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Representations of the Poincaré group associated with unstable particles

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The problem of a relativistically covariant description of unstable particles is reexamined. We follow the approach which associates a unitary reducible representation of the Poincaré group with a larger isolated system, and compare it with the one ascribing a nonunitary irreducible representation to the unstable particle alone. It is shown that the problem originates in the choice of the subspace $\mathscr{H}_u$ of the state Hilbert space which could be related to the unstable particle. Translational invariance of $\mathscr{H}_u$ is proved to be incompatible with unitarity of the boosts. Further, we propose a concrete choice of $\mathscr{H}_u$ and argue that in most cases of the actual experimental arrangements this subspace is effectively one dimensional. A correct slow-down for decay of the moving particles is obtained.

I. INTRODUCTION

The Poincaré group $\mathscr{P}$ of special-relativistic space-time transformations plays undoubtedly a central role in high-energy physics. In particular, its unitary irreducible representations may be used for classification of the (stable) elementary particles according to their mass and spin.[1,2] A relativistically covariant description is needed for unstable particles too. For practical purposes, it is frequently sufficient to describe them as classical point particles which decay exponentially in their proper time. Maybe this is the reason why some quantum aspects of the problem are not yet fully understood.

One is naturally tempted to generalize the idea of stable-particle classification and associate suitable nonunitary irreducible representations of $\mathscr{P}$ with the unstable particles. Such representations were actually constructed and used by many authors.[3–8] Typically the homogeneous Lorentz transformations are represented by unitary operators, while the space-time translations are nonunitary and characterized by some complex four-momentum vector.[9] The generalization from stable to unstable particles should not be taken too literally, otherwise one is faced with interpretative difficulties like growing norms for negative times. It seems reasonable to associate the direct physical meaning with the operators representing the subset $\mathscr{P}_+ \subset \mathscr{P}$ which consists of the homogeneous Lorentz group and translations to the forward light cone; it is called sometimes the Poincaré semigroup.[10] Other authors tried to bypass the difficulty by modifying basic postulates of the quantum theory.[7,11]

In fact, there is no *a priori* reason why there should be nonunitary representations of $\mathscr{P}$ associated with unstable particles. Explained in a standard way, the principle of relativistic invariance means that the state Hilbert space of any *isolated* quantum system is the carrier space of some unitary (strongly continuous) representation of $\mathscr{P}$, under which dynamical variables of the system transform in a specific way. In particular, some important observables are identified directly with generators of the corresponding representation of $L_{\mathscr{P}}$, the Lie algebra of $\mathscr{P}$: the total Hamiltonian $H \equiv P_0$ with the generator of time translations, components P_j of the momentum with the generators of space translations, etc.

Hence one should start with a larger isolated system which contains the unstable particle under consideration as well as its decay products, and to choose on its state Hilbert space $\mathscr{H}$ a suitable unitary representation U: $\mathscr{P} \to \mathscr{B}(\mathscr{H})$. This representation is presumably reducible but it should be characterized by a sharp value of spin; examples of such representations are known.[12,13] Having determined U, one may return to the subspace $\mathscr{H}_u \subset \mathscr{H}$ which belongs to the unstable particle alone, and study the operator-valued function V: $\mathscr{P} \to \mathscr{B}(\mathscr{H}_u)$ defined by

$$V(\Lambda,a) = pr_u U(\Lambda,a) \tag{1}$$

for all elements $(\Lambda,a) \in \mathscr{P}$. The following questions arise naturally:

(i) Do the operators $V(\Lambda,a)$ fulfill the composition law of $\mathscr{P}$, at least for some subgroup or subset of elements?

(ii) If so, what can be said about the relations between such a representation and the corresponding restriction of the above-mentioned nonunitary representations?

The only serious attempt to find an answer, and to reconcile thereby the two approaches, was undertaken by Williams[14]; but he failed on the well-known difficulty of an energy spectrum unbounded from below. Our aim in this paper is to clarify the matter.

II. THE BOOSTS SHOULD NOT BE REPRESENTED UNITARILY

To begin with, let us recall a few basic facts about the Hilbert-space kinematics of decay processes.[14–19] Assume that the Hilbert spaces $\mathscr{H}_u$, and $\mathscr{H}$ referring to the unstable particle and a larger isolated system, respectively, and a strongly continuous unitary representation U of $\mathscr{P}$ on $\mathscr{H}$ are given. Let U_t denote the operators which represent the one-parameters subgroup of time translations, $U_t = \exp(-iHt)$. A natural requirement implied by the nonstability is

$$U_t \mathscr{H}_u \not\subset \mathscr{H}_u, \; t > 0 , \tag{2}$$

or more explicitly, there is no $t > 0$ for which $\mathscr{H}_u$ is invariant under U_t.

