PHYSICAL REVIEW D VOLUME 28, NUMBER 10 15 NOVEMBER 1983

Numerical study of flux patterns in non-Abelian lattice gauge theory

Joe Kiskis
Department of Physics, University of California at Davis, Davis, California 95616
(Received 15 July 1983)

Flux patterns in four-dimensional SU(2) lattice gauge theory are studied with Monte Carlo techniques. Through a certain flux loss from elementary cubes, the theory shows significant non-Abelian effects at scales much shorter than the correlation length. The shape of the remaining flux has an Abelian character until the correlation length drops to the cube size.

I. INTRODUCTION

The finite-lattice approximation to continuum gauge field theory is comprehensible to fast digital computers. For slow analog brains it is less appropriate. As a consequence, the lattice-gauge-theory numerical calculations have produced some interesting numbers but little physical insight. Nevertheless, it may be possible to extract enlightening qualitative information from machine-generated field configurations. Some initial results of this type are presented here.

The existing numerical calculations strongly suggest that four-dimensional, non-Abelian lattice gauge theories confine quarks.[1] Also, Tomboulis[2] has presented the outline of a proof that the SU(2) lattice theory confines for all β. Contrasting these solid results is a folklore which holds that confinement may be due to superconducting magnetic sources. One may wonder if the numerical calculations support this or some other picture.

To deal with this type of question, it is necessary to study the configurations in detail. The first thing that can be tried is to look directly at a typical configuration. The link variables are not of interest since they have no gauge-invariant meaning. The plaquette variables are of more interest. One can display them or some combinations of them. It turns out that very little can be seen. The correlations of plaquette variables are much smaller than the fluctuations of each even for adjacent plaquettes at large β. Plaquettes appear to be operating independently. Of course, they are not really independent. They are constrained kinematically by the Bianchi identity and dynamically by the action. To reveal the correlations, some averaging is required. Interesting averages that filter a rough picture out of the noise will be discussed.

The Bianchi identity has strongly influenced this work. In terms of link variables, it is simply an identity. In terms of plaquette variables, it is more of a constraint. Independent plaquettes are not consistent with the Bianchi identity. Nevertheless, there is still great freedom in satisfying the identity. The action exerts its influence here.

Section II begins with an intuitive discussion of the Bianchi identity and flux. This suggests a choice of correlations for study. Lowest-order weak-coupling results for these are given. The calculations represent an essentially Abelian and relatively understandable limit of the non-Abelian theory. Thus, they serve as a baseline that is helpful in interpreting the numerical results of Sec. III. The conclusion presented in Sec. IV has two parts. Through a certain kind of flux loss from elementary cubes, the theory shows significant non-Abelian effects at scales much shorter than the correlation length. However, the shape of the remaining flux is Abelian by one measure until the correlation length drops to the cube size.

A few papers[3] have reported qualitative results from Monte Carlo calculations. The work of DeGrand and Toussaint on Abelian theories is probably the most clearly related. The discussion[4] of the role of "topological" objects in the crossover region is somewhat related, but it deals with lattice effects that are not believed to be important in the continuum limit. The discussions[5] of "fat" flux tubes are more relevant.

Our work represents a rather modest investment in computer time, and the statistics could certainly be improved. Nevertheless, it shows that interesting qualitative information can be extracted from the configurations.

II. THEORETICAL DISCUSSION

This section begins with an intuitive discussion of the Bianchi identity. This leads to the study of certain flux correlations. Lowest-order weak-coupling contributions to these correlations are given. They are used as a baseline in interpreting the numerical results of Sec. III. The model is four-dimensional Euclidean SU(2) lattice gauge theory.

A. Bianchi identity

The continuum, Abelian Bianchi identity has a simple interpretation: The magnetic flux leaving a volume is zero. The continuum, non-Abelian Bianchi identity is not as easy to interpret. It states that the covariant divergence of the field is zero. Thus, the ordinary divergence of the magnetic field need not vanish, and magnetic monopoles can be accomodated without singularities.

