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Effects of quantum fields on singularities and particle horizons in the early universe

Paul Anderson*
Physics Department, University of California, Santa Barbara, California 93106
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The back-reaction problem for conformally invariant free quantum fields in spatially flat homogeneous and isotropic spacetimes containing classical radiation is solved. The solutions depend upon two regularization parameters which we call α and β. Only solutions which at late times approach the appropriate solution to the field equations of general relativity are considered. For all α and β with $\alpha>0$ there are many such solutions, while for all α and β with $\alpha<0$ there is only one such solution. For $\beta>3\alpha>0$, there is always one solution which undergoes a "time-symmetric bounce" and which contains no singularities or particle horizons. For $\alpha>0$ there is always at least one solution which begins with an initial singularity and which has no particle horizons. For all α and β there is always at least one solution which begins with an initial singularity.

I. INTRODUCTION

The classical theory of general relativity predicts that our universe began with an initial singularity.[1] This means that at very early times the density of matter and curvature of spacetime were arbitrarily large. For densities and curvatures on the order of or larger than l^{-2}, where $l=(16\pi G)^{1/2}$ is the Planck length,[2] quantum effects cannot be ignored and general relativity must be modified in order to take them into account.

We are interested in applying such a modified theory of gravity to the early universe. In order to do so, we must have some model to work with. The most successful classical models of the universe are the Friedmann models. They are homogeneous and isotropic spacetimes containing classical radiation and/or nonrelativistic matter. These models have been so successful in their description of the early universe that a whole set of questions has arisen which are mainly concerned with the explanation of the properties they possess. A study of the effects quantum fields have on homogeneous and isotropic spacetimes promises to address some of these questions.

The line element for homogeneous and isotropic spacetimes is the Robertson-Walker (RW) line element. It has the form

$$ds^2=a^2(\eta)\left[-d\eta^2+\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\right], \qquad (1.1)$$

where $a(\eta)$ is the scale factor, $k=0,\pm1$, and the spatial curvature (intrinsic curvature of a surface of constant η) equals (k/a^2).

Einstein's equations can be solved to give the behavior of the scale factor as a function of time. One finds that in an early radiation-dominated Friedmann universe, $a\propto\eta-\eta_0$, where η_0 is an arbitrary constant and the initial singularity is at $\eta=\eta_0$. If the causal structure of the spacetime is examined, it is found that since the singularity is not at $\eta=-\infty$, there are regions which are not in causal contact with each other. Thus Friedmann universes contain particle horizons. In general, any homogeneous, isotropic spacetime which does not begin at $\eta=-\infty$ has particle horizons.

There are two questions which naturally arise concerning the properties of the Friedmann models. The first asks whether certain initial conditions such as homogeneity and isotropy in the early universe are necessary in order for the universe to evolve into its present state. The second asks whether those properties predicted by general relativity such as the existence of particle horizons and an initial singularity, are retained when quantum effects are taken into account.

It is possible that studies of quantum effects in the early universe will provide negative answers to both these questions. Already, such studies have shown that quantum effects can dissipate anisotropy,[3–5] "soften" singularities, remove singularities, and remove particle horizons.[6] In this paper we shall undertake a more complete investigation than has previously been done of quantum effects in spatially flat, i.e., $k=0$, RW spacetimes containing classical radiation. In the process, we will find new evidence for the "softening" of singularities and the removal of both singularities and particle horizons. We will consider quantum effects when the spatial curvature and cosmological constant are nonzero in a subsequent paper.

We now discuss the assumptions of our models in detail. We are considering the effects of conformally invariant free quantum fields. This is because the Green's function can be found by making a conformal transformation to Minkowski spacetime, finding the appropriate Green's functions there, and transforming them back to the original spacetime.

Massive fields are not conformally invariant and neither is the gravitational field. However, massive fields may be approximately conformally invariant at energy scales characteristic of the very early universe. The neglect of quantum effects due to gravitons is a more serious matter. Nevertheless, if there were many fields in the early universe, then a "$1/N$" expansion[7,8] of quantum gravity can be performed, where N is the number of quantum fields present. In the first order of such an expansion it is consistent to neglect effects due to gravitons. Thus there is reason to believe that our results will be approximately correct even if massive fields and gravitons were present in the very early universe.

There is no particle production in flat spacetime if the only fields present are free fields, so there can be none in conformally flat spacetime if the only fields present are conformally invariant free fields. Thus there is no particle production in our models.

We include classical radiation to support the expansion of the universe in the classical epoch. We neglect classical matter because it makes a negligible contribution to the stress energy tensor compared with that of the radiation when the scale factor is small.

Because of the large amount of symmetry in our models, only the expansion rate and spacetime curvature are influenced by quantum effects. It is just these quantities, however, which need to be changed in order to have particle horizons and singularities removed.

In the beginning of this section, we pointed out that general relativity has to be modified in order to take quantum effects into account. Our final assumption is that the semiclassical theory of gravity is an appropriate way to do this. We will now give a brief review of this theory.[9]

If an expansion in $\hbar$ is performed on the full quantum theory of gravity and if only the classical order is kept for the gravitational field, while first order after the classical is kept for other quantum fields, then Einstein's equations are modified to read

$$G_{ab} = \frac{l^2}{2}(T^{\text{cl}}_{ab} + \langle 0 \,|\, T^{\text{QM}}_{ab} \,|\, 0 \rangle) \;, \tag{1.2}$$

where G_{ab} is the Einstein tensor, T^{cl}_{ab} the stress energy tensor for classical fields, T^{QM}_{ab} the stress energy tensor operator for quantum fields, and $\langle 0 \,|\, T^{\text{QM}}_{ab} \,|\, 0 \rangle$ the regularized vacuum expectation value of T^{QM}_{ab}. The term containing $\langle 0 \,|\, T^{\text{QM}}_{ab} \,|\, 0 \rangle$ is $O(\hbar)$ while other terms are $O(\hbar^0)$.

For conformally invariant free quantum fields, there are no higher-order terms in $\hbar$, so (1.2) is exact. In this case, one can find an explicit form for the Green's functions and these Green's functions can then be used to define the appropriate vacuum states. Once these are known, the unregularized vacuum expectation value of T^{QM}_{ab} can be computed. It is divergent and must be regularized.

Although the quantum theory of gravity is not a renormalizable theory, it is not an unreasonable procedure to subtract off, in an invariant way, the divergences in whatever order of perturbation theory one is working. For conformally invariant fields in conformally flat spacetimes, various authors find the following expression for $\langle 0 \,|\, T^{\text{QM}}_{ab} \,|\, 0 \rangle$ after regularization[10,11]:

$$\begin{aligned}\langle 0 \,|\, T^{\text{QM}}_{ab} \,|\, 0 \rangle = {} & \frac{\alpha}{3}(g_{ab} R^{;c}{}_{;c} - R_{;ab} + R R_{ab} - \tfrac{1}{4} g_{ab} R^2) \\ & + \beta(\tfrac{2}{3} R R_{ab} - R_a{}^c R_{bc} + \tfrac{1}{2} g_{ab} R_{cd} R^{cd} \\ & - \tfrac{1}{4} g_{ab} R^2) \;, \end{aligned} \tag{1.3}$$

where R_{ab} is the Ricci tensor, $R = R^a{}_a$ is the scalar curvature, and α and β are constants.

The constants α and β come from the regularization process and their values depend on the number and types of fields present as well as on the method of regularization. For example, dimensional regularization gives[12]

$$\begin{aligned} \alpha &= \frac{1}{2880\pi^2}(N_S + 6N_\nu + 12N_V) \;, \\ \beta &= \frac{1}{2880\pi^2}(N_S + 11N_\nu + 62N_V) \;, \end{aligned} \tag{1.4}$$

where N_S is the number of scalar fields, N_ν is the number of four-component neutrino fields, and N_V is the number of Maxwell fields included in one's theory.

Because the method of regularization influences the values of α and β and because it is as yet uncertain what fields were present in the very early universe, it is unclear what values of α and β should be used when applying (1.2) to the early universe. For this reason, we will consider all values of α and β.

Several investigations of this problem have been undertaken. For example, Wald[13] investigated the case $\alpha = 0$, Ruzmaikina and Ruzmaikin[14] and Gurovich and Starobinsky[15] began an investigation of the case $\beta=0$, and Fischetti, Hartle, and Hu[6] (FHH) began an investigation of the cases $\alpha\leq 0$, all values of β and $\alpha>0$, $\beta<3\alpha$. Each of these was done for spatially flat RW spacetimes containing classical radiation and having no cosmological constant. Starobinsky[16] investigated the case $\alpha<0$, $\beta>0$ for RW spacetimes with no classical radiation or matter and no cosmological constant.

In this paper, we will examine the cases $\alpha>0$, $\beta\geq 3\alpha$ and find that there are significant qualitative differences between them and those investigated by the above authors. We also complete the investigation begun by FHH by examining those cases in more detail.

