Covariant quantization of the string in dimensions $D \leq 26$ using a Becchi-Rouet-Stora formulation

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The conventional open string theory is cast into the Becchi-Rouet-Stora (BRS) formulation of Fradkin and Vilkovisky. Upon quantizing this theory, the nilpotency condition $\hat{Q}^2=0$ of the BRS charge requires $D=26$ and $\alpha_0=1$ for the space-time dimension D and intercept parameter α_0. Unitarity is proved by relating this theory to the old covariant quantization. The string theory is then generalized as suggested by Polyakov, taking into account the conformal or trace anomaly by $\mathscr{L}=\mathscr{L}_{\text{string}}+C\mathscr{L}_1$, where $\mathscr{L}_1$ yields the Liouville equation in the orthonormal gauge. Under the assumption that the exact quantization of Liouville's equation does not yield any additional anomalies, we show that the condition $\hat{Q}^2=0$ implies $C=(D-26)/48\pi$, in agreement with Polyakov's result, and the "intercept parameter" $\beta=(D-2)/24$.

I. INTRODUCTION

Due to the work of Polyakov,[1] a consistent quantum theory of the bosonic string might be found even when the dimension of space-time, D, is less than 26. It is the purpose of this paper to cast some further light upon this subject, by making a covariant quantization in phase space in a Becchi-Rouet-Stora (BRS) formalism. The starting point for our treatment is the Lagrangian density

$$\mathscr{L}(\tau,\sigma)=\frac{1}{2N}\sqrt{-g}\,g^{\alpha\beta}\partial_\alpha y^\mu\partial_\beta y_\mu+C\mathscr{L}_1\ , \tag{1.1}$$

where y^μ are the coordinates (in Minkowski space) of the string; $g^{\alpha\beta}$ is the metric tensor of the parameter space $\alpha,\beta=0,1$; $\partial_0\equiv\partial_\tau$, $\partial_1\equiv\partial_\sigma$; $g\equiv\det g_{\alpha\beta}$, and N is a normalization constant. $C\mathscr{L}_1$, where C is a constant, is the term needed in order to take into account the conformal or trace anomaly. We use the explicit expression for $\mathscr{L}_1$ found in Ref. 2, which is a local form of Ref. 1. We will first consider the old string theory, i.e., $C\equiv 0$. The BRS quantization there will be shown to be consistent if $D=26$ and α_0, the intercept parameter, equals one. With the formalism developed we generalize to arbitrary C and find that only for a specific value of C, $C=(D-26)/48\pi$, can we make a consistent quantization.

In Sec. II we will first give a short review of the BRS formalism used, namely, that of Fradkin and Vilkovisky.[3] In Sec. III we develop this formalism for the ordinary string and quantize it in Sec. IV. The generalization to arbitrary C is made in Sec. V and the quantization in Sec. VI. Summary and conclusions are given in Sec. VII. In an appendix we show the connection of our BRS formulation to the old covariant quantization, which also proves unitarity.

II. THE BRS FORMALISM IN PHASE SPACE

Kato and Ogawa[4] showed that a BRS quantization of the ordinary string is only possible for $D=26$. They use the formalism of Kugo and Ojima[5] to show this. We will instead use here the formalism developed by Fradkin and Vilkovsky[3] and arrive at the same result. This formalism starts in phase space, which turns out to simplify the calculations. Also the question of unitarity and equivalence to the standard theory is here more easily established (see Appendix).

Fradkin and Vilkovisky showed that for a system of first-class constraints ψ_a in a general gauge theory, satisfying the closed algebra

$$\{\psi_a,\psi_b\}_\pm=\psi_c U^c_{ab}\ , \tag{2.1}$$

$$\{H_0,\psi_a\}_\pm=\psi_b V^b_a\ , \tag{2.2}$$

where $H=H_0+\lambda^a\psi_a$ is the Hamiltonian of the system (λ^a are arbitrary functions), we have a BRS charge and BRS-invariant Hamiltonian satisfying $\{Q,Q\}_+=\{H_\phi,Q\}_\pm=0$ given by

$$Q=\psi_a\eta^a+\tfrac{1}{2}(-1)^{n_a}\mathscr{P}_c U^c_{ab}\eta^b\eta^a\ , \tag{2.3}$$

$$H_\phi=H_0+\mathscr{P}_a V^a_b\eta^b+\{\phi,Q\}_\pm\ , \tag{2.4}$$

where

$\eta^a,\mathscr{P}_a$ are phase-space variables of the opposite Grassmann type to ψ_a, satisfying $\{\eta^a,\mathscr{P}_b\}_\pm=\delta^a_b$,

$$n_a=\begin{cases}0 \text{ for } \psi_a \text{ bosonic}\ ,\\ 1 \text{ for } \psi_a \text{ fermionic}\ ,\end{cases}$$

ϕ is a gauge-fixing function .