From an extremely local viewpoint, the Abelian and non-Abelian identities are not so different. The ordinary divergence of the non-Abelian magnetic field is linear in the gauge potential. Given any point it is possible to find a gauge in which the potential vanishes at that point. Thus in that gauge, the ordinary divergence of the magnetic field will also vanish at that point as is true for the Abelian field.

To get an interpretation that is meaningful in finite regions a somewhat different approach is required. The discussion now applies to both the lattice and continuum because a lattice can be thought of as sitting in a continuum. This will be an intuitive discussion. Complete detail is available in Ref. 6.

Let r stand for some route in space (lattice or continuum). Let $U(r)$ be the gauge group element associated in the usual way with that route. Now consider a three-dimensional cube in space bounded by six faces. Choose a vertex of the cube, and associate with each of the six faces a route that begins and ends at the chosen vertex as indicated in Fig. 1. To obtain expressions with gauge-invariant meaning, it is necessary to refer quantities to the same vertex. This explains the tails on r_2, r_4 and r_6. Applied to this cube the non-Abelian Bianchi identity states that

$$U(r_6)U(r_5)U(r_4)U(r_3)U(r_2)U(r_1)=1 \ . \tag{2.1}$$

In the Abelian case, all the group elements commute so that the tails can be dropped and the order in (2.1) does not matter. Then (2.1) states that the flux leaving the cube is quantized, and for a smooth field in the continuum, it must be zero.

In the non-Abelian case, this statement cannot be made. However, given a cube, it is always possible to find a gauge in which the group elements are the identity on the three links on which there are tails. Then the tails can be dropped and (2.1) will resemble the Abelian version.

Precisely, the identity states that the product of the group elements associated with the six faces of the cube is the identity. Equivalently, if the six elements are multiplied in sequence, then the path obtained in the group begins and ends at the identity. Although this is not ordinary flux conservation, still flux cannot simply disappear in an arbitrary way. Thus in a loose sense, (2.1) is a generalization of flux conservation.

Equation (2.1) is an identity that follows from the kinematic structure of the gauge theory. It is satisfied by any configuration. It reflects the structure of the theory, but it places no restrictions on the configurations beyond those already implied by the defining gauge theory structure. There is a great deal of freedom in the manner in which the identity can be satisfied.

If configurations are discussed in terms of link variables, (2.1) is not particularly important. The link variables are independent and unrestricted and (2.1) is automatically satisfied. However, no gauge-invariant significance can be attached to the value of a particular link variable. It is more physical to discuss configurations in terms of plaquette variables and arrangements of flux.

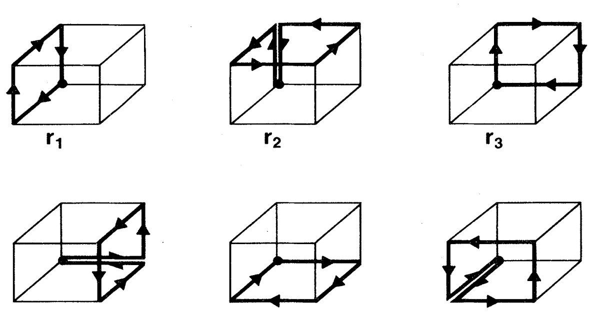

FIG. 1. An elementary cube and the routes r_i associated with the ith faces.

The group elements associated with plaquettes are not independent variables. They are constrained by Bianchi identities such as (2.1) and its generalizations.[6] Thus, not all conceivable arrangements of flux are consistent with the gauge-theory structure. Only those that satisfy the Bianchi identities are allowed. The pattern is determined by the dynamics and thus depends on the value of β.

The flux that enters the cube through a face cannot simply disappear. What tendencies does it have to change orientation in color space and in ordinary space? What is the pattern? Information can be obtained from some simple correlations.