For fixed values of α and β, there is in general a two-parameter family of solutions to Eq. (1.2). Equation (1.2) is a fourth-order equation, but some truly insignificant parameters can be eliminated. When Eqs. (1.1) and (1.3) are substituted into (1.2) and the "00" component is taken, the result is a third-order ordinary differential equation for the scale factor as a function of time. This can be reduced to a second-order differential equation by a change of variables. In order to single out the appropriate solution for our univere, it is necessary to know the boundary conditions. It is possible the semiclassical theory of gravity or the full quantum theory of gravity can supply these boundary conditions, but to date no one has discovered exactly what they are. Therefore, we take a phenomenological approach and require that solutions to (1.2) approach the appropriate solution to Einstein's equations at late times, i.e., times which are large compared to the Planck time. We shall call solutions which do this asymptotically classical solutions (ACS).

This approach allows us to single out a unique solution to Eq. (1.2) for given values of α and β if $\alpha\leq 0$. If $\alpha>0$, for each value of α and β there are many ACS, sometimes with very different types of initial behaviors. At present, we do not know which if any of these actually represents the behavior of our universe. It may be that studies of quantum effects in universes which contain conformally noninvariant fields or in universes which are slightly inhomogeneous or anisotropic will allow a solution to be singled out. In this paper, we will attempt to discover and catalog the ACS for all values of α and β.

We find that there is always one ACS which describes a universe that begins with an initial singularity. This is true for all values of α and β. For $\alpha\leq 0$ there is one such ACS, while for $\alpha>0$ there is a one-parameter family of such ACS.

In addition, we find that if $\beta>3\alpha>0$, one ACS always exists which describes a universe that has no singularities. It begins at $\eta=-\infty$ with an infinite scale factor and initially collapses like a collapsing Friedmann universe. The scale factor reaches a minimum size and then the universe begins to expand, approaching the classical solution at late times. We call solutions which describe this behavior "time-symmetric bounce solutions."

In Sec. II, Eqs. (1.1), (1.2), and (1.3) are used to derive a differential equation which describes the dynamical behavior of spatially flat RW spacetimes containing classical radiation, when quantum effects due to conformally invariant free fields are taken into account. The solutions to this equation which are asymptotically classical are then found and some of their physical properties are examined. Section III contains a discussion of these physical properties and what they imply about the singularity and particle-horizon issues. It also contains a brief discussion of the probability that slightly conformally noninvariant particles will be produced in the universe described by our solutions.

II. THE DYNAMICAL EQUATION OF MOTION AND ITS ASYMPTOTICALLY CLASSICAL SOLUTIONS

The homogeneity and isotropy of RW spacetimes imply that their dynamical behavior consists of uniform expansions and contractions which can be completely described by the scale factor $a(\eta)$. Our goal is to derive an equation for $a(\eta)$ for spatially flat spacetimes, using Eqs. (1.1), (1.2), and (1.3). We will then examine the asymptotically classical solutions to this equation. The early time behavior of these ACS is qualitatively different for different values of α and β. For this reason our discussion of the ACS is broken up into the cases $\alpha>0$, $\alpha=0$, and $\alpha<0$ as well as several subcases which depend on the value of (β/α). All possible values of α and β are considered and the results are summarized in Table I.

To derive an equation for $a(\eta)$, we must have expressions for the stress energy tensors on the right-hand side of (1.2). That for $\langle 0\,|\,T_{ab}^{\rm QM}\,|\,0\rangle$ is given by (1.3). The stress energy tensor for classical radiation has the form

$$T_{ab}^{\rm cl}=(\rho_r+p_r)u_a u_b+p_r g_{ab}\ , \qquad (2.1)$$

where u_a is the four-velocity of the radiation, ρ_r is its energy density, and p_r its pressure. The equation of state is $p_r=\frac{1}{3}\rho_r$ and ρ_r varies with a as

$$\rho_r=\frac{\tilde{\rho}_r}{a^4}\ , \qquad (2.2)$$

where $\tilde{\rho}_r$ is a constant.

TABLE I. Summary of the asymptotically classical solutions and their physical properties.

α	β	Number of parameters needed to specify a solution	Number of solutions without a singularity	Number of solutions without a horizon	Equations describing initial behavior
$\alpha>0$	$\beta\geq 3\alpha$	1	Family	Family	(2.16), (2.19), (2.20)
	$0<\beta<3\alpha$	1	Family	1	(2.16), (2.21), (2.22)
	$\beta<0$	1	None	1	(2.21), (2.22), (2.23), (2.24)
$\alpha=0$	$\beta\geq 0$	None	None	None	(2.25a), (2.25b), (2.26a), (2.26b)
	$\beta<0$	None	None	1	(2.25c), (2.26c)
$\alpha<0$	$\beta>3\alpha$	None	None	None	(2.32)
	$\beta=3\alpha$	None	None	1	(2.31)
	$\beta<3\alpha$	None	None	1	

Because we are considering spatially flat spacetimes, the equation we derive for $a(\eta)$ must be invariant under scale transformations of a and η which preserve the metric. These transformations have the form $a\to\lambda a, \eta\to\lambda^{-1}\eta, r\to\lambda^{-1}r$. Equation (2.2) implies that under such a transformation $\tilde{\rho}_r\to\lambda^4\tilde{\rho}_r$, since scale changes in the coordinates cannot change the density. It is useful, therefore, to define two dimensionless quantities which are invariant under the above transformations. They are

$$b=l^{-1}\tilde{\rho}_r^{\,-1/4}a\ ,\qquad X=6^{-1/2}\tilde{\rho}_r^{\,1/4}\eta\ . \tag{2.3}$$

Since the only variable is $a(\eta)$, all of the nontrivial components of (1.2) must be linearly dependent. For convenience we choose the "00" component, which, combined with Eqs. (1.1), (1.3), (2.1), (2.2), and (2.3), gives the following equation for the scale factor:

$$b'^2=1+\frac{\alpha}{3}\left[\frac{b'''b'}{2b^2}-\frac{b''b'^2}{b^3}-\frac{1}{4}\left[\frac{b''}{b}\right]^2\right]+\frac{\beta}{12}\left[\frac{b'}{b}\right]^4 , \tag{2.4}$$

where $b'\equiv db/dX$. This is a third-order ordinary differential equation which does not explicitly depend on the independent variable X. This implies that its solutions will be invariant under translations in X. It is also easy to check that Eq. (2.4) is invariant under the transformations $X\to -X$, although its solutions in general are not.

Because (2.4) is explicitly independent of X, it can be reduced to a second-order differential equation. One way to do this[14,17] is to define the new variables

$$y=|\alpha|^{-3/4}b^3\ ,\qquad f=|b'|^{3/2}\ . \tag{2.5}$$

In terms of f and y, Eq. (2.4) becomes

$$\frac{d^2f}{dy^2}=-\frac{\beta}{12\alpha}\frac{f}{y^2}+\frac{\alpha}{|\alpha|}\frac{1}{y^{2/3}}\left[\frac{1}{f^{1/3}}-\frac{1}{f^{5/3}}\right] . \tag{2.6}$$

Both (2.4) and (2.6) are independent of $\tilde{\rho}_r$. This comes from the fact that b is invariant under changes of scale, while $\tilde{\rho}_r$ is not. Once b is known, the scale can be set by $\tilde{\rho}_r$ and $a(\eta)$ can be determined.

In terms of b and X, the classical solution to Einstein's equations is $b=X$, while in terms of f and y it is $f=1$. Therefore, the ACS approach $b=X$ and $f=1$ at large values of X.

The rest of this section consists of solving (2.4) and (2.6) for the ACS and discussing their early time behaviors and physical properties. Only for $\alpha=0$ were we able to find general analytic solutions to Eqs. (2.4) and (2.6). In this case there exists for each β one ACS. These solutions are displayed in Eq. (2.25) and their physical properties are discussed in Sec. II B.

For $\alpha>0$, FHH found a one-parameter family of ACS. In order to discover their early time behavior, we numerically integrated the ACS backward in time. Figures 1–5 show the results for $\alpha=(2880\pi^2)^{-1}$ and $(\beta/\alpha)=6,3,1,0,-1$, respectively.

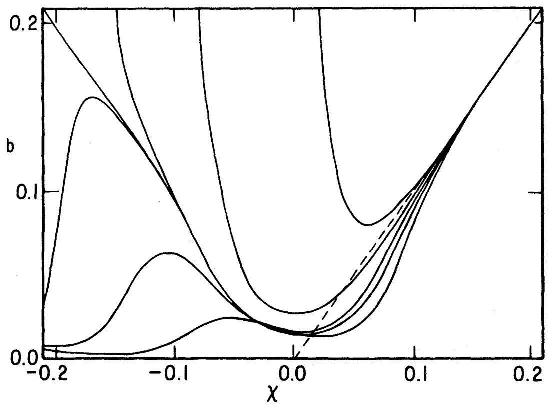

FIG. 1. This figure shows selected ACS for $\beta=6\alpha=6(2880\pi^2)^{-1}$. The dashed line is the classical solution $b=\chi$. There are three types of ACS plotted. The upper curves are a family of solutions which begin with $b=\infty$ at a finite value of χ. At early times, they behave like contracting de Sitter universes with effective cosmological constants equal to $2880\pi^2 l^{-2}$. They bounce once and have particle horizons, but no singularities. Their early time behavior is given by Eq. (2.16). The second type is a single solution which initially behaves like a contracting radiation-dominated Friedmann universe with $b=|\chi|$. It undergoes a time-symmetric bounce and has no particle horizons or singularities. The lower curves are a family of solutions which begin with $b=0$ at $\chi=-\infty$. They bounce an infinite number of times and are called multiple-bounce solutions. They have initial singularities, but no particle horizons.