When quantizing this system, we should according to Dirac project out physical states by

$$\hat{\psi}_a\,|\,\text{phys}\rangle=0\ \forall a\ . \tag{2.5}$$

The great advantage of the BRS quantization procedure is that one may exchange all the conditions above with a single one,

$$\hat{Q}\,|\,\text{phys}\rangle=0\ , \tag{2.6}$$

provided that at the quantum level we still have

$$\hat{Q}^2 \,|\, \text{state}\rangle \equiv 0 \,. \tag{2.7}$$

That this covariant quantization using conditions (2.6) and (2.7) is equivalent to the constraint conditions (2.5) has been shown by Marnelius in a simple fashion in Ref. 6, based upon the formulation of Fradkin and Vilkovisky. In the Appendix we will give a review of the basic points.

Since Q in its structure contains all information about the physical states of the system, the nilpotency of Q, (2.7), is a crucial test of the quantum properties of a classical system. We shall verify this for the bosonic string, first for the "old" theory and then for Polyakov's modified theory.

III. THE BRS FORMULATION OF THE ORDINARY BOSONIC STRING

For the ordinary string we have the Lagrangian density (taking $N \equiv 1$ for simplicity)

$$\mathscr{L}_0(\tau,\sigma) = \tfrac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\partial_\alpha y^\mu \partial_\beta y_\mu \,, \tag{3.1}$$

where $g^{\alpha\beta}$ are to be treated as three dynamical variables ($g^{\alpha\beta} = g^{\beta\alpha}$). $\alpha = 0,1$, $\partial_0 \equiv \partial_\tau, \partial_1 \equiv \partial_\sigma$, and $g \equiv \det g_{\alpha\beta}$. This Lagrangian density implies the constraints

$$\begin{aligned} &p_{g\alpha\beta} = 0 \,, \\ &p\cdot y' \equiv p\cdot\partial_1 y = 0 \,, \\ &p^2 + y'^2 = 0 \,, \end{aligned} \tag{3.2}$$

where p_μ are conjugate to y^μ satisfying

$$\{y^\mu(\sigma), p_\nu(\sigma')\}_{\tau=\tau'} = \delta^\mu_\nu \delta(\sigma-\sigma')$$

and $p_{g\alpha\beta}$ are conjugate to $g^{\alpha\beta}$. The last two constraints may be rewritten as

$$\begin{aligned} L_+(\tau,\sigma) &\equiv \tfrac{1}{4}(p+y')^2 = 0 \,, \\ L_-(\tau,\sigma) &\equiv \tfrac{1}{4}(p-y')^2 = 0 \,. \end{aligned} \tag{3.3}$$

L_+, L_-, and $p_{g\alpha\beta}$ constitute a set of first-class constraints satisfying the algebra

$$\begin{aligned} \{L_+(\sigma), L_+(\sigma')\} &= [L_+(\sigma) + L_+(\sigma')]\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') \,, \\ \{L_-(\sigma), L_-(\sigma')\} &= -[L_-(\sigma) + L_-(\sigma')]\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') \,. \end{aligned} \tag{3.4}$$

All other Poisson brackets are zero.

Using the general expression (2.3), we may now construct a BRS charge Q. One slight difficulty here is that the structure coefficients U^a_{bc} are differential operators, as can be seen from (3.4). The BRS charge may anyway be found to be

$$Q = \int_0^\pi d\sigma (L_+\eta^1 + L_-\eta^2 + \mathscr{P}_1\partial_1\eta^1\eta^1 - \mathscr{P}_2\partial_1\eta^2\eta^2) \tag{3.5}$$

with the desired property $\{Q,Q\}_+ = 0$. The coordinates η^1, η^2, $\mathscr{P}_1$, and $\mathscr{P}_2$ are of odd Grassmann type since $L_\pm$ are even. They satisfy thus

$$[\eta^i,\eta^j]_+ = [\eta^i,\mathscr{P}_j]_+ = [\mathscr{P}_i,\mathscr{P}_j]_+ = 0 \tag{3.6}$$