B. Correlations

To study flux, it is necessary to define it. Elements of SU(2) can be parametrized by a vector of angles $\vec{\theta}$,

$$U=e^{i\vec{\theta}\cdot\vec{\sigma}}=\cos\theta+i\hat{\theta}\cdot\vec{\sigma}\sin\theta \tag{2.2}$$

with

$$0\le\theta=|\vec{\theta}|\le\pi \ . \tag{2.3}$$

Associated with each plaquette p there is a group element $U(\partial p)$. It is tempting to call the corresponding $\vec{\theta}$ the flux through p. While it is useful to associate θ with the magnitude of the flux through p, it is not so useful to think of $\hat{\theta}$ as the color direction of the flux. The problem is that $\hat{\theta}$ is only gauge covariant and not gauge invariant. Even a comparison of the relative orientation of gauge covariants based at different lattice vertices has no gauge-invariant meaning. To construct a gauge invariant from gauge covariants, it is necessary that they be based at the same lattice vertex. Thus it is possible to measure the relative orientation of the fluxes of different plaquettes only after parallel transport to a common vertex. [This is one way of understanding the tails on $U(r_2)$, $U(r_4)$, and $U(r_6)$.] The correlations to be studied will be constructed accordingly. They contain more detail than the standard plaquette-plaquette correlation.

The procedure admits some arbitrariness in the selection of the base vertex and routes to it. To fix this, we take our lead from the Bianchi identity itself. Notice that each of the six routes in Fig. 1 is associated with a particular face of the cube and all are based at the same vertex. Thus for the purpose of comparing the fluxes out of faces 1 and 4, take for the flux out of 1 $\vec{\theta}_1$ and for that out of 4 $\vec{\theta}_4$ where

$$U(r_1)=e^{i\vec{\theta}_1\cdot\vec{\sigma}} \text{ and } U(r_4)=e^{i\vec{\theta}_4\cdot\vec{\sigma}} \ . \tag{2.4}$$

It follows that

$$\vec{\theta}_1\cdot\vec{\theta}_4 \tag{2.5}$$

is a gauge-invariant measure of the relative orientation of the fluxes. Objects such as (2.5) will be our major interest.

The Bianchi identity can be expressed in terms of these fluxes. For weak coupling a small-θ approximation of (2.1) can be made and it gives

$$\sum_{i=1}^{6} \theta_i = 0 \tag{2.6}$$

in the Abelian case and

$$\sum_{i=1}^{6} \vec{\theta}_i = 0 \tag{2.7}$$

in the non-Abelian case. It is no surprise that the non-Abelian fluxes act like three independent Abelian fluxes in this approximation. In the Abelian case, the exact result is

$$\sin\left[\sum \theta_i\right] = 0 \tag{2.8}$$

or

$$\sum \theta_i = 0 \mod 2\pi . \tag{2.9}$$

In the non-Abelian case, there is no simple expression such as (2.8) and no monopole quantization condition such as (2.9).

The left-hand side of (2.7) is an interesting quantity. It is gauge covariant. It vanishes in leading order at weak coupling for both Abelian and non-Abelian theories. In Abelian theories, it vanishes in all higher orders also. The nonvanishing contributions are from quantized, nonperturbative, monopole configurations. In the non-Abelian theory, it can be used as a sensible measure of the divergence of the flux. It can also be thought of loosely as a density of magnetic sources.

Use the gauge freedom to orient $\vec{\theta}_1$ to the 3-direction. Then the 3-component of the left-hand side of (2.7) is

$$\sum_i \theta_{i3} . \tag{2.10}$$

In the weak-coupling limit, (2.10) is zero. This means that the flux that enters the cube through face 1 must exit with the same orientation through the other five faces. Away from the weak-coupling limit, this will no longer be true. We will obtain a measure of the extent to which this Abelian picture is valid as a function of β.