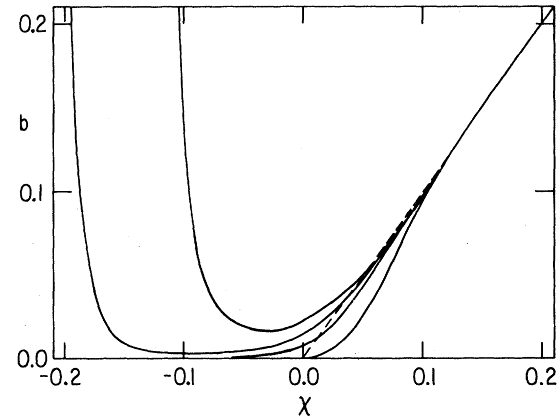

FIG. 2. This figure shows selected ACS for $\beta=3\alpha=3(2880\pi^2)^{-1}$. The dashed line is the classical solution $b=\chi$. There are two types of ACS plotted. The upper curves are a family of solutions which begin with $b=\infty$ at a finite value of χ. At early times they behave like contracting de Sitter universes with effective cosmological constants equal to $5760\pi^2 l^{-2}$. They bounce once and have particle horizons but no singularities. Their initial behavior is given by Eq. (2.16). The lower curves are a family of solutions which begin with $b=0$, at $\chi=-\infty$ and do not bounce. They have initial singularities but no particle horizons. Their early time behaviors are given by Eqs. (2.19) and (2.20).

In Sec. II A we examine the physical properties of the ACS in these figures and generalize to other values of (β/α).

For $\alpha<0$, FHH found one ACS for each value of α and β. Figure 6 shows the result of numerical integrations of these ACS backward in time for $\alpha=-(2880\pi^2)^{-1}$ and $(\beta/\alpha)=6,3,1,0$. In Sec. II C we examine the physical properties of the ACS for $\alpha<0$.

We now begin a detailed and thorough discussion of the ACS and their early time behaviors for the cases outlined above, beginning with the case $\alpha>0$.

A. $\alpha>0$

For $\alpha>0$, FHH found a one-parameter family of ACS whose late time behavior is given by the following asymptotic power series:

$$f=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\psi_{nm}e^{-n\sqrt{3}y^{2/3}}y^{n/6-2m/3},\quad y\to\infty\ . \tag{2.7a}$$

The first few terms of this series are

$$f=\{1+\frac{\beta}{16\alpha}y^{-4/3}+\cdots\}+\psi_{10}y^{1/6}e^{-\sqrt{3}y^{2/3}}\{1+\cdots\}-\tfrac{1}{2}\psi_{10}{}^2y^{1/3}e^{-2\sqrt{3}y^{2/3}}\{1+\cdots\}+\cdots, \tag{2.7b}$$

where ψ_{10} is an arbitrary constant. Equation (2.7b) can be integrated with the result that

$$b=\chi+(\beta/72)\chi^{-3}+\cdots,\quad \chi\to\infty\ . \tag{2.7c}$$

In (2.7c) the integration constant has been set equal to zero in order to define the origin of the coordinate χ.

These are the only ACS we know of for $\alpha>0$ and it is plausible that there are no others. Our reasoning is as follows. Equation (2.6) is a second-order differential equation, so we expect a two-parameter family of solutions at large y. If (2.6) is linearized about $f=1$, then the general solution at large y is

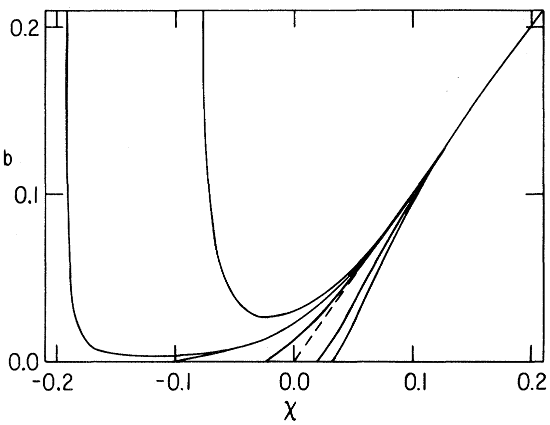

FIG. 3. This figure shows selected ACS for $\beta=\alpha=(2880\pi^2)^{-1}$. The dashed line is the classical solution $b=\chi$. There are two types of ACS plotted. The upper curves are a family of solutions which begin with $b=\infty$ at a finite value of χ. At early times they behave like contracting de Sitter universes with effective cosmological constants equal to $17280\pi^2 l^{-2}$. They bounce once and have particle horizons but no singularities. Their initial behavior is given by Eq. (2.16). The lower curves are a family of solutions which begin with $b=0$ at a finite value of χ and do not bounce. They have particle horizons and initial singularities. Their early time behaviors are given by Eq. (2.21). A third type of ACS is hinted at and actually exists. It begins with $b=0$ at $\chi=-\infty$ and does not bounce. It has an initial singularity but no particle horizons. Its early time behavior is given by Eq. (2.22).

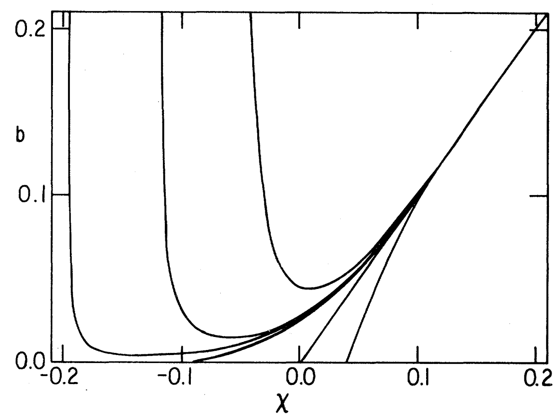

FIG. 4. This figure shows selected ACS for $\beta=0,\alpha=(2880\pi^2)^{-1}$. There are two types of ACS plotted. The upper curves are a family of solutions which begin with $b=\infty$ at a finite value of χ. They bounce once and have initial singularities and particle horizons. Their early time behavior by Eq. (2.23). The lower curves are a family of solutions which begin with $b=0$ at finite value of χ and do not bounce. They have initial singularities and particle horizons. Their early time behavior is given by Eq. (2.21). The classical solution $b=\chi$, is one of these ACS. A third type of ACS is hinted at and actually exists. It begins with $b=0$ at $\chi=-\infty$ and does not bounce. It has an initial singularity but no particle horizons. Its early time behavior is given by Eq. (2.22).

$$f_{\text{linearized}}=1+\frac{\beta}{16\alpha}y^{-4/3}+c_1 y^{1/6}\exp(-\sqrt{3}y^{2/3})+c_2 y^{1/6}\exp(\sqrt{3}y^{2/3}),\quad y\to\infty\ , \tag{2.8}$$

where c_1 and c_2 are arbitrary constants. The solutions which approach $f=1$, i.e., those with $c_2=0$, form a one-parameter family which has the same form as (2.7b) at large y. From counting and form it is plausible that for $\alpha>0$, all ACS are enumerated in Eq. (2.7a). Having discussed the late time behavior of the ACS, we now turn to their early time behavior. Before examining the numerical integrations it is useful to discuss the small-f behaviors of all solutions to (2.6) because many of the ACS approach $f=0$ at small values of y.

In the limit $f\to 0$, Eq. (2.6) reduces to

$$\frac{d^2f}{dy^2}=-\frac{\beta}{12\alpha}\frac{f}{y^2}-\frac{1}{y^{2/3}f^{5/3}}\ . \tag{2.9}$$

With the change of variables $f=y^{1/2}|v|^{3/2}$, $y=e^w$ this can be integrated once with the result that

$$\frac{dv}{dw}=\pm\frac{2}{3v}(\sigma^2v^4+cv+3)^{1/2}\ , \tag{2.10}$$

where $\sigma^2=\frac{1}{4}(1-\beta/3\alpha)$ and c is an arbitrary constant.

The qualitative behavior of solutions which approach $w=-\infty$ depends on the value of σ^2 and is discussed in the subcases below. The qualitative behavior of solutions for which $v=0$ at $w=w_0$, where w_0 is some constant, is independent of σ^2, and we discuss it next.