and the fundamental Poisson bracket: $\{\eta^i(\sigma), \mathscr{P}_j(\sigma')\}_{+\tau=\tau'} = \delta^i_j\delta(\sigma-\sigma')$. Here we have imposed the conformal or orthonormal (ON) gauge

$$g^{\alpha\beta} = \rho(\tau,\sigma)\eta^{\alpha\beta} \,, \quad \eta^{00} = -\eta^{11} = 1 \,, \quad \eta^{01} = 0 \,, \tag{3.7}$$

where ρ is an arbitrary function, so that the constraints $p_{g\alpha\beta}$ are eliminated.

From the Lagrangian density (3.1) we find the Hamiltonian

$$H_0(\tau) = 0 \tag{3.8}$$

which implies that in order to find a BRS-invariant Hamiltonian, we need only the last term of (2.4),

$$H_\phi = \{\phi, Q\}_+ \tag{3.9}$$

which is automatically BRS invariant due to the Jacobi identity. Within the ON gauge the ordinary part of the Hamiltonian is given by

$$H(\tau) = \tfrac{1}{2}\int_0^\pi d\sigma (p^2 + y'^2) = \int_0^\pi d\sigma (L_+ + L_-) \,. \tag{3.10}$$

Therefore we must specify ϕ so that the bosonic part of H_ϕ is H. Taking

$$\phi = \int_0^\pi d\sigma (\mathscr{P}_1 + \mathscr{P}_2) \tag{3.11}$$

we find

$$\begin{aligned} H_\phi = \{\phi, Q\} &= \int_0^\pi d\sigma \int_0^\pi d\sigma' \{(\mathscr{P}_1 + \mathscr{P}_2)(\sigma), (L_+\eta^1 + L_-\eta^2 + \mathscr{P}_1\eta^{1'}\eta^1 - \mathscr{P}_2\eta^{2'}\eta^2)(\sigma')\}_+ \\ &= \int_0^\pi d\sigma (L_+ + L_- + 2\mathscr{P}_1\eta^{1'} + \mathscr{P}_1'\eta^1 - 2\mathscr{P}_2\eta^{2'} - \mathscr{P}_2'\eta^2) \\ &= H + (\text{ghost terms}) \,. \end{aligned} \tag{3.12}$$

From (3.12) we get the equations of motion:

$$\begin{aligned} &\ddot{y}^\mu - y''^\mu = 0 \,, \\ &\dot{\eta}^1 = \eta^{1'} \,, \quad \dot{\eta}^2 = -\eta^{2'} \,, \\ &\dot{\mathscr{P}}_1 = \mathscr{P}_1' \,, \quad \dot{\mathscr{P}}_2 = -\mathscr{P}_2' \,. \end{aligned} \tag{3.13}$$

These equations imply for the ghost coordinates

$$\begin{aligned} &\eta^1(\tau,\sigma) = \eta^1(\xi) \,, \quad \eta^2(\tau,\sigma) = \eta^2(\zeta) \,, \\ &\mathscr{P}_1(\tau,\sigma) = \mathscr{P}_1(\xi) \,, \quad \mathscr{P}_2(\tau,\sigma) = \mathscr{P}_2(\zeta) \,, \end{aligned} \tag{3.14}$$

where $\xi \equiv \tau + \sigma, \zeta \equiv \tau - \sigma$. The action in phase-space coordinates becomes

$$S=\int d\tau \int_0^{\pi} d\sigma(p\cdot\dot{y}+\mathscr{P}\cdot\dot{\eta}-\mathscr{H}_{\phi}) , \tag{3.15}$$

where

$$H_{\phi}=\int_0^{\pi} d\sigma \mathscr{H}_{\phi} .$$

Varying S we find the boundary conditions

$$\partial_1 y^{\mu}=0 , \quad \sigma=0,\pi ,$$

$$(2\mathscr{P}_1 d\eta^1-2\mathscr{P}_2 d\eta^2+d\mathscr{P}_1\eta^1-d\mathscr{P}_2\eta^2)\,|_{\sigma=0,\pi}=0 .$$

In order to satisfy these, we must take

$$\eta^1=\pm\eta^2 ,$$

$$\mathscr{P}_1=\pm\mathscr{P}_2 ,$$

at $\sigma=0,\pi$, to get nontrivial η^1, η^2, $\mathscr{P}_1$, and $\mathscr{P}_2$. This, together with the solutions of the equations of motion (3.13), implies that

$$\eta^1=\eta(\xi) , \quad \eta^2=_{(-)}^{\,+}\eta(\zeta) ,$$

$$\mathscr{P}_1=\mathscr{P}(\xi) , \quad \mathscr{P}_2=_{(-)}^{\,+}\mathscr{P}(\zeta) ,$$

where η and $\mathscr{P}$ are periodic functions of period 2π. We may disregard the minus sign without any loss of generality.