The *reduced propagator* is defined by $V_t = pr_u U_t$

$\equiv E_u U_t \restriction \mathscr{H}_u$, where E_u is the projection referring to $\mathscr{H}_u$. It is easy to see[17–20] that the function $t \to V_t$ is positive definite and continuous (weakly or strongly, it amounts to the same here), and fulfills $V_0 = I_u$. On the other hand, it appears that these properties of $\{V_t\}$ are sufficient to ensure existence of solution to the *inverse decay problem*, i.e., to reconstruct a triple $\{\mathscr{H}, U_t, E_u\}$ such that $V_t = pr_u U_t$ for all t, and moreover, that this solution is essentially unique under a natural minimality condition.[15] Technically these results are achieved by means of the unitary-dilations theory.[20]

Experience suggests that the operators V_t might fulfill the *semigroup condition* $V_t V_s = V_{t+s}$ for all $t,s \geq 0$. Unfortunately, in such a case the Hamiltonian H referring to the solution of the inverse decay problem contains the entire real axis in its spectrum.[21] Nonetheless, the semigroup reduced propagators represent a very useful approximation. The unphysical character of the energy spectrum causes no harm, since it has no observable consequences[18]; it may be removed when preparation of the unstable particle is completed by an energy-filtering procedure.[23,24] In fact, the inevitable deviations from the semigroup behavior are likely to be unobservable even if they are amplified by repeated measurements performed on the particle and an artifical energy filtering.[25]

Let us finally mention the definition of the *decay laws.* For an unstable particle which is described initially by a density matrix ρ, Ran $\rho \subset \mathscr{H}_u$, the nondecay probability equals $P_\rho(t) = \mathrm{Tr}(\rho V_t^* V_t)$. In particular, if the initial state is pure and described by a unit vector $\psi \in \mathscr{H}_u$, its decay law is

$$P_\psi(t) = \| V_t \psi \|^2 = \| E_u U_t \psi \|^2 . \tag{3a}$$

The situation is especially simple in the case of a one-dimensional $\mathscr{H}_u$ (spanned by ψ) when

$$P_\psi(t) = | v(t) |^2, \quad v(t) = (\psi, U_t \psi) ; \tag{3b}$$

the semigroup condition imposed on $\{V_t\}$ now requires the decay law (3b) to be exponential.

Now we shall return to the Poincaré group. Its space-time transformations are given by

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu , \tag{4}$$

where Λ belongs to SO(3,1) and a is a four-vector. For simplicity, we shall consider the connected component of $\mathscr{P}$ only avoiding discussion of the space and time inversions. The composition law of the transformation (4) implies

$$U(\Lambda,a) U(\Lambda',a') = U(\Lambda\Lambda', a + \Lambda a') \tag{5}$$

for all $(\Lambda,a),(\Lambda',a') \in \mathscr{P}$. Unitarity of U together with the definition (1) yields the relation

$$V(\Lambda,a)^* = V(\Lambda^{-1}, -\Lambda^{-1}a) \tag{6}$$

for all $(\Lambda,a) \epsilon \mathscr{P}$. Suppose that V fulfills the group law analogous to (5), then $V(\Lambda,a)^* V(\Lambda,a) = V(\Lambda,a) \times V(\Lambda,a)^* = I_u$ so $V(\Lambda,a)$ is unitary. However, this is equivalent to the fact that $U(\Lambda,a)$ commutes with E_u; particularly for the time translations, it would mean that the condition (2) was violated. Thus the operators $V(\Lambda,a)$ cannot fulfill the group composition law for all $(\Lambda,a) \in \mathscr{P}$, i.e., *$V$ cannot be a (nonunitary) representation of $\mathscr{P}$.*

This conclusion is not yet disastrous. Motivated by the above-sketched description of the time evolution, we are ready to accept the following possibility: there is a nonunitary representation $\tilde{V}$ of $\mathscr{P}$, presumably some of the ones mentioned in the Introduction, such that $\tilde{V}(\Lambda,a) = V(\Lambda,a)$ within some reasonable subset of $\mathscr{P}$, say $\mathscr{P}_+$. Unfortunately even this point of view cannot be retained. The reason is that *it does not respect the Euclidean invariance.* It is quite natural to assume that two observers, whose reference frames are obtained one from the other by space translations and rotations, will determine exactly the same decay law and other characteristics for a given unstable particle. Hence, in particular, the operators $V(I,a)$ with $a = (0, \vec{a})$ should be unitary, and this is not true for the representations which we have in mind.