In terms of f and y and b' and b, the point $v=0,w=w_0$ corresponds to the points $f=0$, $y=y_0=\ln w_0$, and $b'=0$, $b=b_0=\alpha^{1/4}y_0{}^{1/3}$, respectively. Thus at $b=b_0$ the scale factor reaches an extremum. If this is a minimum, then we call the solution a "bounce" solution. Near an extremum, (2.4) and (2.6) can be solved without neglecting any terms. The result is

$$b=b_0\pm 3^{1/2}\alpha^{-1/2}b_0(\chi-\chi_0)^2\pm 2^{-1/2}3^{5/8}\alpha^{-21/16}b_0D(\chi-\chi_0)^3+\cdots\ , \tag{2.11a}$$

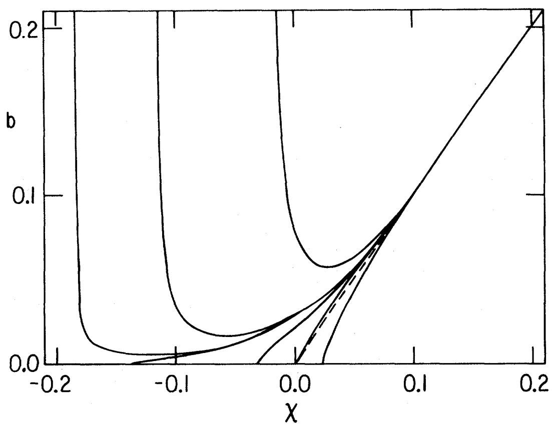

FIG. 5. This figure shows selected ACS for $\beta=-\alpha=-(2880\pi^2)^{-1}$. The dashed line is the classical solution $b=\chi$. There are two types of ACS plotted. The upper curves are a family of solutions which begin with $b=\infty$ at a finite value of χ. They bounce once and have initial singularities and particle horizons. Their initial behavior is given by Eq. (2.24). The lower curves are a family of solutions which begin with $b=0$ at a finite value of χ and do not bounce. They have initial singularities and particle horizons. Their early time behavior is given by Eq. (2.21). A third type of ACS is hinted at and actually exists. It begins with $b=0$ at $\chi=-\infty$ and does not bounce. It has an initial singularity but no particle horizons. Its early time behavior is given by (2.22).

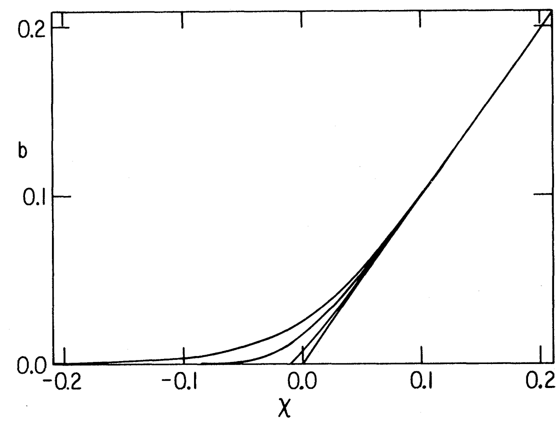

FIG. 6. This figure shows selected ACS for $\alpha=-(2880\pi^2)^{-1}$. From top to bottom they are the ACS for $\beta=6\alpha$, $\beta=3\alpha$, $\beta=\alpha$, and $\beta=0$. There is only one ACS for each value of α and β if $\alpha<0$. Each of the ACS plotted begins with $b=0$ and has an initial singularity. For $\beta=6\alpha$, the ACS begins at $\chi=-\infty$ and has no particle horizons. For $\beta=3\alpha$, the ACS also begins at $\chi=-\infty$ and has no particle horizons. Its early time behavior is given by Eq. (2.31). For $\beta=\alpha$, the ACS begins at a finite value of χ and has particle horizons. Its early time behavior is given by Eq. (2.32). For $\beta=0$, the ACS is the classical solution, $b=\chi$, and it has particle horizons.

$$f=\left[\frac{16}{3}\right]^{3/8} y_0{}^{-1/4}|y-y_0|^{3/4}-\operatorname{sign}(\chi-\chi_0)D\,|y-y_0|^{5/4}+\cdots,\quad y_0{}^{-1}|y-y_0|<<1\,, \tag{2.11b}$$

where D is an arbitrary constant and the plus (minus) sign corresponds to b reaching a minimum (maximum) value of b_0 at time $\chi=\chi_0$.

When $D=0$, all of the terms containing odd powers of $(\chi-\chi_0)$ in (2.11a) vanish, so these solutions are symmetric about the extremum and we call them "time-symmetric bounce solutions" if they reach a minimum. When $D\neq 0$, the second term on the right-hand side of (2.11b) changes sign at the extremum.

We next discuss the subcases $\beta>3\alpha$, $\beta=3\alpha$, $0<\beta<3\alpha$, $\beta=0$, and $\beta<0$. Each of the plots in Figs. 1–5 corresponds to one of these subcases. The plots were obtained by numerically integrating (2.4) backward in time, using (2.7) to find starting values for b' and b''. It should be noted that since (2.6) depends only on the sign of α and on the value of (β/α), solutions to (2.6) with the same value of (β/α) but different values of α will correspond to the same solution of (2.6). Thus the behaviors for different values of α and the same ratio of (β/α) can be found by scaling.

Since the subcases $(\beta/\alpha)\geq 3$ have not been discussed by any authors, we begin with them.

1. $\beta>3\alpha$

Figure 1 shows ACS for $\beta=6\alpha=6(2880\pi^2)^{-1}$. There are three types of solutions shown in this figure. Those which bounce many times and originate at $b=0$, we call "multiple-bounce" solutions. The others begin at $b=\infty$ and bounce only once. One of these has $b=|\chi|$ initially and it undergoes a time-symmetric bounce. The others collapse exponentially as a function of proper time, at early times, and do not undergo time-symmetric bounces. We shall discuss each of these types of solutions separately.

The ACS which are multiple-bounce solutions begin at $b=0$, so their initial behavior can be discovered by examining solutions to (2.6) in the

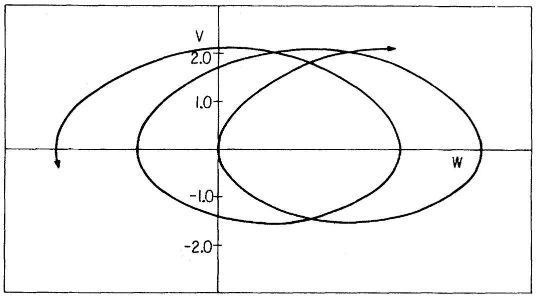

FIG. 7. This figure shows part of a solution to Eq. (2.10), for $\sigma^2=-0.25, c=1$. The solution "spirals" around the w axis, beginning at $w=-\infty$ and ending at $w=\infty$. Each crossing of the w axis corresponds to $b'=0$ and therefore to an extremum in the $(\mathcal{X},b)$ plane. That implies that this is a multiple-bounce solution.

limit $y\to 0$. It is not hard to show that for $\beta>3\alpha$, solutions to (2.6) approach $f=0$ in the limit $y\to 0$, so (2.10) is a good approximation in this limit. It should be noted that all solutions for which $f\to 0$ as $y\to 0$ have some value of c. Since $\sigma^2<0$, dv/dw is imaginary at large values of $|v|$. A phase-plane analysis shows that solutions spiral around the "w" axis if $c\neq 0$, see Fig. 7. Since $f=\exp(w/2)|v|$, each intersection of the "v" axis corresponds to an extremum in the $(\mathcal{X},b)$ plane. Thus these solutions are multiple-bounce solutions. If (2.10) is integrated near a bounce, it can be seen that solutions with $c=0$ are time-symmetric bounce solutions.

Multiple-bounce solutions begin with an initial singularity in the sense that the curvature becomes infinite in the limit $w\to-\infty$. This can be seen directly from the equation for the scalar curvature,

$$\begin{aligned} R&=l^{-2}b''b^{-3}\\ &=2l^{-2}|\alpha|^{-1}y^{-1/3}f^{1/3}\frac{df}{dy}\\ &=l^{-2}|\alpha|^{-1}\exp(-2w/3)\left[v^2+3v\frac{dv}{dw}\right]. \end{aligned} \tag{2.12}$$

Clearly, $R\to\infty$ in the limit $w\to-\infty$ if v is bounded, as it is for the multiple-bounce solutions.

The singularity is located at $\mathcal{X}=-\infty$ because the equation $b'=f^{2/3}$ can be integrated once with the result that

$$\mathcal{X}=\alpha^{-1/4}\int^{b}\frac{db}{vb}+\text{constant}. \tag{2.13}$$

Since v is bounded, this integral diverges in the limit $b\to 0$. Thus the multiple-bounce solutions do not have particle horizons. The proper time from the singularity is given by

$$t=6^{1/2}l\int^{\mathcal{X}}b\,d\mathcal{X}. \tag{2.14}$$

Substitution of (2.13) into (2.14) shows that the singularity occurs at a finite proper time in the past.