We shall now Fourier decompose the BRS charge. The constraints $L_{\pm}$ become just

$$L_{+}(\tau,\sigma)=L(\xi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} L_n e^{-in\xi} ,$$

$$L_{-}(\tau,\sigma)=L(\zeta)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} L_n e^{-in\zeta} , \tag{3.16}$$

where

$$L_n=\int_{-\pi}^{\pi} d\xi\, e^{in\xi}L(\xi) \tag{3.17}$$

are the Virasoro coefficients satisfying the algebra

$$\{L_n,L_m\}=-i(n-m)L_{n+m} . \tag{3.18}$$

For the ghost variables we get

$$\eta(\xi)=\sum_{n=-\infty}^{\infty}\eta_n e^{-in\xi} ,$$

$$\mathscr{P}(\xi)=\frac{i}{2\pi}\sum_{n=-\infty}^{\infty}\mathscr{P}_n e^{-in\xi} , \tag{3.19}$$

where

$$\eta_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\xi\, e^{in\xi}\eta(\xi) ,$$

$$\mathscr{P}_n=-i\int_{-\pi}^{\pi} d\xi\, e^{in\xi}\mathscr{P}(\xi) , \tag{3.20}$$

$$\{\eta(\xi),\mathscr{P}(\xi')\}_{+}=\delta(\xi-\xi') \Longrightarrow \{\eta_n,\mathscr{P}_m\}_{+}=-i\delta_{m,-n} . \tag{3.21}$$

For all coefficients we have

$$L_{-n}=L_n^{*} , \quad \eta_{-n}=\eta_n^{*} , \quad \mathscr{P}_{-n}=\mathscr{P}_n^{*} . \tag{3.22}$$

Using (3.17) and (3.20) we get a decomposition of Q:

$$Q=\sum_{n=-\infty}^{\infty} L_n\eta_{-n}+\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} m\mathscr{P}_n\eta_m\eta_{-n-m} . \tag{3.23}$$

We turn now to the quantization.

IV. QUANTIZATION OF THE ORDINARY STRING

Quantizing our system by $i\{\ \}_{\pm}\rightarrow[\ ]_{\pm}$ ($\hbar\equiv 1$), we get the fundamental anticommutator for the ghost-mode coefficients

$$[\hat{\eta}_m,\hat{\mathscr{P}}_n]_{+}=\delta_{m,-n} . \tag{4.1}$$

In terms of mode coefficients, we obtain an expression for Q (: : means normal ordering):

$$\hat{Q}=:\hat{Q}:+A\hat{\eta}_0 ,$$

where A is an infinite constant. We will renormalize this expression by setting $A\hat{\eta}_0=-\beta\hat{\eta}_0$, where β is an arbitrary finite constant. This may be realized by the replacement

$$\hat{L}_n\rightarrow\hat{L}_n'=\hat{L}_n-\beta\delta_{n,0} . \tag{4.2}$$

It is well known that the Virasoro algebra (3.18) does not close at the quantum level due to the normal ordering. Instead we get

$$[\hat{L}_n',\hat{L}_m']=(n-m)\hat{L}_{n+m}'+\left[\frac{D}{12}n(n^2-1)+2\beta n\right]\delta_{n,-m} . \tag{4.3}$$