Furthermore, the translational invariance implies that *the operators $V(\Lambda,0)$ referring to the pure Lorentz transformations (boosts) must not be unitary.* In order to see it,[26,38] notice that the relation (5) yields the identity

$$U(I,\Lambda a) U(\Lambda^{-1},0) U(I,-a) U(\Lambda,0) = U(I, \Lambda a - \Lambda^{-1} a) . \tag{7}$$

Let $\Lambda = \Lambda(\vec{\beta})$ be a boost with a velocity $\vec{\beta}$ and $a = (0,\vec{a})$, where $\vec{a}$ is parallel to $\vec{\beta}$. In such a case, one has

$$\Lambda a - \Lambda^{-1} a = (-2\xi |\vec{a}| \sinh |\vec{\beta}|, \vec{0}) , \quad \xi = \mathrm{sgn} \vec{\beta} \cdot \vec{a} . \tag{8}$$

We have pointed out that $V(\Lambda,a)$ is unitary for some (Λ,a) if and only if the corresponding $U(\Lambda,a)$ commutes with E_u. Thus if the boosts were represented unitarily, the same would be true for the right-hand side of (7). Since $\xi, |\vec{\beta}|, |\vec{a}|$ may be chosen arbitrarily, the relation (8) shows that E_u must commute with the operators representing time translations. Of course, this contradicts (2), so the conclusion is proved.

Notice finally that up to now no requirement specific for unstable particles was used. The above considerations apply therefore by the same right to free unstable nuclei and other decaying objects for which a relativistically covariant description is appropriate.

III. THE REPRESENTATIONS U RELATED TO UNSTABLE PARTICLES

Since the unstable particles may be characterized by spin quantum numbers, the most natural choice for U is a *direct integral over mass of the unitary irreducible representations* $U^{(m,s,+)}$.[1,27] The carrier space of such a representation is given by

$$\mathscr{H} = L^2 \left[[m_0, \infty) \times \mathrm{R}^3, dm \otimes \frac{d^3p}{2(m^2 + \vec{p}^{\,2})^{1/2}} \right] \otimes C^{2s+1} , \tag{9}$$

where m_0 is a threshold mass. It is useful sometimes to separate fully the kinematical variables from the mass. To this end, one has to employ the four-velocity $k = p/m$, i.e., to introduce the Hilbert space

$$\tilde{\mathscr{H}}=L^2([m_0,\infty))\otimes L^2(\mathrm{R}^3,d^3k/2k_0)\otimes C^{2s+1}\ , \qquad (10)$$

where $k_0=(1+\vec{k}^2)^{1/2}$; the two spaces are isomorphic by means of the relation

$$\tilde{\psi}_j(m,\vec{k})=m\psi_j(m,m\vec{k}) \qquad (11)$$

valid for all $j=-s,-s+1,\ldots,s$, $m\in[m_0,\infty)$, and $\vec{k}\in\mathrm{R}^3$.

The representation U acts on the space (9) according to the following prescription:

$$(U(\Lambda,a)\psi)(m,\vec{p})=e^{-ip\cdot a}S(m,s;\Lambda)\psi(m,\vec{p}_\Lambda)\ , \qquad (12)$$

where $a\cdot p=a_\mu p^\mu$, further $\vec{p}_\Lambda$ is the three-vector part of $\Lambda^{-1}p$, and the matrix S expresses by means of representations of the little group SU(2). For the space-time translation on $x=(t,\vec{x})$, we have $S=I$ so

$$(\varphi,U(I,x)\psi)=\sum_{j=-s}^{s}\int_{m_0}^{\infty}dm\int_{\mathrm{R}^3}\frac{d^3p}{2(m^2+\vec{p}^{\,2})^{1/2}}\exp\{-i[t(m^2+\vec{p}^{\,2})^{1/2}-\vec{x}\cdot\vec{p}]\}\bar{\varphi}_j(m,\vec{p})\psi_j(m,\vec{p})\ . \qquad (13)$$

In particular, for the pure time translations and $\varphi=\psi$ we have

$$(\psi,U_t\psi)=\sum_{j=-s}^{s}\int_{m_0}^{\infty}dm\int_{R^3}\frac{d^3p}{2(m^2+\vec{p}^{\,2})^{1/2}}\exp\{-it(m^2+\vec{p}^{\,2})^{1/2}\}\,|\psi_j(m,\vec{p})|^2\ . \qquad (14a)$$

Changing the variables (m,p) to (λ,p) with $\lambda=p_0=(m^2+\vec{p}^{\,2})^{1/2}$, we may rewrite the last expression in the form

$$(\psi,U_t\psi)=\int_{m_0}^{\infty}d\lambda\, e^{-i\lambda t}\left\{\sum_{j=-s}^{s}\int_{V_\lambda}\frac{d^3p}{2(\lambda^2-\vec{p}^{\,2})^{1/2}}\,|\psi_j((\lambda^2-\vec{p}^{\,2})^{1/2},\vec{p})|^2\right\}, \qquad (14b)$$

where $V_\lambda=\{\vec{p}:|\vec{p}|\le(\lambda^2-m_0{}^2)^{1/2}\}$.

IV. EFFECTIVE ONE-DIMENSIONALITY OF $\mathscr{H}_u$

Now the crucial point lies in the choice of the subspace $\mathscr{H}_u$ which would be ascribed to the unstable particle alone. If this space were one dimensional (spanned by some $\psi\in\mathscr{H}$), then (14) would yield according to (3b) the nondecay amplitude. However, we have argued above that $\mathscr{H}_u$ should be invariant particularly with respect to the space translations. This is impossible for a one-dimensional $\mathscr{H}_u$, because the momentum operators P_j have purely continuous spectra so ψ cannot be their eigenvector. Nevertheless, we are going to formulate an argument which shows that in most cases the relations (14) may be accepted as expressions of the nondecay amplitude in a reasonable approximation.