Since (2.7) is gauge covariant and not gauge invariant, it cannot be measured directly. To make measurements, we follow (2.10) and introduce the gauge-invariant measurment

$$D \equiv \left\langle \vec{\theta}_1 \cdot \sum \vec{\theta}_i \right\rangle \tag{2.11}$$

Three quantities in addition to D will be considered,

$$A \equiv \langle \vec{\theta}_1^{\,2} \rangle , \tag{2.12}$$

$$B \equiv -\langle \vec{\theta}_1 \cdot \vec{\theta}_3 \rangle , \tag{2.13}$$

$$C \equiv -\langle \vec{\theta}_1 \cdot \vec{\theta}_4 \rangle . \tag{2.14}$$

A is the average of the square of the flux entering the cube through face 1. B measures the tendency of the flux that enters through face 1 to exit with the same orientation through an adjacent face. C measures the tendency of the flux that enters through face 1 to exit with the same orientation through the opposite face. It is then easy to see that

$$A - 4B - C = D . \tag{2.15}$$

This measures the extent to which flux that enters with some orientation does not exit with that orientation. It is a measure of lost or at least reoriented flux.

Another quantity of interest is

$$R = \frac{C}{B} . \tag{2.16}$$

This is a measure of the average shape of the flux configurations that removes the effect of lost flux. Consider again a gauge with

$$\vec{\theta}_1 = \theta_1 \hat{3} . \tag{2.17}$$

Not all of the 3-type flux that enters the cube will leave it in the 3-direction. But for that that does, R gives the ratio of the amount that goes out of an opposite versus an adjacent face. This gives information on the shape of the flux.

C. Baseline calculations

To interpret the measurements of A, B, C, D, and R it is helpful to have some theoretical guidance. The relative importance of different configurations is determined by the integrand of the partition function

$$e^{-\beta S} \text{ with } S = -\sum_p \tfrac{1}{2}\mathrm{TrU}(\partial p) . \tag{2.18}$$

The typical arrangement of flux will change with β. It is to be expected that at large β the typical configurations of the non-Abelian theory will resemble those of the Abelian theory. Lowest-order weak-coupling results, which are essentially the same for the two cases, should be accurate as $\beta \to \infty$.

The purpose of these calculations is not to predict the Monte Carlo data. Indeed we would not expect the lowest-order calculations to be very good in the moderate β regions where data is taken. Rather the calculations provide a framework that can be used to interpret the data. In this way, qualitative and intuitively useful information can be obtained.

First note that in an adequate approximation, A, B, C, and D can be expressed in more familiar notation. Consider

$$l_B \equiv -\tfrac{1}{2}\mathrm{Tr}[U(r_1)U(r_3)] + \tfrac{1}{2}\mathrm{Tr}[U(r_1)]\tfrac{1}{2}\mathrm{Tr}[U(r_3)] . \tag{2.19}$$

In terms of angles, this is

$$l_B = \hat{\theta}_1 \sin\theta_1 \cdot \hat{\theta}_3 \sin\theta_3 . \tag{2.20}$$

So

$$l_B = \vec{\theta}_1 \cdot \vec{\theta}_3 [1 + O(\theta^2)] , \tag{2.21}$$

and in a lowest-order weak-coupling calculation

$$\langle l_B \rangle = -B . \tag{2.22}$$

Similar results apply to A, C, and D.

Standard weak-coupling techniques[7] are used to calculate the lowest-order contribution. The calculation was done on an infinite lattice. In the four-dimensional momentum integrations, the first integration was done analytically and the remaining three numerically. The results at order $1/\beta$ are

$$\begin{aligned} A &= 1.500/\beta\ , \\ B &= 0.324/\beta\ , \\ C &= 0.206/\beta\ , \\ D &= 0\ , \\ R &= 0.636\ . \end{aligned} \tag{2.23}$$

The fact that D vanishes is just a statement of the Bianchi identity in this weak-coupling limit. R less than one indicates that flux is somewhat more likely to exit through a side face than the opposite face in a typical weak-coupling configuration.

D. Possible problem

For a given configuration, the computation of the $\vec{\theta}$'s is unambiguous. Nevertheless, if some θ is near π, then a small shift in certain links can change $\vec{\theta}$ (but not θ) by a large amount. Thus if there are many plaquettes with $\theta \cong \pi$, then the intuitive picture breaks down.