This together with the quantized version of Q

$$\begin{aligned}\hat{Q}=&\sum_{n=-\infty}^{\infty}\hat{L}_n'\hat{\eta}_{-n}+\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} m:\hat{\mathscr{P}}_n\hat{\eta}_m\hat{\eta}_{-n-m}:\\
=&\sum_{n=-\infty}^{\infty}\hat{L}_n'\hat{\eta}_{-n}+\sum_{m=1}^{\infty} m(\hat{\mathscr{P}}_m^{\dagger}\hat{\eta}_m+\hat{\eta}_m^{\dagger}\hat{\mathscr{P}}_m)\hat{\eta}_0-2\sum_{m=1}^{\infty} m\hat{\eta}_m^{\dagger}\hat{\eta}_m\hat{\mathscr{P}}_0\\
&-\sum_{n=1}^{\infty}\sum_{m=1}^{n-1}(n\hat{\mathscr{P}}_m^{\dagger}\hat{\eta}_{n-m}^{\dagger}\hat{\eta}_n+n\hat{\eta}_n^{\dagger}\hat{\eta}_{n-m}\hat{\mathscr{P}}_m+m\hat{\mathscr{P}}_n^{\dagger}\hat{\eta}_{n-m}\hat{\eta}_m+m\hat{\eta}_m^{\dagger}\hat{\eta}_{n-m}^{\dagger}\hat{\mathscr{P}}_n)\\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m(\hat{\eta}_{n+m}^{\dagger}\hat{\mathscr{P}}_n=\hat{\eta}_m-\hat{\mathscr{P}}_n^{\dagger}\hat{\eta}_m^{\dagger}\hat{\eta}_{n+m}) ,\end{aligned} \tag{4.4}$$

implies

$$\{\hat{Q},\hat{Q}\}_+=\sum_{n=1}^{\infty}\frac{D}{6}n(n^2-1)\hat{\eta}_n^\dagger\hat{\eta}_n - \sum_{n=1}^{\infty}[\tfrac{1}{3}n(n^2-1)+4n^3]\hat{\eta}_n^\dagger\hat{\eta}_n + 4\beta\sum_{n=1}^{\infty}n\hat{\eta}_n^\dagger\hat{\eta}_n ,$$

where the first sum originates from the anomaly in the Virasoro algebra, the second from normal ordering the ghost part in (4.4), and the third from adding the suitable constant to L_0. Taking the terms together we find

$$\hat{Q}^2=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{D-26}{6}n^3+\frac{2-D+24\beta}{6}n\right]\hat{\eta}_n^\dagger\hat{\eta}_n . \tag{4.5}$$

Hence the nilpotency of Q at the quantum level demands

$$D=26 ,\quad \beta=1 , \tag{4.6}$$

where β may be identified with α_0, the intercept parameter of the conventional theory.[7]

The result (4.6), which coincides with Ref. 4 shows that the covariant quantization in the BRS formulation is only consistent if $D=26$. This result is well known in the old treatment of the bosonic string.[7] However, the standard way of deriving it is to check the closure of the Lorentz algebra in the physical subspace, i.e., by specifying the gauge completely like the transverse gauge. Thus the BRS result is more general since it only requires the ON-gauge fixing. Accordingly the theory still has gauge degrees of freedom left, which are represented by the different ways to span the genuine physical state space. The result is also consistent with Polyakov's covariant quantization, namely, that it is only for $D=26$ the conformal or trace anomaly is not present in the theory. Therefore any procedure which can probe the consistency of the quantization and is sensitive enough should end up with this result. We shall now turn to the modified theory, taking care of the conformal anomaly.

V. BRS FORMULATION OF POLYAKOV'S MODIFIED STRING THEORY

We will consider the full Lagrangian density

$$\mathscr{L}(\tau,\sigma)=\tfrac{1}{2}\sqrt{-g}\,g^{\alpha\beta}\partial_\alpha y^\mu\partial_\beta y_\mu+C\mathscr{L}_1 ,$$

where C is a suitable constant different from zero. It has been shown by Marnelius[2] that the following local form of $\mathscr{L}_1$ gives an equivalent formulation to Polyakov:

$$\mathscr{L}_1(\chi)=\tfrac{1}{2}\sqrt{-g}\,g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi+\sqrt{-g}R\chi+\mu^2\sqrt{-g} \tag{5.1}$$

if in the ON gauge $g^{\alpha\beta}=\rho(\tau,\sigma)\eta^{\alpha\beta}$ we take

$$\chi_0\equiv\chi-\ln\rho=0 . \tag{5.2}$$

Here χ is an auxiliary scalar field, R is the curvature scalar, and μ is a constant. The condition (5.2) breaks, as we shall see, the reparametrization invariance. But at the quantum level it may be restored.