We shall consider first the scalar particles $s=0$. Our most important hypothesis is that there is a state of the unstable particle described by a wave function which factorizes,

$$\psi(m,\vec{p})=f(m)g(\vec{p})\ . \qquad (15)$$

Next we adopt various simplifying assumptions. First of all, we set

$$\mathrm{supp}\, f=(M-\eta,M+\eta)\subset[m_0,\infty)\ , \qquad (16a)$$

$$\int_{m_0}^{\infty}|f(m)|^2dm=\int_{M-\eta}^{M+\eta}|f(m)|^2dm=1\ , \qquad (16b)$$

where η is supposed to be a positive number much less than M. Further we assume

$$\mathrm{supp}\, g=B_\epsilon=\{\vec{p}:|\vec{p}|<\epsilon\} \qquad (16c)$$

so the support of g is centered at $\vec{p}=\vec{0}$. For small enough ϵ, this is practically equivalent to the assumption that the particle dwells in its rest system. According to (12), the space translations give $\psi_{\vec{a}}:\psi_{\vec{a}}(m,\vec{p})=e^{i\vec{a}\cdot\vec{p}}\psi(m,\vec{p})$ when acting on $\psi=\psi_{\vec{0}}$. Since $\psi_{\vec{a}}$ should belong to $\mathscr{H}_u$ for all $\vec{a}\in\mathrm{R}^3$, and the exponentials form a complete set in $L^2(B_\epsilon)$, we may set

$$\mathscr{H}_u=\{\psi:\psi(m,\vec{p})=f(m)g(\vec{p}),g\in L^2(B_\epsilon)\}\ . \qquad (17)$$

As a set, this $\mathscr{H}_u$ coincides with $C(f)\otimes L^2(B_\epsilon)$, where $C(f)$ is the complex linear span of f. The scalar product is, however, different: the norm of ψ is according to (9) and (16) given by

$$||\psi||^2=\int_{M-\eta}^{M+\eta}dm\,|f(m)|^2\int_{B_\epsilon}\frac{d^3p}{2(m^2+\vec{p}^{\,2})^{1/2}}\,|g(\vec{p})|^2\ . \qquad (18a)$$

Let $||\cdot||_2$ denote the norm in $L^2(B_\epsilon)$:

$$||g||_2^2=\int_{B_\epsilon}|g(\vec{p})|^2d^3p\ . \qquad (18b)$$

We may use it to estimate the norm (18a) from both sides, or vice versa, to derive the inequalities

$$2(M-\eta)||\psi||^2\le||g||_2{}^2\le2[(M+\eta)^2+\epsilon^2]^{1/2}||\psi||^2\ . \qquad (19)$$

It shows particularly that $\{\psi_n\}\subset\mathscr{H}_u$ is a Cauchy sequence if and only if the same is true for the corresponding sequence $\{g_n\}\subset L^2(B_\epsilon)$; hence $\mathscr{H}_u$ defined by (17) is a (closed) subspace in $\mathscr{H}$. The inequality (33a) below shows that $\epsilon\ll M$ and the same restriction was imposed on η, so the function g corresponding to a unit vector $\psi\in\mathscr{H}_u$ fulfills $||g||_2^2\approx(2M)^{1/2}$.

Let us inspect now the action of the time-translation operators on a unit ψ from the chosen subspace (17). According to (12), they multiply $\psi(m,\vec{p})$ by $\exp\{-it(m^2+\vec{p}^{\,2})^{1/2}\}$. The expression does not factorize, but for ϵ small enough one may try to approximate it by e^{-imt}. Since $\epsilon\ll M$,[28] we may restrict ourselves to the first two terms of the expansion

$$\exp\{-it(m^2+\vec{p}^{\,2})^{1/2}\}=e^{-imt}\left[1-i\frac{\vec{p}^{\,2}t}{2m}+O(\vec{p}^{\,4})\right] . \tag{20}$$

The evolution operator is correspondingly written as $U_t=U_t^{(0)}+U_t^{(1)}$ neglecting the remainder. In order to estimate the influence of the second term, we take an arbitrary unit vector $\varphi\in\mathscr{H}_u$, $\varphi(m,\vec{p})=f(m)h(\vec{p})$, and express

$$(\varphi,U_t^{(1)}\psi)=\int_{M-\eta}^{M+\eta}dm\, e^{-imt}|f(m)|^2\int_{B_\epsilon}\frac{d^3p}{2(m^2+\vec{p}^{\,2})^{1/2}}\left[-i\frac{\vec{p}^{\,2}t}{2m}\right]\bar{h}(\vec{p})g(\vec{p}) . \tag{21}$$