The second type of ACS in Fig. 1 is the one which undergoes a time-symmetric bounce. We verified this by using Eq. (2.11b) to obtain starting values of f and df/dy for time-symmetric bounce solutions and by then numerically integrating (2.6) for several values of (β/α). Our results for $(\beta/\alpha)=6$ are plotted in Fig. 8. The time-symmetric bounces form a continuous family of solutions to (2.6) parametrized by y_0. For $y_0\lesssim 8.92\times 10^{-3}$ the solutions reach $f=0$ at large y, while for $y_0\gtrsim 8.92\times 10^{-3}$ they approach $f=\infty$ in the limit $y\to\infty$. The theorem in the Appendix tells us therefore, that one solution with $y_0\approx 8.92\times 10^{-3}$ is an ACS. We found similar evidence for the existence of an ACS which undergoes a time-symmetric bounce for every $(\beta/\alpha)>3$ which we examined. Table II lists for each of these (β/α), the value of y_0 for the time-symmetric bounce ACS. This solution has no singularities or particle horizons, since it begins at $\mathcal{X}=-\infty$. It is the most attractive type of single-bounce solution we have found because it approaches the classical solution $b=|\mathcal{X}|$, whenever the scale factor is large.

The third type of ACS in Fig. 1, begin at $b=\infty, \mathcal{X}=\mathcal{X}_0$ where $\mathcal{X}_0$ is an arbitrary constant. In the (y,f) plane this corresponds to $y=\infty$, $f=\infty$. In the large-f limit, the term containing $f^{-5/3}$ in (2.6) can be neglected. With the change of variables $f=ry, y=e^{w}, s=dr/dw$ we find the following equation for s:

$$\frac{ds}{dr}=s^{-1}\left[-\frac{\beta}{12\alpha}r+r^{-1/3}\right]. \tag{2.15}$$

A phase-plane analysis of this equation shows that for $\beta>0$, its solutions spiral into the point $r=(12\alpha/\beta)^{3/4}$, $s=0$. This corresponds to the behavior

$$f=(12\alpha/\beta)^{3/4}y,\quad y\to\infty, \tag{2.16a}$$

$$b=(\beta/12)^{1/2}(\mathcal{X}-\mathcal{X}_0)^{-1},\quad \mathcal{X}\to\mathcal{X}_0, \tag{2.16b}$$

where $\mathcal{X}_o$ is an arbitrary constant.

Substituting (2.16a) into (2.12), we find in the limit $y\to\infty$

$$R\to 24l^{-2}\beta^{-1}. \tag{2.17}$$

Although these solutions do not have an initial singularity, they do begin with curvatures which are of the order of l^{-2}, so it is not surprising that quantum effects are important even though the scale fac-

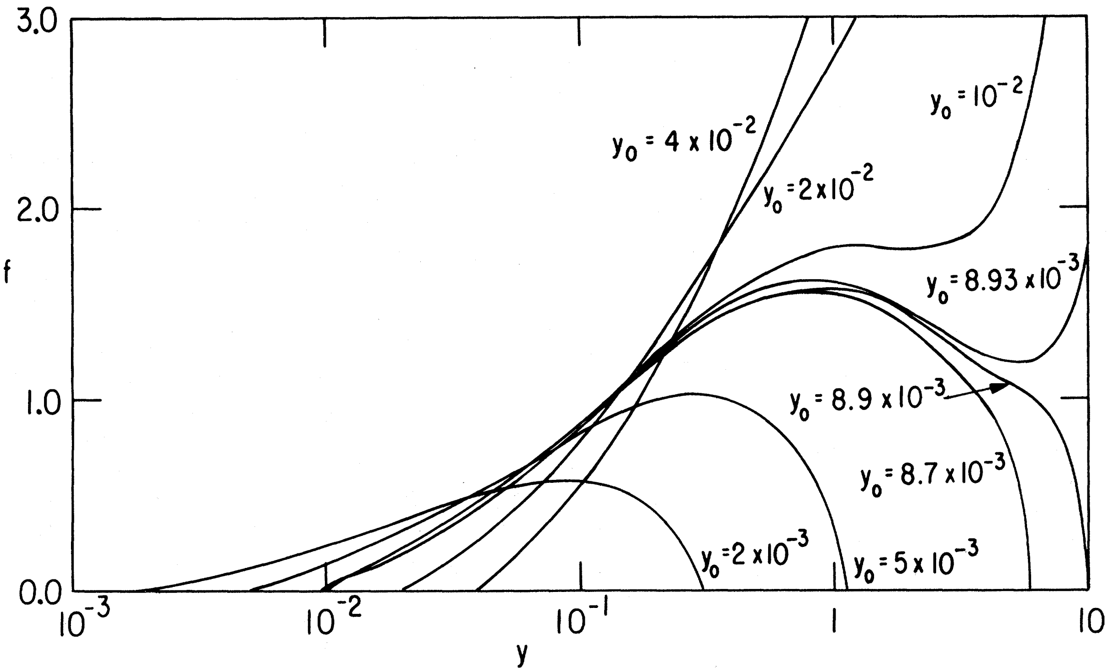

FIG. 8. This figure shows selected time-symmetric bounce solutions for $\beta=6\alpha$. The bounces occur when the scale factor reaches a minimum at $b_0=\alpha^{1/4}y_0^{1/3}$. For $y_0<8.92\times10^{-3}$ the solutions approach $f=0$ at large y, while for $y_0>8.92\times10^{-3}$ they approach $f=\infty$ as $y\to\infty$. Thus the theorem in the Appendix implies that a time-symmetric bounce solution exists which is an ACS. It bounces at $y_0\approx8.92\times10^{-3}$.

tor is large. Since they begin at $\chi=\chi_0$, they contain particle horizons.

If (2.16b) is written in terms of the proper time in (2.14) it is apparent that these solutions appear like contracting de Sitter universes at early times. Their effective cosmological constants are

$$\Lambda_{\text{effective}}=6l^{-2}\beta^{-1} . \quad (2.18)$$

This concludes our discussion of the ACS for $(\beta/\alpha)>3$.

2. $\beta=3\alpha$

Figure 2 shows some ACS for $\beta=3\alpha=3(2880\pi^2)^{-1}$. There are two types of solutions shown. The first type begins at $b=\infty$, $\chi=\chi_0$, and

TABLE II. Asymptotically classical time-symmetric bounce solutions.

(β/α)	y_0
4	2.22×10^{-4}
6	8.92×10^{-3}
12	0.115
120	10.03

bounce once. Their initial behavior is given by Eqs. (2.16). They have particle horizons but no singularities.

The second type begin at $b=0$ and do not bounce. To discover their physical properties, we must examine (2.6) in the limit $y\to0$. As before, $f\to0$ for solutions in this limit and (2.6) reduces to (2.10). Since $\sigma^2=0$, (2.10) can be integrated with the result that for $c\neq0$

$$f=|c|^{-3/2}y^{1/2}q\{[1+(1-27q^{-2})^{1/2}]^{2/3} +[1-(1-27q^{-2})^{1/2}]^{2/3} +3q^{-2/3}\}^{3/2},\quad y\to0 , \quad (2.19a)$$

$$b=\text{const}\times\exp[3^{-1/2}\alpha^{-3/4}(\chi-\chi_1)^3],\quad \chi\to\infty , \quad (2.19b)$$

where $q=(c_1-\frac{1}{2}c^2\ln y)$ and c_1 and χ_1 are arbitrary constants. For $c=0$

$$f=y^{1/2}\left[c_1-\frac{4}{\sqrt{3}}\ln y\right]^{3/4},\quad y\to0 , \quad (2.20a)$$

$$b=\text{const}\times\exp[-\alpha^{-1/2}(\chi-\chi_1)^2],\quad \chi\to-\infty . \quad (2.20b)$$

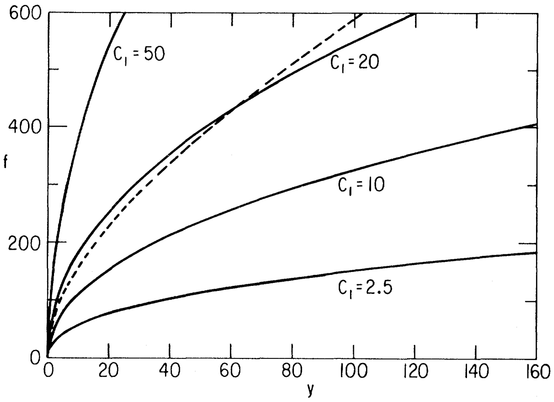

FIG. 9. The solid curves in this figure are selected solutions of Eq. (2.9). Their behavior is given by Eq. (2.19a) with $c=1$. The dashed curve is an exact solution to Eq. (2.6). Its small-y behavior is given by Eq. (2.19a) with $c=1$, $c_1=5$. Note that for all $y>0$, the dashed curve lies above the solid curve labeled $c_1=5$.

These solutions begin at $\chi=-\infty$, so they have no particle horizons. Substitution of (2.19) and (2.20) into (2.12) and (2.14) shows that they begin with an initial singularity at a finite proper time in the past.