Imposing the ON gauge, we may write the action

$$S=\int d\tau\int_0^\pi d\sigma[\tfrac{1}{2}\partial_\alpha y^\mu\partial^\alpha y_\mu+C(\tfrac{1}{2}\partial_\alpha\chi_0\partial^\alpha\chi_0-\tfrac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi+\mu^2e^\varphi)]+(\text{boundary terms}) , \tag{5.3}$$

where

$$\varphi\equiv\ln\rho,\quad \chi_0\equiv\chi-\varphi .$$

We also find the following two first-class constraints:

$$\begin{aligned}\tilde{L}_+(\tau,\sigma)&\equiv\tfrac{1}{4}(p+y')^2+C[F_+(\chi_0)-G_+(\varphi)] ,\\ \tilde{L}_-(\tau,\sigma)&\equiv\tfrac{1}{4}(p-y')^2+C[F_-(\chi_0)-G_-(\varphi)] ,\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}F_+(\chi_0)&\equiv\tfrac{1}{4}\left[\frac{P_\chi}{C}+\chi_0'\right]^2+\left[\frac{P_\chi'}{C}+\chi_0''\right] ,\\ F_-(\chi_0)&\equiv\tfrac{1}{4}\left[\frac{P_\chi}{C}-\chi_0'\right]^2-\left[\frac{P_\chi'}{C}-\chi_0''\right] ,\\ G_+(\varphi)&\equiv\tfrac{1}{4}\left[\frac{\pi}{C}-\varphi'\right]^2+\left[\frac{\pi'}{C}-\varphi''\right]+\tfrac{1}{2}\mu^2e^\varphi ,\\ G_-(\varphi)&\equiv\tfrac{1}{4}\left[\frac{\pi}{C}+\varphi'\right]^2-\left[\frac{\pi'}{C}+\varphi''\right]+\tfrac{1}{2}\mu^2e^\varphi .\end{aligned} \tag{5.5}$$

P_χ is conjugate to χ_0 and π is conjugate to φ satisfying the fundamental Poisson bracket relations

$$\begin{aligned}\{\chi_0(\sigma),P_\chi(\sigma')\}_{\tau=\tau'}&=\delta(\sigma-\sigma') ,\\ \{\varphi(\sigma),\pi(\sigma')\}_{\tau=\tau'}&=\delta(\sigma-\sigma') .\end{aligned} \tag{5.6}$$

The constraints (5.4) satisfy the algebra of reparametrization invariance

$$\begin{aligned}\{\tilde{L}_+(\sigma),\tilde{L}_+(\sigma')\}&=(\tilde{L}_+(\sigma)+\tilde{L}_+(\sigma'))\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') ,\\ \{\tilde{L}_-(\sigma),\tilde{L}_-(\sigma')\}&=-(\tilde{L}_-(\sigma)+\tilde{L}_-(\sigma'))\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') ,\end{aligned} \tag{5.7}$$

analogous to (3.4). This holds as long as the constraint (5.2) is not imposed. We will not for the time being consider this constraint. It will be imposed later.

Now because of the similarities with the previous sections, it is straightforward to do the BRS formulation. The BRS charge of (3.5) is replaced by

$$Q=\int_0^\pi d\sigma(\tilde{L}_+\eta^1+\tilde{L}_-\eta^2+\mathscr{P}_1\partial_1\eta^1\eta^1-\mathscr{P}_2\partial_1\eta^2\eta^2) \tag{5.8}$$

and the Hamiltonian (3.12) by

$$H_\phi(\tau)=\int_0^\pi d\sigma(\tilde{L}_+ +\tilde{L}_- +2\mathcal{P}_1\eta^{1'}+\mathcal{P}_1'\eta^1 -2\mathcal{P}_2\eta^{2'}-\mathcal{P}_2'\eta^2) . \quad (5.9)$$

The equations of motion for y^μ, p_μ, and the ghost variables are identical to those in Sec. III. For the additional variables we find from (5.9)

$$\dot{\chi}_0=\frac{1}{C}P_\chi,\quad \dot{\varphi}=-\frac{1}{C}\pi ,$$
$$\dot{P}_\chi=C\chi_0'',\quad \dot{\pi}=-C\varphi''+\mu^2e^\varphi , \quad (5.10)$$
$$\Longrightarrow \Box\chi_0=0,\quad \Box\varphi=\mu^2e^\varphi,\quad \Box\equiv\partial_0{}^2-\partial_1{}^2 .$$

So we see that χ_0 satisfies the same equation of motion as the y's, while φ satisfies Liouville's equation.