The relations (16), (19), and (21) yield the following inequalities:

$$\begin{aligned}|(\varphi,U_t^{(1)}\psi)| &\le \frac{1}{2(M-\eta)}\frac{\epsilon^2 t}{2(M-\eta)}\|h\|_2\|g\|_2\\ &\le\frac{[(M+\eta)^2+\epsilon^2]^{1/2}}{M-\eta}\frac{\epsilon^2 t}{2(M-\eta)} .\end{aligned} \tag{22}$$

Hence we may estimate the norm

$$\|E_u U_t^{(1)}\psi\|=\sup\{\,|(\varphi,U_t^{(1)}\psi)|:\varphi\in\mathscr{H}_u,\|\varphi\|=1\} . \tag{23}$$

Since both ϵ,η are supposed to be much less than M, we find (23) to be $\lesssim\epsilon^2 t/2M$; the approximation mentioned above is therefore possible under the condition

$$\frac{\epsilon^2 t}{2M}\ll 1 . \tag{24}$$

In such a case, a norm of the difference between $E_u U_t\psi=V_t\psi$ and $E_u U_t^{(0)}\psi$ is very small, and we are allowed to write $V_t\psi\approx E_u U_t^{(0)}\psi$.

In the next step, we shall verify that the last expression is close to $(\psi,U_t^{(0)}\psi)\psi$. To this end, we take an arbitrary unit vector $\varphi\in\mathscr{H}_u,\varphi(m,\vec{p})=f(m)h(\vec{p})$, which is perpendicular to ψ. The orthogonality of φ,ψ together with (16b) makes it possible to estimate $(h,g)_2$ from the identity

$$\frac{1}{2M}(h,g)_2=\int_{M-\eta}^{M+\eta}dm\,|f(m)|^2\int_{B_\epsilon}\left[\frac{1}{2M}-\frac{1}{2(m^2+\vec{p}^{\,2})^{1/2}}\right]\bar{h}(p)g(p)d^3p . \tag{25}$$

Since ϵ,η are much less than M, we have the following estimate:

$$\left|\frac{1}{2M}-\frac{1}{2(m^2+\vec{p}^{\,2})^{1/2}}\right|\lesssim\frac{1}{2M}\left[\frac{\eta}{M}+\frac{\epsilon^2}{2M^2}\right] \tag{26}$$

(up to higher-order terms). Combining it with the Hölder inequality, we obtain

$$|(h,g)_2|\lesssim\left[\frac{\eta}{M}+\frac{\epsilon^2}{2M^2}\right]\|h\|_2\|g\|_2 . \tag{27}$$

Now we are able to estimate the scalar product $(\varphi,U_t^{(0)}\psi)$:

$$\begin{aligned}|(\varphi,U_t^{(0)}\psi)| &\le\left|\frac{1}{2M}\int_{M-\eta}^{M+\eta}dm\,|f(m)|^2e^{-imt}(h,g)_2\right|\\ &\quad+\int_{M-\eta}^{M+\eta}dm\,|f(m)|^2\int_{B_\epsilon}\left|\frac{1}{2(m^2+\vec{p}^{\,2})^{1/2}}-\frac{1}{2M}\right||h(\vec{p})|\,|g(\vec{p})|d^3p .\end{aligned} \tag{28}$$

Applying (25) and (26) to the second term and the first term on the right-hand side of (28), respectively, and using the Hölder inequality again, we get

$$|(\varphi,U_t^{(0)}\psi)|\lesssim\frac{2}{2M}\left[\frac{\eta}{M}+\frac{\epsilon^2}{2M^2}\right]\|h\|_2\|g\|_2\int_{M-\eta}^{M+\eta}|f(m)|^2dm .$$

However, φ and ψ are assumed to be unit vectors so $\|h\|_2\approx\|g\|_2\approx(2M)^{1/2}$. Finally, the normalization condition (16b) yields

$$|(\varphi,U_t^{(0)}\psi)|\lesssim 2\frac{\eta}{M}+\frac{\epsilon^2}{M^2} . \tag{29}$$

Since φ is an arbitrary unit vector from $\mathscr{H}_u$ orthogonal to ψ, we see that $U_t^{(0)}\psi$ stays nearly parallel to ψ. Hence we may write

$$(V_t\psi)(m,\vec{p})\approx\psi(m,\vec{p})\int_{m_0}^{\infty}d\mu\, e^{-i\mu t}|f(\mu)|^2\int_{B_\epsilon}\frac{d^3\kappa}{2(m^2+\vec{\kappa}^2)^{1/2}}|g(\vec{\kappa})|^2 . \qquad (30a)$$

Moreover, the inequality (26) allows us to replace the denominator in the last integral by $2M$; the corresponding error is again at most comparable with the right-hand side of (27). Thus we have also

$$(V_t\psi)(m,\vec{p})\approx\psi(m,\vec{p})\int_{m_0}^{\infty}e^{-i\mu t}|f(\mu)|^2 d\mu . \qquad (30b)$$

Concluding the above discussion, we may say that *if the three-momentum spread of ψ is sufficiently narrow, the decay goes effectively as if $\mathscr{H}_u$ would be one dimensional.* In that case, the nondecay amplitude is given by (14), and it may be approximated by the integrals appearing in (30). Of course, the approximation also needs $\eta \ll M$ but this can be achieved as we shall see in a while.

