *only* when all logarithmically decreasing terms $[\alpha_s(Q^2)/\alpha_s(Q_0{}^2)]^{-2P^{(0)}/\beta_0}$ are neglected as commonly done.[4,6,8,11] Otherwise the $[\alpha_s(Q^2)/\alpha_s(Q_0{}^2)]^{-2P^{(0)}/\beta_0}$ term itself must be Mellin inverted yielding a complicated x- *and* Q^2-dependent function which must then furthermore be convoluted with the above-mentioned $a(x)$ and $b(x)$ according to Eqs. (2.6) and (2.8).

Obviously this procedure is not too practical or advantageous as compared to a direct numerical integration of the evolution equations (2.1) in Bjorken-x space which more explicitly read

$$\frac{dq_{NS}^\gamma(x,Q^2)}{d\ln Q^2}=\frac{\alpha}{2\pi}\left[k_{NS}^{(0)}(x)+\frac{\alpha_s}{2\pi}k_{NS}^{(1)}(x)\right] +\frac{\alpha_s}{2\pi}\left[P_{qq}^{(0)}+\frac{\alpha_s}{2\pi}P_{NS}^{(1)}\right]*q_{NS}^\gamma \tag{3.3}$$

for the nonsinglet case, whereas for the singlet photonic parton distributions they result in a coupled set of evolution equations:

$$\frac{d\Sigma^{\gamma}(x,Q^2)}{d\ln Q^2}=\frac{\alpha}{2\pi}\left[k_q^{(0)}(x)+\frac{\alpha_s}{2\pi}k_q^{(1)}(x)\right]+\frac{\alpha_s}{2\pi}\left[P_{qq}^{(0)}+\frac{\alpha_s}{2\pi}P_{qq}^{(1)}\right]*\Sigma^{\gamma}+\frac{\alpha_s}{2\pi}\left[P_{qg}^{(0)}+\frac{\alpha_s}{2\pi}P_{qg}^{(1)}\right]*G^{\gamma}\,,$$
$$\frac{dG^{\gamma}(x,Q^2)}{d\ln Q^2}=\frac{\alpha\alpha_s}{(2\pi)^2}k_G^{(1)}(x)+\frac{\alpha_s}{2\pi}\left[P_{gq}^{(0)}+\frac{\alpha_s}{2\pi}P_{gq}^{(1)}\right]*\Sigma^{\gamma}+\frac{\alpha_s}{2\pi}\left[P_{gg}^{(0)}+\frac{\alpha_s}{2\pi}P_{gg}^{(1)}\right]*G^{\gamma}\,, \tag{3.4}$$

with the convolutions defined by

$$P*q\equiv\int_x^1\frac{dy}{y}P\left[\frac{x}{y}\right]q(y,Q^2) \tag{3.5}$$

and $\alpha_s(Q^2)$ given by Eq. (2.11). The mixed photon-parton splitting functions are given by[13,16–18]

$$k_{\rm NS}^{(0)}(x)=3f(\langle e^4\rangle-\langle e^2\rangle^2)2[x^2+(1-x)^2]\,,$$
$$k_q^{(0)}(x)=3f\langle e^2\rangle 2[x^2+(1-x)^2]\,,$$
$$k_{\rm NS}^{(1)}(x)=3f(\langle e^4\rangle-\langle e^2\rangle^2)k(x)\,, \tag{3.6}$$
$$k_q^{(1)}(x)=3f\langle e^2\rangle k(x)\,,$$
$$k_G^{(1)}(x)=3f\langle e^2\rangle\tfrac{4}{3}\left[-16+8x+\tfrac{20}{3}x^2+\frac{4}{3x}-(6+10x)\ln x-2(1+x)\ln^2x-\delta(1-x)\right],$$

with

$$k(x)=\tfrac{4}{3}\{4-9x-(1-4x)\ln x-(1-2x)\ln^2x+4\ln(1-x)$$
$$+[4\ln x-4\ln x\ln(1-x)+2\ln^2x-4\ln(1-x)+2\ln^2(1-x)-\tfrac{2}{3}\pi^2+10][x^2+(1-x)^2]\}\,.$$