Next, we must determine which of these are ACS. Equation (2.19a) is a two-parameter family of solutions, while equation (2.20a) is a one-parameter family of solutions. The ACS condition gives one condition between the two parameters in (2.19a) and the one parameter in (2.20a). Thus we expect one ACS will have its early time behavior given by (2.20a) and the rest which approach $b=0$, will have their early time behavior given by (2.19a).

We can in fact prove that for each value of c, including $c=0$, there is at least one ACS. The above counting argument then implies that there is exactly one ACS for each value of c.

Our proof is accomplished by showing first that for large values of c_1, for a given value of c, solutions to (2.6) whose small-y behavior is given by (2.19a) and (2.20a), approach $f=\infty$ at large y. Second, it is shown that for large negative values of c_1, solutions approach $f=0$ at large y. Then the theorem in the Appendix tells us that for at least one value of c_1, the solution is an ACS.

We begin our proof with the case $c_1 \gg 0$. First note that (2.19a) and (2.20a) are exact solutions of (2.9). We shall label solutions of (2.9) with the subscript T in order to distinguish them from solutions to (2.6). As $y\to 0$, (2.19a) and (2.20a) approach the exact solutions of (2.6), so that $f_T(y,c_1)\to f(y,c_1)$ for all c_1. It follows from the fact that $d^2f/dy^2 > d^2f_T/dy^2$ for all y and f, that near $y=0$, $df/dy(y,c_1) > df_T/dy(y,c_1)$ and that $f(y,c_1) > f_T(y,c_1)$.

Let us choose some specific value of c_1 and denote it by ξ. Then if we plotted the solutions in (2.19a) or (2.20a) in the (f_T,y) plane for various values of c_1, those curves with $c_1>\xi$ would lie above the curve $f_T(y,\xi)$ for all $y>0$, see Fig. 9. If we also plotted the curve $f(y,\xi)$, the exact solution to (2.6), then near $y=0$, it too must lie above the $f_T(y,\xi)$ curve and have a larger slope than it. At larger values of y, the $f(y,\xi)$ curve must continue to stay above the $f_T(y,\xi)$ curve. To see this, suppose the $f(y,\xi)$ curve were to cross the $f_T(y,\xi)$ curve. Then the $f(y,\xi)$ curve must first become tangent to some $f_T(y,c_1)$ curve with $c_1>\xi$. If it did become tangent to such a curve, then $d^2f/dy^2(y,\xi) > d^2f_T/dy^2(y,c_1)$ so that the $f(y,\xi)$ curve would have to stay above the $f_T(y,c_1)$ curve. This is a contradiction, since the $f_T(y,c_1)$ curve lies above the $f_T(y,\xi)$ curve. So the $f(y,\xi)$ curve always lies above the $f_T(y,\xi)$ curve.

The $f_T(y,\xi)$ curve reaches a maximum height which increases as ξ increases. Thus by choosing ξ large enough, we can force the $f(y,\xi)$ curve to have positive slope at arbitrarily large values of f and y. The phase-plane analysis of solutions with positive slope at large f and y [see discussion above Eq. (2.15)] shows that they must approach $f=\infty, y=\infty$. Therefore for large enough ξ, $f(y,\xi)\to\infty$ as $y\to\infty$.

The case $c_1 \ll 0$ is much simpler. For large enough negative values of c_1, $df_T/dy<0$ at arbitrarily small values of y. But at small y, $f(y,c_1)\approx f_T(y,c_1)$ so $df/dy(y,c_1)\approx df_T/dy(y,c_1)$. Thus $df/dy(y,c_1)<0$. Also, at small y, $f(y,c_1)<1$. Thus there exist values of c_1 such that $f(y,c_1)<1$ and $df/dy(y,c_1)<0$ for small values of y. Inspection of (2.6) shows that $d^2f/dy^2<0$ whenever $f<1$. This forces $f(y,c_1)$ to approach zero at some value of y, for large negative values of c_1.

This ends our proof and completes the discussion of the subcase $\beta=3\alpha$.

3. $0<\beta<3\alpha$

Figure 3 shows some ACS for $\beta=\alpha=(2880\pi^2)^{-1}$. There again are two types of solutions, those which begin at $b=\infty$ and bounce once and those which begin at $b=0$ and do not bounce. The initial behavior of the former is given by (2.16). They have particle horizons but no initial singularities.

To find the physical properties of the solutions which begin at $b=0$, we must examine the small-y behavior of (2.6). FHH did this and found the following two-parameter family of solutions for $\sigma^2>0$:

$$f=y^{1/2-\sigma}\left[C+C'y^{2\sigma}-\tfrac{9}{16}C^{-5/3}\sigma^{-2}y^{8\sigma/3}+\cdots\right],$$

$$y\to 0\,, \tag{2.21a}$$

$$b=\text{const}(\mathcal{X}-\mathcal{X}_0)^{1/2\sigma}+\cdots, \quad \mathcal{X}\rightarrow\mathcal{X}_0 , \tag{2.21b}$$

where C, C', and $\mathcal{X}_0$ are arbitrary constants with the restriction $C>0$. Substituting these expressions into (2.12), we see that these solutions begin with an initial singularity at $\mathcal{X}=\mathcal{X}_0$. Therefore, they contain particle horizons.

FHH also found a one-parameter family of solutions with the following small-y behavior for $\sigma^2>0$:

$$f=\sigma^{-3/4}y^{1/2}(1+c_2y^{2\sigma/3}+\cdots), \quad y\rightarrow 0 , \tag{2.22a}$$

$$b=\text{const}\times\exp(\sigma^{-1/2}\alpha^{-1/4}\mathcal{X}), \quad \mathcal{X}\rightarrow\infty , \tag{2.22b}$$

where c_2 is an arbitrary constant. These solutions begin at $\mathcal{X}=-\infty$, so they do not have particle horizons. Substituting (2.22) into (2.12) and (2.14), we see that they begin with an initial singularity at a finite proper time in the past.

In terms of Eq. (2.10), these solutions correspond to the case $c=-4\sigma^{1/2}$. A phase-plane analysis shows that for large positive values of c_2, $v\rightarrow\infty$ in the limit $w\rightarrow\infty$. Thus solutions to (2.9) with initial behavior given by (2.22a) have $f\rightarrow\infty$ in the limit $y\rightarrow\infty$. By a similar proof to the one given in Sec. II A 2, it can be shown that at least one ACS exists whose initial behavior is given by (2.22a).

It is clear from Fig. 3 that a one-parameter family of ACS have their initial behavior given by (2.21). This completes our discussion of the subcase $0<\beta<3\alpha$.

4. $\beta=0$

This subcase was originally examined by Ruzmaikina and Ruzmaikin[14] and was later examined by FHH. Figure 4 shows some of the ACS. There are again two types of solutions, those which begin at $b=\infty$ and bounce once and those which begin at $b=0$ and do not bounce.

In terms of f and y, those solutions which begin at $b=\infty$, begin at $f=\infty$, $y=\infty$. A phase-plane analysis of (2.15) with $\beta=0$ shows that all solutions must approach the curve $s=r^{-1/3}$ at large r. This corresponds to the behavior

$$f=(\tfrac{4}{3})^{3/4}y(\ln y)^{3/4}, \quad y\rightarrow\infty , \tag{2.23a}$$

$$b(\ln b)^{1/2}=[2(\mathcal{X}-\mathcal{X}_0)]^{-1} , \tag{2.23b}$$

where $\mathcal{X}_0$ is an arbitrary constant. Since they begin at $\mathcal{X}-\mathcal{X}_0$, these solutions contain particle horizons. Substitution of (2.23) into (2.12) shows that there is an initial singularity at $\mathcal{X}=\mathcal{X}_0$. Because these solutions start out with infinite curvature, it is not surprising that quantum effects are important when the scale factor is large.

The analysis of the solutions which begin at $b=0$ is exactly the same as in Sec. II A 3. Thus there is a one-parameter family of ACS with particle horizons and an initial singularity whose early time behavior is given by (2.21). There is at least one ACS with an initial singularity and no particle horizons whose early time behavior is given by (2.22). The classical solution $f=1$ is also an exact solution of (2.6) in this subcase. This completes our discussion of the subcase $\beta=0$.

5. $\beta<0$

Figure 5 shows some ACS for this subcase. There are two types of solutions, those which begin at $b=\infty$ and bounce once and those which begin at $b=0$ and do not bounce. In terms of f and y, those solutions which begin at $b=\infty$ and bounce once, begin at $f=\infty$, $y=\infty$. A phase-plane analysis of (2.15) shows that solutions approach $s=(-\frac{1}{2}+\sigma)r$ as $r\rightarrow\infty$. So the initial behavior of the solutions which bounce once is given by

$$f=c_1y^{1/2+\sigma}, \quad y\rightarrow\infty , \tag{2.24a}$$

$$b=\text{const}(\mathcal{X}-\mathcal{X}_0)^{-1/2\sigma}, \quad \mathcal{X}\rightarrow\mathcal{X}_0 , \tag{2.24b}$$

where c_1 and $\mathcal{X}_0$ are arbitrary constants. Substitution of (2.24) into (2.12) shows that these solutions begin with an initial singularity at $\mathcal{X}=\mathcal{X}_0$. Thus they contain particle horizons.