The presented argument generalizes easily for particles with a nonzero spin. One has only to use the rotational invariance of $\mathscr{H}_u$ too, then the following choice is natural:

$$\mathscr{H}_u=\{\psi:\psi(m,\vec{p})=f(m)g(\vec{p}),g\in L^2(B_\epsilon)\otimes C^{2s+1}\} . \qquad (31)$$

Mimicking the above reasoning, we arrive again at the approximation (30b).

Hence we must ask under which circumstances the conditions (24) and $\eta \ll M$ are valid. In any realistic description of unstable particles, the function $|f(\cdot)|^2$ should have a sharp peak of more or less Breit-Wigner shape. Its position may be identified with the mass M of the particle. On the other hand, the mean life is defined by

$$T=\int_0^{\infty}P_\psi(t)dt ; \qquad (32)$$

its inverse Γ characterizes width of the peak. For all real unstable particles, M is much larger than Γ: the ratio M/Γ varies from 1.06×10^5 for Σ^0 to 1.31×10^{27} for neutrons (with the exception of π^0, η, and Σ^0, its lower bound is 10^{11}). Hence we can choose the parameter η so the inequalities $\Gamma \ll \eta \ll M$ hold. The first one of them ensures that truncation of the mass distribution $|f(\cdot)|^2$ to the interval $(M-\eta, M+\eta)$ will cause a negligible change in the decay law.[29,30]

Of course, the condition (24) cannot hold for all values of t, but it is reasonable to demand its validity in the region where the decay law is actually measured, i.e., up to few T. Thus the three-momentum spread $\Delta p\equiv\epsilon$ must obey $(\Delta p)^2 \ll M\Gamma$ or

$$\Delta p \ll c^{-1}(M\Gamma)^{1/2} \qquad (33a)$$

when we return to the conventional system of units.[28] In order to appreciate this restriction, let us rewrite it by the uncertainty relation to the form[39]

$$\Delta q \gg \hbar c(M\Gamma)^{-1/2} . \qquad (33b)$$

Thus we come to the following result: the conclusion about the effectively one-dimensional $\mathscr{H}_u$ is applicable *provided the unstable particle is not spatially localized so sharply that (33b) is violated.* However, this condition is fulfilled almost always in actual experimental arrangements as shown in the listed values shown[31] in Table I.

Notice finally, that the above considerations apply to the "coordinate" part of the wave function only. If the part of the decay problem related to internal degrees of freedom cannot be decomposed completely, we have $\dim\mathscr{H}_u>1$ even in the sense of the discussed approximation. So for the neutral kaons, e.g., the space $\mathscr{H}_u$ is effectively two dimensional provided the conditions (33) are valid.

TABLE I. Values of $\hbar c\ (M\Gamma)^{-1/2}$ for the metastable particles.

Particle	$\hbar c(M\Gamma)^{-1/2}$ (cm)
μ	1.11×10^{-4}
τ	$1.2\ \times10^{-8}$
$\pi^{\pm}$	1.05×10^{-5}
π^0	6.03×10^{-10}
η	$1.3\ \times10^{-11}$
$K^{\pm}$	3.85×10^{-6}
K_S	3.26×10^{-7}
K_L	7.85×10^{-6}
$D^{\pm}$	$1.7\ \times10^{-8}$
D^0	$1.2\ \times10^{-8}$
F	$8.0\ \times10^{-8}$
n	0.763
Λ	3.74×10^{-7}
Σ^+	3.53×10^{-7}
Σ^0	$5.4\ \times10^{-12}$
Σ^-	2.71×10^{-7}
Ξ^0	$3.6\ \times10^{-7}$
Ξ^-	2.71×10^{-7}
Ω^-	1.70×10^{-7}
Λ_c^+	$5.3\ \times10^{-9}$

V. DECAY OF A MOVING PARTICLE

We are obliged to show that the proposed description by means of the representation (12) and its restriction to a subspace of the type (31) will yield a correct result for an unstable particle which is not at rest. Let a reference frame S belong to the observer, and suppose the rest system of the particle moves with a velocity $\vec{\beta}$, relative to S (Fig. 1). Of course, we may not only sandwich the propagator between $U(\Lambda(\pm\vec{\beta}),0)$; similarly as a simple-minded look on the factor which multiplies the time variable in

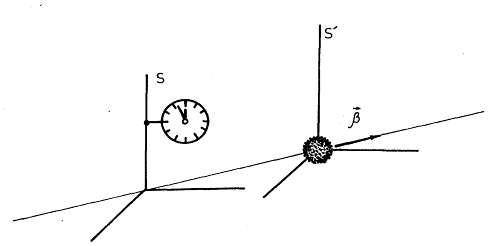

FIG. 1. Decay of a moving particle.