The purely hadronic splitting functions P_{ij} in Eqs. (3.3) and (3.4) are well known[16,17] and, for completeness, are summarized in the Appendix.[19]

Having solved for $q_{\rm NS}^{\gamma}(x,Q^2)$, $\Sigma^{\gamma}(x,Q^2)$, and $G^{\gamma}(x,Q^2)$ one finally obtains the measured photon structure function $(1/x)F_2^{\gamma}(x,Q^2)$ according to Eq. (2.13) where the simple products of moments of the coefficient functions and parton distributions become now convolutions defined by Eq. (3.5) with[17,20]

$$B_{\rm NS}(x)=B_q(x)=\frac{8}{3}\left[\frac{9+5x}{4}-\frac{1+x^2}{1-x}\ln x-\frac{3}{4}\frac{1+x^2}{(1-x)_+}+(1+x^2)\left[\frac{\ln(1-x)}{1-x}\right]_+-\left[\frac{9}{2}+\frac{\pi^2}{3}\right]\delta(1-x)\right],$$
$$B_G(x)=2f\left[(1-2x+2x^2)\ln\frac{1-x}{x}-1+8x(1-x)\right], \tag{3.7}$$
$$B_{\gamma}(x)=\frac{2}{f}B_G(x)\,.$$

It is now straightforward to calculate $F_2^{\gamma}(x,Q^2)$ by solving Eqs. (3.3) and (3.4) iteratively for the required values of x and Q^2 and using appropriate input parton distributions (either measured or guessed from VMD) at Q_0^2. We turn to a more detailed and comparative discussion in the next section.

Finally, in order to obtain the leading-order predictions, we have to solve Eqs. (3.3) and (3.4) with all two-loop quantities $k^{(1)}$ and $P^{(1)}$ set equal to zero and by taking $\alpha_s(Q^2)$ from Eq. (2.4′); $(1/x)F_2^{\gamma}(x,Q^2)$ follows then simply from Eq. (2.12) without any further convolution to be performed.

IV. DISCUSSION AND CONCLUSIONS

The solution of the inhomogeneous evolution equations (2.1) for photonic parton distributions can be expressed analytically only for the moments of these distributions and is given, to leading order in perturbation theory, by Eq. (2.6) which in general depends of course on the input quantities at some finite Q_0^2 (boundary condition). Here $q^{\gamma}(n,Q_0^2)$ refers to the hadronic nonpointlike photon structure function which is commonly estimated on grounds of VMD ideas[7] and *assumed* to be negligible for large $n(\geq 3)$, i.e., for large values of $x(\geq 0.4)$, since counting rule arguments suggest[7] $q_{\rm VMD}^{\gamma}(x,Q_0^2)\sim(1-x)$. Typically $Q_0^2\simeq 1$ GeV^2 where VMD is in good agreement with the measured total $\gamma\gamma$ cross section, but for $Q^2>Q_0^2\simeq 1$ GeV^2 there is a clear deviation[21] from the VMD expectations which points toward the onset of the pointlike partonic contribution given by the first term proportional to $a(n)$ in Eq. (2.6). As we can see this latter contribution still depends on the choice of Q_0^2 even if $q^{\gamma}(n,Q_0^2)$ is assumed to be negligible for large n. It has become customary to avoid this $Q_0{}^2$ dependence by assuming that Q^2 is so large as to justify the omission of the logarithmically decreasing term $[\alpha_s(Q^2)/\alpha_s(Q_0{}^2)]^{1-2P^{(0)}/\beta_0}$ leaving us with the widely celebrated unique asymptotic solution[2,5,7]

$$q^\gamma(n,Q^2)=4\pi a(n)/\alpha_s(Q^2)\ ,$$

which is often referred to as a unique and absolute test of QCD. The x dependence of this simple asymptotic solution can be most easily extracted by solving the evolution equations (3.3) and (3.4) in leading order ($k^{(1)}=P^{(1)}=0$) directly in x space with the ansatz that the Q^2 dependence of the solution is solely given by $\ln Q^2/\Lambda^2$; this has been extensively studied in Ref. 6 and essentially amounts to solving Eq. (3.1) iteratively for $a(x)$.