Fortunately, plaquettes with $\theta \approx \pi$ are suppressed for

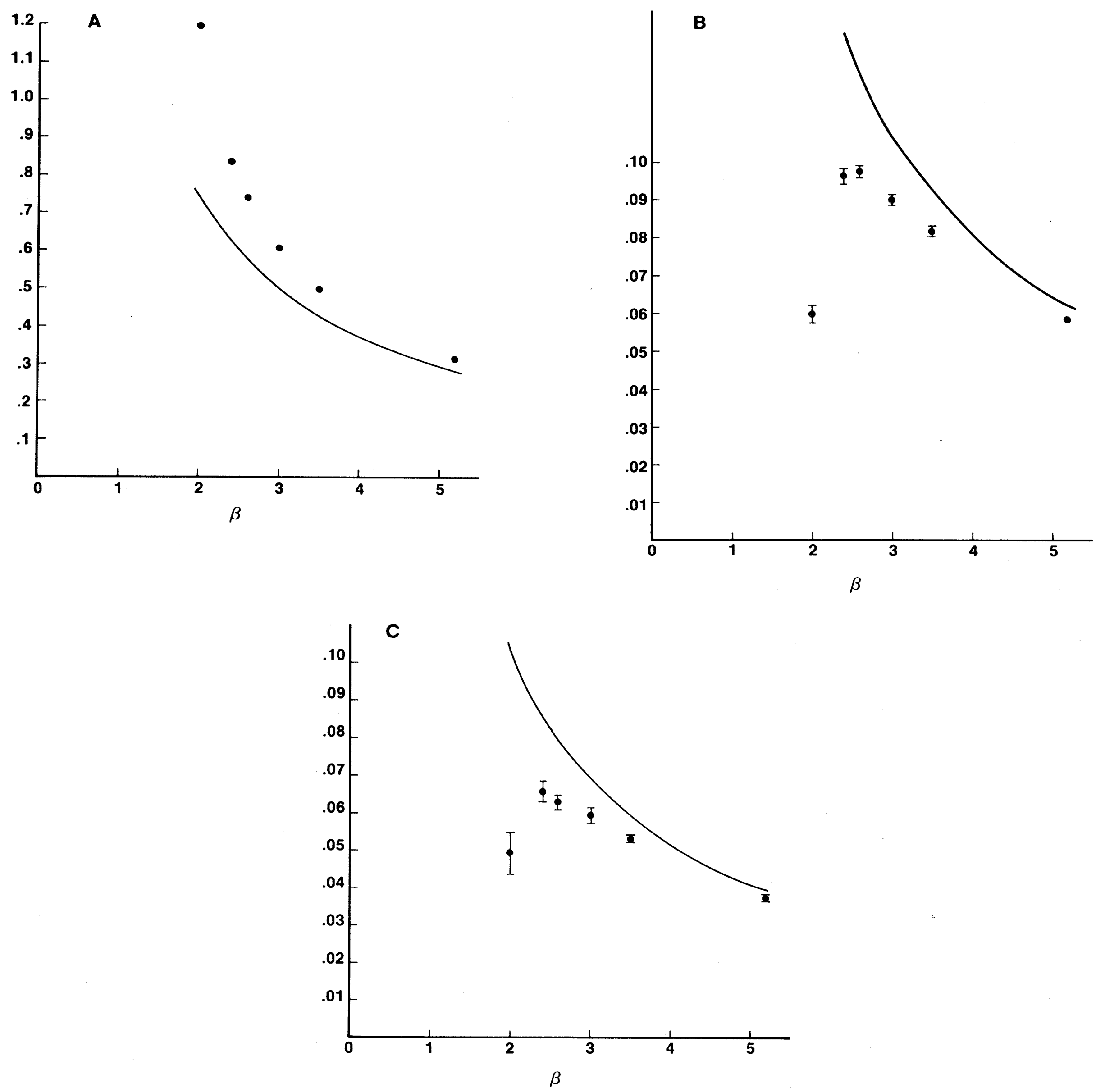

FIG. 2. The measurements A, B, and C with curves from (2.23).

two reasons. First, the action is unfavorable for such large fluxes. Second, the measure on the group is small around $\theta \approx 0$ and $\theta \approx \pi$. Thus the problem may be insignificant at the values of β under investigation.