Lorentz transformation does not yield the time dilatation. From the viewpoint of the reference frame S, we are interested in the space-time shift on $x=(t,\vec{\beta}t)$. If the condition (33a) is valid, i.e., if we are allowed to characterize the particle by a single vector $\psi\in\mathscr{H}$ which refers to its rest system, then the observer will ascribe to it the vector $U(\Lambda(\vec{\beta}),0)^{-1}\psi$. The corresponding nondecay amplitude equals

$$v(t;\vec{\beta})=(U((\vec{\beta}),0)^{-1}\psi,U(I,x)U(\Lambda(\vec{\beta}),0)^{-1}\psi) . \tag{34}$$

Using the relations (5) and (13), we may rewrite (34) as follows:

$$v(t;\vec{\beta})=(\psi,U(I,\Lambda(\vec{\beta})x)\psi)=\sum_{j=-s}^{s}\int_{m_0}^{\infty}dm\int_{\mathrm{R}^3}\frac{d^3p}{2(m^2+\vec{p}^{\,2})^{1/2}}\exp\{-ip\cdot\Lambda(\vec{\beta})x\}\,|\psi_j(m,\vec{p})|^2 .$$

However, the Lorentz transformation gives $\Lambda(\vec{\beta})x=(t(1-\vec{\beta}^{\,2})^{1/2},\vec{0})$ so

$$v(t;\vec{\beta})=v(t(1-\vec{\beta}^{\,2})^{1/2};\vec{0}) . \tag{35}$$

This provides us with the relation

$$P_\psi(t;\vec{\beta})=P_\psi(t(1-\vec{\beta}^{\,2})^{1/2};\vec{0}) , \tag{36}$$

which is valid as far as the approximation identifying the decay law with the square of (34) may be used. The relation (36) is, of course, the desired result. It is tested by numerous experiments; and it was even used for a direct proof of the relativistic time dilatation from cosmic-ray muons 30 years ago.[32]

VI. CONCLUSIONS

Let us compare the above-discussed description of unstable particles with the one based on nonunitary representations of $\mathscr{P}$. We have already mentioned the Williams' construction[14] of minimal unitary dilation for the nonunitary representation proposed by Zwanziger.[3] He obtained the Hilbert space (10) with $m_0=-\infty$ and a unitary representation $\tilde{U}$ of $\mathscr{P}$ on $\tilde{\mathscr{H}}$ which coincides with (12) when transformed by means of (11). The principal difference concerns the choice of $\mathscr{H}_u$: Zwanziger's representation is recovered by projection of $\tilde{U}$ to the subspace

$$\tilde{\mathscr{H}}_u^Z=C(f)\otimes L^2(\mathrm{R}^3,d^3k/2k_0)\otimes C^{2s+1} , \tag{37}$$

where $f(m)=(2\pi/\Gamma)^{1/2}(m-M+\frac{1}{2}i\Gamma)^{-1}$.

Williams himself regarded the below-unbounded mass spectrum as the main defect, but it can be rectified by a mass-filtering procedure without any observable consequence[23–25]; essentially the same argument we used in condition (16a). Excepting that, in a theory pretending for completeness the function f should be obtained as a solution to the dynamical problem, with the Breit-Wigner shape of $|f(\cdot)|^2$ resulting from the pole approximation to this solution. However, it seems that we will not have such a theory soon. In spite of substantial progress achieved in the perturbation theory of embedded eigenvalues during the last decade,[33] one can hardly proceed beyond the Fermi golden rule since even finding the "unperturbed" eigenvalues represents a difficult problem for the theory of strong interactions.

A difference between the two approaches is now obvious. In both of them, it is only the mass distribution which is essential for expression of the decay law, while effect of the momentum (velocity) dependence of the wave function is suppressed. In the approach treated here, this conclusion is obtained by realizing that the momentum distribution is actually very narrow.[34] On the contrary, with the choice (37) the mentioned independence is achieved because it makes all velocity distributions possible. Both the approaches yield the same decay laws as far as a fixed reference frame is considered, simply because they have been constructed so. However, the first one has the advantage of producing the translationally invariant description.