However, at moderate values of Q^2 where present data are taken, the logarithmically decreasing terms in (2.6) are not negligible and thus the photonic parton distributions still depend on the boundary conditions at $Q_0{}^2$ and are therefore *not* uniquely predicted by the evolution equations alone, even when assuming $q^\gamma(n,Q_0{}^2)\simeq 0$ for $n\geq 3$. Taking[21] $Q_0{}^2\simeq 1$ GeV^2, a simple calculation shows that the subasymptotic logarithmically decreasing term $[\alpha_s(Q^2)/\alpha_s(Q_0{}^2)]^{1-2P^{(0)}/\beta_0}$ in Eq. (2.6) modifies the unique asymptotic solution $4\pi a(n)/\alpha_s(Q^2)$ for $4\leq n\leq 12$ by as much as 40% at $Q^2=5$ GeV^2 and by about 25% at $Q^2=20$ GeV^2. These sizable corrections imply that $q^\gamma(x,Q^2)$ will be softer than the asymptotic solution $4\pi a(x)/\alpha_s(Q^2)$ at large x.

Clearly, the easiest and most reliable way to extract and study the effects of such corrections is to solve directly Eqs. (3.3) and (3.4) in leading order ($k^{(1)}=P^{(1)}=0$) but *without* assuming that the whole Q^2 dependence of the resulting solution is solely given by $\ln Q^2/\Lambda^2$. Choosing a given value for $Q_0{}^2$, the input can be either $q_i^\gamma(x,Q_0{}^2)=0$, which results in a particular solution of the inhomogeneous evolution equations corresponding to the first term in (2.6), or $q^\gamma(x,Q_0{}^2)\neq 0$ which will yield the general solution of the homogeneous part of the evolution equations (3.3) and (3.4) and corresponds to the second term in Eq. (2.6). This latter solution is obviously identical to the one obtained from the standard (homogeneous) Altarelli-Parisi equations.

The same discussion and remarks apply of course also when higher-order corrections are included, but here the logarithmically decreasing boundary terms $[\alpha_s(Q^2)/\alpha_s(Q_0{}^2)]^{-2P^{(0)}/\beta_0}$ of the most general solution in Eq. (2.8) are not only sizable but also *absolutely necessary* for providing us with a meaningful and finite solution in the small-x region (or for $n=2$) as discussed in Sec. II. The unjustified neglect of all $Q_0{}^2$-dependent boundary terms in Eq. (2.8) generates an apparent problem[8,9] [because of the singularity appearing in the singlet b at $n=2$ in Eq. (2.9)] which requires some *ad hoc* nonperturbative regularization[10,11] for its resolution in order to avoid the photon structure function becoming negative[8] at small x. In addition it should be remembered that, when higher orders are taken into account, even this regularized asymptotic (i.e., $Q_0{}^2$-independent) solution has a further arbitrariness due to an unknown constant coming from the photon matrix element of the hadronic energy-momentum tensor[4] (which survives asymptotically because one eigenvalue of the singlet matrix $P_S(n,Q^2)$ vanishes due to energy-momentum conservation). Having estimated this perturbatively unknown hadronic component of the photon on grounds of VMD arguments,[7,8] one usually fits[4,8,10,11] the parameters of some assumed form of $(1/x)F_2^\gamma(x,Q^2)$ to the predicted moments in Eq. (2.8), with the logarithmically decreasing terms suppressed, in order to extract the relevant x dependence to be compared with experiment. This indirect procedure is necessary since the singularity structure of the solution (2.8) in moment n space is too complicated for a direct (numerical) Mellin inversion.