This has been checked directly. The data can be taken with the contribution from θ's near π explicitly removed. Over most of the β range the effect is invisible and even at $\beta = 2.0$ it is not significant.

III. NUMERICAL RESULTS

In the previous section, a technique for investigating flux was discussed. In this section, the Monte Carlo data are presented. The lowest-order weak-coupling results are helpful in interpreting the data.

The lattice is of size 10^4 and the β values are 2.0, 2.4, 2.6, 3.0, 3.5, and 5.2. The configurations were generated by the heat bath method with a program published by Creutz.[8] At each value of β, the lattice was thermalized with 30—35 sweeps. Then three subsequent configurations separated by five sweeps were saved on magnetic tape. The data came from these saved configurations.

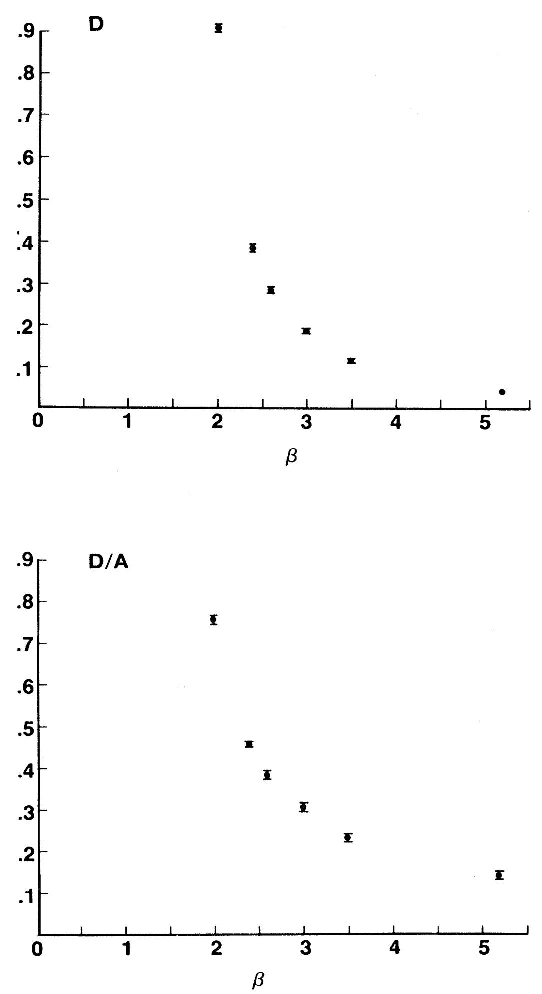

FIG. 3. Plots of D and D/A derived from the A, B, and C data.

$1 \times 1 \times 1$ cubes in the lattice are specified by a vertex and a set of three positive directions. There are four different choices for the set of three directions. For each choice, there are 10^4 such cubes. In each cube, there are three ways to choose pairs of opposite faces. Thus, there are ways to divide up the measurement of any quantity into partial measurements. The error bars in the graphs are the standard deviation of 12 (four on each of three configurations) such partial measurments of the quantity. The errors on the points without bars are too small to display.

In Figs. 2(a), 2(b), and 2(c) the measurements of A, B, and C are presented. The curves are the lowest-order results (2.23). As expected, the lowest-order theory is reasonably close to the data at the larger β values. Perhaps it is a little surprising that the lowest-order calculations are not wildly off even at β as low as 2.4.

A is above the calculation and B and C are below. To interpret this, consider D. The results for A, B, and C indicate that D will increase as β decreases. [Recall (2.14).] D as a function of β is shown in Figs. 3(a) and $D/\langle \vec{\theta}_1{}^2 \rangle$ in 3(b). These show that even at rather short distances the field takes advantage of the non-Abelian structure that allows $D \neq 0$. This effect increases substantially and smoothly as β decreases. It occurs at β values well above the crossover ($\beta \approx 2.2$) and in regions where the correlation length is still much larger than the $1 \times 1 \times 1$ cubes under study.