The analysis of solutions which begin at $b=0$, is exactly the same as in Sec. II A 3. Thus there is a one-parameter family of ACS with particle horizons and an initial singularity whose early time behavior is given by (2.21). There is also at least one ACS with an initial singularity and no particle horizons whose early time behavior is given by (2.22). This completes our discussion of the subcase $\beta<0$.

B. $\alpha=0$

In this case, it is useful to define the variables f and x so that $b'=f^{2/3}$ as before and $b=x^{1/3}$. Substituting these variables into Eq. (2.4) with $\alpha=0$ and solving for f, one finds the ACS are given by

$$f=(6/\beta)^{3/4}x[1-(1-(\beta/3)x^{-4/3})^{1/2}]^{3/4} , \quad \beta>0 , \tag{2.25a}$$

$$f=1, \quad \beta=0, \tag{2.25b}$$

$$f=(6/|\beta|)^{3/4}x\{[1+|\beta|/(3x^{4/3})]^{1/2}-1\}^{3/4} , \quad \beta<0 . \tag{2.25c}$$

The subcase $\beta>0$ was first discussed by Wald[13] and later by FHH. Evaluation of the scalar curvature R shows that the solution begins with an initial singularity at $x=(\beta/3)^{3/4}$. Near the singularity, it has the form

$$b=(\beta/3)^{1/4}[1-(12/\beta)^{1/4}(\chi-\chi_0)]^{-1}\,, \qquad \chi\to\chi_0\,, \quad (2.26a)$$

where χ_0 is an arbitrary constant. Thus this solution contains particle horizons.

For $\beta=0$, there are no quantum effects and the solution is the classical solution

$$b=\chi-\chi_0\,, \quad (2.26b)$$

where χ_0 is an arbitrary constant. As discussed in the Introduction, this solution has an initial singularity at $\chi=\chi_0$ and contains particle horizons.

For $\beta<0$, evaluation of the scalar curvature R, shows that the solution begins at $x=f=0$ with an initial singularity. Near the singularity it has the form

$$b=\text{const}\times\exp[(12/|\beta|)^{1/4}\chi],\ \ \chi\to-\infty\ . \quad (2.26c)$$

Since this solution begins at $\chi=-\infty$, it has no particle horizons. Substitutions of (2.26c) into (2.14) shows that the singularity occurs at a finite amount of proper time in the past. This ends our discussion of the case $\alpha=0$.

C. $\alpha<0$

In this case, FHH found a single ACS whose late time behavior is given by

$$f(y)=1+\frac{\beta}{16|\alpha|}y^{-4/3}+\cdots,\ \ y\to\infty\ , \quad (2.27a)$$

$$b(\chi)=\chi+(\beta/72)\chi^{-3}+\cdots,\ \ \chi\to\infty\ . \quad (2.27b)$$

By an argument similar to that in the $\alpha>0$ case, we find it very likely that there are no other ACS.

Having exhibited the late time behavior of the ACS, we now turn to their early time behavior. It was pointed out by FHH that there are no bounce solutions for $\alpha<0$. This is because at a bounce (2.4) reduces to

$$b''=(12/\alpha)^{1/2}b,\ \ b'=0\ . \quad (2.28)$$

Thus all solutions begin at $b=y=0$.

Many solutions to (2.6) also begin at $f=0$. In fact for $\beta<0$, it is not hard to show that all solutions to (2.6) begin at $f=0$. With the change of variable $f=y^{1/2}v^{3/2}, y=e^{w}$, (2.6) can be integrated once in the limit $f\to0$, with the result that

$$\frac{dv}{dw}=\pm\tfrac{2}{3}v^{-1}[\sigma^2v^4+cv-3]^{1/2}\ , \quad (2.29)$$

where c is an arbitrary constant and $\sigma^2=\frac{1}{4}(1-\beta/3\alpha)$ as before. To go farther, we must examine the subcases $\sigma^2<0,\sigma^2=0$ and $\sigma^2>0$ separately. Since the subcases $\sigma^2\le0$, i.e., $\beta\le3\alpha$, have not been examined by other authors, we begin with them.

1. $\beta<3\alpha$

In this subcase, the initial behavior of all solutions including the ACS, is given by (2.29). Inspection of (2.29) shows that dv/dw is imaginary at both large and small values of v if $\sigma^2<0$. A phase-plane analysis shows that solutions oscillate between two positive values of v which correspond to roots of dv/dw. The solutions begin with an initial singularity at $w=-\infty$ as can be seen by substituting (2.29) into (2.12). The singularity is located at $\chi=-\infty$ because the equation $b'=f^{2/3}$ can be integrated with the result that

$$\chi=-|\alpha|^{-1/4}\int^{b}\frac{db}{vb}+\text{const}\ . \quad (2.30)$$

So these solutions do not have particle horizons. Substitution of (2.30) into (2.14) shows that the singularity occurs at a finite proper time in the past. In Fig. 6 we show a plot of the ACS for $\beta=6\alpha=-6(2880\pi^2)^{-1}$.

2. $\beta=3\alpha$

In this subcase, the initial behavior of all solutions, including the ACS is again given by (2.29). Inspection of (2.29) shows that since $\sigma^2=0$, dv/dw is imaginary for $c\le0$. For $c>0$, (2.29) can be integrated with the result that

$$f=y^{1/2}(c_1-c^{1/2}\ln y),\ \ y\to0\ , \quad (2.31a)$$

$$b=\text{const}\times\exp[\tfrac{1}{3}|\alpha|^{-3/4}(\chi-\chi_1)^3],\ \ \chi\to-\infty\ , \quad (2.31b)$$

where c_1 and χ_1 are arbitrary constants. Since they begin at $\chi=-\infty$, these solutions have no particle horizons. Substitution of (2.31b) into (2.12) and (2.14) shows that they begin with an initial singularity at a finite proper time in the past. The ACS for $\beta=3\alpha=3(2880\pi^2)^{-1}$ is shown in Fig. 6.

3. $\beta>3\alpha$

The particular subcase $\beta=0$, has been examined in some detail by Ruzmaikina and Ruzmaikin[14] and by Gurovich and Starobinsky.[15] For $\beta>3\alpha$, Eq. (2.6) can be solved exactly near $y=0$ and the result is

$$f=y^{1/2-\sigma}(C+C'y^{2\sigma}+\cdots],\ \ y\to 0\ , \tag{2.32a}$$

$$b=\text{const}(\chi-\chi_0)^{1/2\sigma},\ \ \chi\to\chi_0\ , \tag{2.32b}$$

where C,C', and χ_0 are arbitrary constants with the restriction that $C>0$. Since these solutions begin at $\chi=\chi_0$, they contain particle horizons. Substitution of (2.32) into (2.12) shows that they begin with an initial singularity at $\chi=\chi_0$ if $\beta\neq 0$. If $\beta=0$, evaluation of the Ricci tensor R_{ab} in some orthonormal frame will give the same result. Equation (2.32) gives the initial behavior of all solutions to (2.6) including the ACS. The ACS for $\beta=\alpha=-(2880\pi^2)^{-1}$ and $\beta=0$, $\alpha=-(2880\pi^2)^{-1}$ are plotted in Fig. 6. Note that the ACS for $\beta=0$ is the classical solution $b=\chi$. This concludes our discussion of the case $\alpha<0$.

III. SOME PHYSICAL PROPERTIES OF THE ACS

In the previous section we found a large variety of possible behaviors for the early universe. They are summarized in Table I. To know which is ours, we would need to know α and β. This in turn requires that we know which regularization procedure is the correct one and what fundamental fields were present in the early universe. Even then, if $\alpha>0$, there are many solutions and further boundary conditions must be extracted from somewhere.

Since we cannot , at present, determine which behavior if any our universe underwent, we are left with a variety of predictions for two important issues in cosmology: particle horizons and singularities. We find that the ACS have no particle horizons if $\alpha=0$ and $\beta<0$ or $\alpha<0$ and $\beta\leq 3\alpha$. The ACS do have particle horizons if $\alpha=0$ and $\beta\geq 0$ or $\alpha<0$ and $\beta>3\alpha$. For $\alpha>0$, there is always at least one ACS with no particle horizons and at least one family of ACS with particle horizons.

It has been suggested by Davies[18] that the trace anomaly may be able to remove singularities. We find that this is the case and that for $\beta>3\alpha>0$, there is one time-symmetric bounce ACS with no singularities or particle horizons. For $\alpha>0$, $\beta>0$, there is a family of ACS which bounce once, have no singularities, and which do have particle horizons. These are the only ACS we have found which do not begin with an initial singularity.