One might say that in a subspace $\mathscr{H}_u$ of the type (37) a lot of space is left unemployed. The presented quantitative considerations show that what one really needs is[35]

$$\tilde{\mathscr{H}}_u'=C(f)\otimes L^2(B_\kappa,d^3k/2k_0)\otimes C^{2s+1} , \tag{38}$$

where $\kappa\approx\epsilon/M$. The subspace $\mathscr{H}_u'\subset\mathscr{H}$ isomorphic to (38) though (11) is "intermediate" in a sense between (31) and $\mathscr{H}_u^Z$ referring to (37). For $\mathscr{H}_u'$, one can derive a conclusion similar to that of Sec. IV with more ease. On the other hand, (38) is no longer translationally invariant, though the violation is manifested on large distances only.[36]

Finally, let us mention that frequently the possibility of neglecting the $\vec{p}$ spread of the wave function is even better than the condition (33b) together with the data shown in Table I. We have in mind the situation when the unstable particle suffers repeated nondecay measurements,[22,25] e.g., by monitoring its track. Since the decay starts anew after each measurement (which has given the positive result), we need not require (24) to hold for times comparable with Γ^{-1} but merely with the mean time between the neighboring measurements which is usually a few orders of magnitude shorter. As an example, consider the decay of charged kaons in a bubble chamber treated in Ref. 25: there the mean time between the measurements is $\sim10^{-4}\Gamma^{-1}$. Instead of (33b), we obtain then the condition $\Delta q\gg10^{-8}$ cm, but actually the kaons are localized within the range of bubble diameter, i.e., about 10^{-2} cm. Similar conclusions may be obtained for the other unstable particles and track-monitoring devices too. On the other hand, the conclusion about the effective one dimensionality of $\mathscr{H}_u$ can be used to justify the basic reduction postulate of the repeated-measurements theories.

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[1]Classification of the unitary irreducible representations of $\mathscr{P}$ started from the paper by E. P. Wigner, Ann. Math. 40, 149 (1939). For their description and application to classification of elementary particles see, e.g., Ref. 2, Sec. 17.2.

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[9]Besides the mentioned irreducible representations, some nonunitary indecomposable representations (i.e., reducible but not completely reducible) were proposed for description of unstable particles: see Refs. 10 and 11 below and Ref. 2, Sec. 17.4. However, these attempts aimed mainly to yield decay laws related to higher-order poles which have never been observed.

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[21]The semigroup condition is equivalent to the absence of regeneration. The conclusion about the spectrum remains valid even if the regeneration ceases after a finite time, see Ref. 17. A necessary and sufficient condition for energy semiboundedness was given in Ref. 19. The problem has been discussed many times; for a more complete biblography see Ref. 22.

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[26]A formal Lie-algebraic version of the following argument was given in Ref. 18.

[27]The same U was used in Refs, 12 and 13, but it was not accompanied there by a Euclidean-invariant choice of $\mathscr{H}_u$.

[28]According to (33a), we have $\epsilon^2 \ll M\Gamma$, and therefore in most cases $(\epsilon/M)^2 \ll, 10^{-11}$.

[29]See, e.g., Refs. 22 and 25. Simple estimates similar to those performed there can show that for all practical purposes it is enough to choose $\eta \approx 10^2\Gamma$. Thus we may assume $\eta/M \lesssim 10^{-9}$ in most cases. In fact, truncation of the mass distribution might change T substantially, because the modified decay law has asymptotically a powerlike decrease, eventually as t^{-1}. However, from the practitioner's point of view the infinity may be replaced in the integral (32) by the range where the decay law is actually measured, say $10\Gamma^{-1}$, so the tail effect is suppressed. Notice that validity of the condition (24) may be discussed under a similar restriction on t.

[30]We restrict our attention to real unstable (metastable) particles. Our assumptions may not be fulfilled for scattering resonances. However, time evolution of a resonance as a separate object cannot be studied experimentally (as noticed by many authors, e.g., in Ref. 12) and even associating some $\mathscr{H}_u$ with it is a speculative matter. On the other hand, there are many similarities between the unstable particles and resonances, mainly because the underlying dynamical mechanism is mostly of the same type. See Refs. 15 and 33 or F. Coester and L. Schlessinger, Ann, Phys. (N.Y.) 78, 90 (1973). A more detailed discussion may be found in a forthcoming monograph—Ref. 37.

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[34]This fact was already noticed, particularly in Ref. 12. However, the main problem is to use this observation to prove that the decay law defined naturally by (3a) may be approximated by the much more simpler expression which follows from (3b) and (30b).

[35]Of course, the condition (33b) does not require the momentum (velocity) distribution to be supported by some ball. In order to take the possible tails of these distributions into account (preserving at the same time the translational invariance of $\mathscr{H}_u$), a mathematically more sophisticated treatment is needed; we hope to discuss it elsewhere.

[36]Assume again the condition (16a) to be fulfilled. Then it is easy to see that the noninvariance becomes essential for $\hbar^{-1}(\eta/M)a\Delta p \gtrsim 1$, i.e., $a \gtrsim (M/\eta)\Delta q$. Thus if $M/\eta \gtrsim 10^9$ and $\Delta q = 10^{-2}$ cm, we get $a \gtrsim 100$ km.

[37]P. Exner, *Open Quantum System and Feynman Integrals* (Reidel, Dordrecht, to be published).

[38]Since the argument presented here employs, in fact, the group composition law of $\mathscr{P}$ only, it is not surprising that similar conclusions can appear in a different context. See F. Coester and W. N. Polyzou, Phys. Rev. D 26 1348 (1982).

[39]It is clear from the above considerations that the position spread Δq here refers to the rest system of the particle.