Such a fitting procedure for extracting the x dependence from predicted moments (with integer n) is certainly not very reliable or adequate especially for the small-x region. This is so because the small-x region is entirely dominated by *low-n* moments, in particular by the rightmost singularities in the n plane, which are not adequately accounted for by the low-integer $n=2$ and 3 moments usually considered.[8,10,11] Clearly a better way to study the x dependence of $b(x)$ is to solve the integral equation (3.2) with the leading-order quantity $a(x)$ given by Eq. (3.1).

In view of all these considerations it is clear that the only reliable and correct way to find the x dependence of the general boundary condition respecting solution (2.8) is to solve the evolution equations (3.3) and (3.4) *directly* in Bjorken-x space. Clearly the solution will depend on the specific choice of $Q_0{}^2$ (VMD suggests[21] $Q_0{}^2\cong 1$ GeV^2) and on the theoretically unknown input distributions $q_{NS}^\gamma(x,Q_0{}^2)$, $\Sigma^\gamma(x,Q_0{}^2)$, and $G^\gamma(x,Q_0{}^2)$. Thus the theory can only be genuinely tested by first measuring these input quantities, i.e., $F_2^\gamma(x,Q_0{}^2)$, and then performing further measurements at some other Q^2 *differing* from $Q_0{}^2$. Needless to say that the presently available data do not provide the required testing conditions unless one makes specific *assumptions* for these input quantities at $Q^2=Q_0{}^2$. Setting all $q_i^\gamma(x,Q_0{}^2)=0$ will yield the particular solution of the inhomogeneous equations (3.3) and (3.4), corresponding to the first two terms in Eq. (2.8), and allows us to study the dependence of these theoretical predictions on the logarithmically decreasing terms, in particular in the small-x region, as compared to the boundary-condition-independent but ill defined[8] or regularized[10,11] asymptotic terms $4\pi a/\alpha_s(Q^2)+b$ in Eq. (2.8) studied thus far. Solving for a finite input $q_i^\gamma(x,Q_0{}^2)\neq 0$, which at present could be estimated on grounds of VMD,[7] will provide us with the additional solution corresponding to the homogeneous part of the evolution equations (3.3) and (3.4), i.e., to the third term in Eq. (2.8). The x and Q^2 evolution of this latter solution is therefore obviously identical to the one obtained by solving the standard Altarelli-Parisi equations to the two-loop accuracy[22] for hadronic distributions. More quantitative results and a detailed comparison with forthcoming data will appear in a future publication.

APPENDIX

For completeness we summarize the hadronic splitting functions required in Eqs. (3.3) and (3.4). The leading-order ones are standard and given by

$$
\begin{aligned}
P_{qq}^{(0)}(x)&=\frac{4}{3}\left[\frac{1+x^2}{1-x}\right]_+ ,\\
P_{qg}^{(0)}(x)&=2f\tfrac{1}{2}[x^2+(1-x)^2] ,\\
P_{gq}^{(0)}(x)&=\frac{4}{3}\frac{1+(1-x)^2}{x} ,\\
P_{gg}^{(0)}(x)&=6\left[\frac{x}{(1-x)_+}+\frac{1-x}{x}+x(1-x)\right.\\
&\qquad\left.+\left[\frac{11}{12}-\frac{f}{18}\right]\delta(1-x)\right] ,
\end{aligned}
\tag{A1}
$$

where distributions like $(1-x)^{-1}{}_+$ are defined by

$$\int_0^1 dx[f(x)]_+ q(x)\equiv\int_0^1 dx\, f(x)[q(x)-q(1)] ,$$

which implies the following useful formula for the actual evaluation of convolutions:

$$
\begin{aligned}
f_+ * q = -q(x)\int_0^x dy\, f(y)\\
+\int_x^1 \frac{dy}{y} f\left[\frac{x}{y}\right]\left[q(y)-\frac{x}{y}q(x)\right] .
\end{aligned}
\tag{A2}
$$