Although most of the ACS begin with an initial singularity, it is possible that the strength of the singularity may be different for different solutions. One measure of the strength is the divergence of the Riemann tensor as a function of proper time t, where $dt=a\,d\eta$. For the solutions to (2.4) and (2.6) which begin with an initial singularity and for which $b=0$ initially, it can be shown that in an orthonormal frame the nonvanishing components of the Riemann tensor diverge as $(1/t^2)$. So by this measure, the singularities have the same strength.

For homogeneous universes, another measure of the strength of a singularity is the divergence of the energy density as a function of proper volume. The total energy density for our models is

$$\rho=\rho_r+\rho_V\ , \tag{3.1}$$

where $\rho_r=-T_0^{0\text{cl}}$ is the contribution from the classical radiation and $\rho_V=-\langle 0\,|\,T_0^{0\text{QM}}\,|\,0\rangle$ is the contribution from the quantum fields. In terms of b and χ these are

$$\rho_r=(lb)^{-4} \tag{3.2a}$$

$$\rho_V=6^{-1}(lb)^{-4}\left\{\alpha\left[6\frac{'''b'}{b^2}-\frac{2b''b'^2}{b^3}-\frac{1}{2}\left[\frac{b''}{b}\right]^2\right]+\frac{\beta}{2}\left[\frac{b'}{b}\right]^4\right\}. \tag{3.2b}$$

In terms of f and y these are

$$\rho_r=|\,\alpha\,|^{-1}l^{-4}y^{-4/3}\ , \tag{3.2c}$$

$$\rho_V=\rho_r(f^{4/3}-1)\ . \tag{3.2d}$$

For solutions with an initial singularity at $b=0$, we can compare the strength of the singularity with the strength of the classical singularity by computing ρ and comparing it with (3.2a) and (3.2c). The result is that for those solutions without particle horizons, the singularity is always weaker than the classical singularity. This is just what one would expect, since to remove particle horizons the expansion rate near the singularity must be slowed. For solutions with particle horizons, the singularity is weaker (stronger) than the classical singularity if $(\beta/\alpha)>0$ $[(\beta/\alpha)<0]$. If $\beta=0$, the solutions with particle horizons have singularities with the same strength as the classical singularity.

One of our main approximations has been to neglect the effects of conformally noninvariant fields on the dynamical equation of motion. Now that the early time behaviors of the ACS are known, one way to check whether this is a good approximation is to compute the probability that conformally noninvariant particles will be produced in the universes described by the ACS. If the probability is finite, then the solutions pass this test of self-consistency, while if it is infinite they do not.

We shall consider only the probability for producing a pair of slightly conformally noninvariant massless scalar particles in a finite volume of spacetime. Hartle[19] has calculated this probability for homogeneous and isotropic spacetimes. He finds

$$P \propto \int d\chi\, b^4(\chi) R^2 \ . \tag{3.3}$$

For a Friedmann universe containing classical radiation, $R=0$ and no particles are produced. However, if any dust is present, Hartle showed that the particle production is infinite. If $\beta=0$, the behavior of the scale factor near the singularity for those ACS whose early time behaviors are given by (2.21) and (2.32), is the same as for a Friedmann universe.

It is not hard to show that for all ACS with no particle horizons, the probability in (3.3) is finite. It is also finite for ACS with particle horizons if $(\beta/\alpha)>0$. The production probability is infinite for ACS with particle horizons if $(\beta/\alpha)<0$. Finally, the production probability is infinite for those ACS which begin with $b=\infty$ and have particle horizons, if $\alpha>0$ and $\beta=0$. Thus for $(\beta/\alpha)>0$ there is a finite production probability for all ACS while for $(\beta/\alpha)\leq 0$, the results are mixed.

We have investigated the back-reaction problem for spatially flat, homogeneous, isotropic spacetimes containing classical radiation when quantum effects due to conformally invariant free fields are taken into account. For all values of the regularization parameters α and β we find at least one ACS. We cannot at present say which ACS, if any, is the correct one for our universe. Nevertheless, we can say that one is no longer constrained to consider models of the early universe which have initial singularities, particle horizons, and scale factors which only increase with time.

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APPENDIX

In this appendix, we prove the following theorem for the existence of asymptotically classical solutions in the case $\alpha>0$. Given a one-parameter family of solutions, $f(y,\lambda)$, to Eq. (2.6) which is parametrized by λ and which has the following properties:

1. $f(y_I,\lambda)$ and $df/dy(y_I,\lambda)$ are continuous functions of λ for some y_I.
2. There exists a λ_1 such that for $\lambda>\lambda_1$ all solutions in the limit $y\to\infty$ approach $f=\infty$.
3. There exists a $\lambda_2\leq\lambda_1$ such that for $-\infty<\lambda<\lambda_2$ some solutions approach $f=0$ at a finite value of y.

Then, there exists at least one solution which approaches $f=1$ in the limit $y\to\infty$. Since the ACS are defined by this property, the above solution is an ACS.

Proof: There are two parts to the proof of this theorem. The first consists of showing that a solution with $\lambda=\lambda_{\rm lub}\geq\lambda_2$ exists which approaches neither $f=0$ nor $f=\infty$ at large y. The second part consists of showing that solutions to (2.6) which do not approach $f=0$ or $f=\infty$ at large y approach $f=1$.

Existence theorems for ordinary differential equations guarantee that if $f(y_I,\lambda)$ and $df/dy(y_I,\lambda)$ are continuous functions of λ, then $f(y,\lambda)$ and $df/dy(y,\lambda)$ are also continuous functions of λ, for all y. Continuity of $f(y,\lambda)$ means that there exist some $\epsilon_1>0$ and $\epsilon_2>0$ such that for any $\delta>0$,

$$\begin{aligned} &|f(y,\lambda)-f(y,\lambda-\epsilon_1)| < \delta \ , \\ &|f(y,\lambda)-f(y,\lambda+\epsilon_2)| < \delta \ . \end{aligned} \tag{A1}$$

Similarly, continuity of $df/dy(y,\lambda)$ implies that

$$\begin{aligned} &\left| \frac{df}{dy}(y,\lambda)-\frac{df}{dy}(y,\lambda-\epsilon_1) \right| < \delta \ , \\ &\left| \frac{df}{dy}(y,\lambda)-\frac{df}{dy}(y,\lambda+\epsilon_2) \right| < \delta \ . \end{aligned} \tag{A2}$$

Because of continuity and the fact that some solutions approach $f=0$ for $\lambda<\lambda_2$ and $f=\infty$ for $\lambda>\lambda_1$, there must be a least upper bound, $\lambda_{\rm lub}$, where $\lambda_{\rm lub}$ is the smallest λ such that for $\lambda>\lambda_{\rm lub}\geq\lambda_2$, no solutions approach $f=0$.

One can now ask whether the solution with $\lambda_{\rm lub}$ can approach $f=0$ for $y>y_I$. Suppose that it does so. Then from (2.11b) it is clear that $df/dy\to-\infty$ in that limit. Now consider a solution with $\lambda=\lambda_{\rm lub}+\epsilon$ where $\epsilon>0$. For arbitrarily small ϵ, these solutions must remain arbitrarily close to the $\lambda_{\rm lub}$ solution. Thus for some $y>y_I$ and for some $\epsilon>0$, $f(y,\lambda_{\rm lub}+\epsilon)<1$ and $df/dy(y,\lambda_{\rm lub}+\epsilon)<0$. However, from (2.6) it can be seen that for all $f<1$, $d^2f/dy^2<0$. Thus the solution corresponding to $\lambda_{\rm lub}+\epsilon$ must approach $f=0$ if the solution corresponding to $\lambda_{\rm lub}$ does. This is a contradiction, so the $\lambda_{\rm lub}$ solution does not approach $f=0$ for $y>y_I$.

A similar argument making use of the fact that for arbitrarily small ϵ, a solution with $\lambda=\lambda_{\rm lub}-\epsilon$ must approach $f=0$ at large y (because of the definition of $\lambda_{\rm lub}$), shows that the solution corresponding to $\lambda_{\rm lub}$ does not approach $f=\infty$ in the limit $y\to\infty$.

We now show that solutions which do not approach $f=0$ for $y>y_I$ and do not approach $f=\infty$ as $y\rightarrow\infty$, approach $f=1$ as $y\rightarrow\infty$. Clearly if $f=\infty$ and $f=0$ are not approached by a solution, then $f=C$ with $0<C<\infty$ is approached by that solution in the limit $y\rightarrow\infty$. To see if this is consistent, substitute $f=C$ into the right-hand side of (2.6) and take the limit $y\rightarrow\infty$. Then

$$\frac{d^2f}{dy^2}\rightarrow y^{-2/3}(C^{-1/3}-C^{-5/3}) . \qquad \text{(A3)}$$

For $C\neq 1$, this has the solution

$$F=\tfrac{9}{4}(C^{-1/3}-C^{-5/3})y^{4/3} \qquad \text{(A4)}$$

which clearly does not approach $f=C$ at large y. For $C=1$, (A3) has the solution $f=1$. Therefore solutions which do not approach $f=\infty$ or $f=0$ at large y must approach $f=1$

*Present address: Department of Physics, University of Florida, Gainesville, Florida, 32611.

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