R gives a measure of the shape of the flux pattern. $R = 0.636$ is the large-β limit and is associated with an Abelian configuration. The data for R are presented in Fig. 4. R stays close to 0.636 down to surprisingly small values of β. At $\beta = 2.4$ the correlation length is about 1.3.

IV. CONCLUSIONS

Three conclusions can be drawn from the data of Sec. III.

(1) For A, B, and C, lowest-order calculations are more

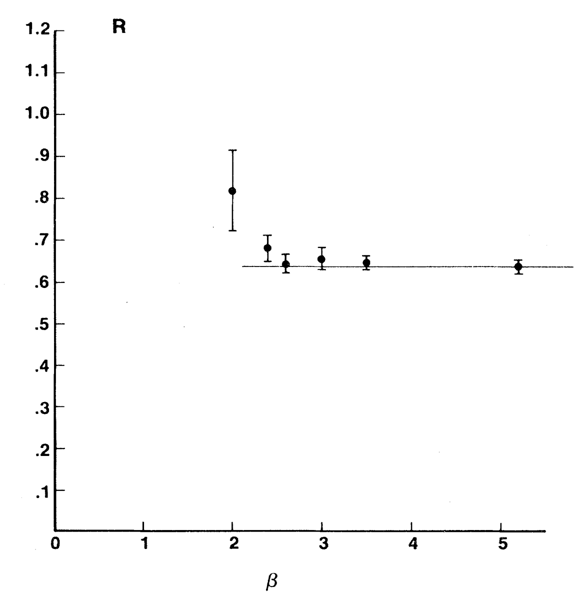

FIG. 4. Plot of R derived from B and C data.

accurate than might be expected down to $\beta=2.4$.

(2) D is significantly different from zero. It increases steeply as β decreases. Thus non-Abelian nonconservation of flux is significant at scales much shorter than the correlation length.

(3) R stays very close to its weak-coupling limit down to surprisingly small β values. By this measure, the remaining flux maintains an Abelian shape until the correlation length approaches the cube size.

It would be interesting to know if the contributions to D come primarily from lattice approximations to point Wu-Yang monopoles or from some more extended sources. This question can be answered with more detailed numerical studies that are in progress. Following DeGrand and Toussaint,[3] it is possible to make rough estimates of the contributions from point monopoles. They do not seem to account for the size of D.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge helpful conversations with M. Creutz, J. Gunion, L. Jacobs, and K. Sparks. The research was supported by the Department of Energy and the Alfred P. Sloan Foundation. Help with the numerical work was generously provided by the Stanford Linear Accelerator Center.

[1]For a recent review of the literature see M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rep. 95, 201 (1983).

[2]E. Tomboulis, Phys. Lett. 50, 885 (1983).

[3]T. DeGrand and D. Toussaint, Phys. Rev. D 22, 2478 (1980); T. Sterling and J. Greensite, Nucl. Phys. B220, 327 (1983); J. Polonyi, Report No. ILL-(TH)-83-8 (unpublished).

[4]I. Halliday and A. Schwimmer, Phys. Lett. 101B, 327 (1981); 102B, 337 (1981); L. Caneschi, I. Halliday, and A. Schwimmer, Nucl. Phys. B200, 409 (1982); R. Brower, D. Kessler, and H. Levine, Phys. Rev. Lett. 47, 621 (1981); Nucl. Phys. B205, 77 (1982).

[5]G. Mack and V. Petkova, Ann. Phys. (N.Y.) 125, 117 (1980); L. Yaffe, Phys. Rev. D 21, 1574 (1980).

[6]J. Kiskis, Phys. Rev. D 26, 429 (1982); G. Batrouni, Nucl. Phys. B208, 467 (1982).

[7]B. Baaquie, Phys. Rev. D 16, 2612 (1977).

[8]M. Creutz, lecture delivered at the 18th Winter School of Theoretical Physics at Karpacz, Poland, 1981 (unpublished).