In order to facilitate future detailed (x,Q^2) analyses we represent the subleading two-loop splitting functions[16,17] by simple analytic expressions[22]

$$
\begin{aligned}
P_{\rm NS}^{(1)}(x)\equiv P_{\rm NS+}^{(1)}&=\tfrac{4}{3}\left[14.48-7.992x-6.497x^2-5.613x^3+0.6465x^{-0.5106}\right.\\
&\qquad\left.-13.71(1-x)^{-0.1779}+8.019\frac{1}{(1-x)_+}+8.218\delta(1-x)\right] ,\\
P_{qq}^{(1)}(x)&=\tfrac{4}{3}\left[7.260+7.360x-1.854x^2-13.71x^3+4.751x^{-1.114}\right.\\
&\qquad\left.-22.25(1-x)^{-0.1288}+8.019\frac{1}{(1-x)_+}+8.218\delta(1-x)\right] ,\\
P_{qg}^{(1)}(x)&=63.63+70.22x+56.31x^2-37.20x^3+13.29x^{-1.122}-94.32(1-x)^{-0.2319} ,\\
P_{gq}^{(1)}(x)&=\tfrac{4}{3}[33.59-74.72x+80.81x^2-43.96x^3-1.747x^{-1.220}+4.479(1-x)^{-0.3471}] ,\\
P_{gg}^{(1)}(x)&=286.7-290.0x+327.6x^2-201.1x^3-14.45x^{-1.111}\\
&\qquad-166.5(1-x)^{-0.1246}+24.06\frac{1}{(1-x)_+}+45.79\delta(1-x) ,
\end{aligned}
\tag{A3}
$$

which refer to the $\overline{\rm MS}$ scheme and to $f=4$ flavors, and are valid for $0.02\leq x\leq 0.99$ where they represent very accurately the exact but more complicated analytic expressions. The corresponding parametrizations for three flavors can be found in Ref. 22.

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[12] The notation and conventions are closely related to the ones of Ref. 4.

[13] The moments of splitting functions are related to the standard anomalous dimensions γ^n and K^n of Ref. 4 by

$$P^{(0)}(n)=-\tfrac{1}{4}\gamma^{(0),n} ,$$

$$P^{(1)}(n)=-\tfrac{1}{8}\gamma^{(1),n} ,$$

$$k^{(0)}(n)=\tfrac{1}{4}K^{(0),n} ,$$

and

$$k^{(1)}(n)=\tfrac{1}{8}K^{(1),n} .$$

[14] Neglecting the subasymptotic (boundary) terms the asymptotic solution reads $q^\gamma(Q^2)=4\pi a/\alpha_s(Q^2)$. It is amusing to note that in an Abelian gluon (fixed-point α^*) theory this solution becomes instead

$$q^\gamma=\frac{\alpha}{2\pi}\frac{1}{-(\alpha^*/2\pi)\frac{3}{4}P^{(0)}}k^{(0)}$$

which predicts a similar n (or x) dependence as in QCD but *no* $\ln Q^2$ increase. Thus an x-dependence measurement alone, at a fixed value of Q^2, does not provide us with a unique test of QCD; only the *additional* observation of the expected $\ln Q^2$ increase would hint toward a non-Abelian gauge theory (QCD).

[15]Such a cancellation has already been observed in Ref. 10 provided one considers *virtual* photon structure functions where also the second photon in $\gamma(Q^2)\gamma(-p^2)\rightarrow$hadrons is off shell, i.e., $-p^2 \gtrsim 1$ GeV^2. As we have demonstrated this cancellation occurs for real ($p^2=0$) structure functions as well.

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[19]For large x (≥ 0.4), where the photonic gluon and sea distributions are negligible, the evolution equations reduce to just *one* single (uncoupled NS) equation where $q_{\rm NS}^{\gamma}/\Sigma^{\gamma} = (\langle e^4\rangle - \langle e^2\rangle^2)/\langle e^2\rangle$.

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