As boundary conditions we take[2,8,9] (these boundary conditions were first considered in Ref. 8)

$$\partial_1\varphi=-\mu\sqrt{2e^\varphi} \text{ at } \sigma=0 ,$$
$$\partial_1\varphi=\mu\sqrt{2e^\varphi} \text{ at } \sigma=\pi , \quad (5.11)$$
$$\partial_1\chi_0=0 \text{ at } \sigma=0,\pi .$$

The set of Eqs. (5.10) together with the boundary conditions (5.11) imply

$$\tilde{L}_+(\tau,\sigma)=\tilde{L}(\xi)=0 ,$$
$$\tilde{L}_-(\tau,\sigma)=\tilde{L}(\zeta)=0 , \quad (5.12)$$

where [$L(\xi)$ is identical to the one in (3.16)]

$$\tilde{L}(\xi)\equiv L(\xi)+C[F(\xi)-G(\xi)],$$
$$\tilde{L}(\xi-\pi)=\tilde{L}(\xi+\pi) ,$$
$$F(\xi)\equiv\tfrac{1}{4}g^2(\xi)+g'(\xi) , \quad (5.13)$$
$$G(\xi)\equiv4[h'(\xi)]^{1/2}\partial_\xi{}^2\{[h'(\xi)]^{-1/2}\} .$$

$g(\xi),h(\xi)$ are arbitrary periodic functions of ξ of period 2π. Following the steps in Sec. III, we now Fourier decompose $\tilde{L}$:

$$\tilde{L}(\xi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\tilde{L}_ne^{-in\xi} , \quad (5.14)$$

where

$$\tilde{L}_n=\int_{-\pi}^{\pi}d\xi\, e^{in\xi}\tilde{L}(\xi) \quad (5.15)$$

satisfying the Virasoro algebra

$$\{\tilde{L}_n,\tilde{L}_m\}=-i(n-m)\tilde{L}_{n+m} . \quad (5.16)$$

We will now impose (5.2), which turns our constraint into

$$\tilde{L}(\xi)=L(\xi)-CG(\xi) . \quad (5.17)$$

Using (5.7) it is straightforward to show that expression (5.16) will get an additional term, so that

$$\{\tilde{L}_n,\tilde{L}_m\}=-i(n-m)\tilde{L}_{n+m}+i4\pi Cn^3\delta_{n,-m} \quad (5.18)$$

which means that the reparametrization invariance is broken. For the BRS charge we find a mode expansion similar to (3.23)

$$Q=\sum_{n=-\infty}^{\infty}\tilde{L}_n\eta_{-n}+\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}m\mathcal{P}_n\eta_m\eta_{-n-m} . \quad (5.19)$$

However, owing to the anomalous term in (5.18), we find

$$\{Q,Q\}_+=C4\pi i\sum_{n=-\infty}^{\infty}n^3\eta_{-n}\eta_n\neq0 . \quad (5.20)$$

Thus the nilpotency of Q is broken. We shall see in the next section that at the quantum level it may again be restored for a specific value of C.

VI. QUANTIZATION OF POLYAKOV'S MODIFIED STRING THEORY

From (5.17) we see that $\tilde{L}_n$ consists of two parts

$$\tilde{L}_n=L_n+M_n , \quad (6.1)$$

where M_n are the Fourier coefficients of $CG(\xi)$. Equation (5.18) then implies

$$\{M_n,M_m\}=-i(n-m)M_{n+m}+i4\pi Cn^3\delta_{n,-m} . \quad (6.2)$$

When we now quantize our model, we must again normal order. For the ordinary Virasoro coefficients this leads to the expression (4.3). As for the coefficients M_n, we unfortunately do not know if (6.2) is modified by the normal-ordering procedure, since the Liouville theory has not yet been quantized exactly. Instead we must make an assumption, namely, that the algebra (6.2) is unchanged at the quantum level:

$$[\hat{M}_n,\hat{M}_m]=(n-m)\hat{M}_{n+m}-4\pi Cn^3\delta_{n,-m} . \quad (6.3)$$

Defining the operators $\hat{\tilde{L}}_n$ by

$$\hat{\tilde{L}}_n=\hat{L}_n+\hat{M}_n-\beta\delta_{n,0} , \quad (6.4)$$

where β again is an arbitrary constant due to the normal-ordering ambiguity, we find from (4.3) and (6.3)

$$[\hat{\tilde{L}}_n,\hat{\tilde{L}}_m]=(n-m)\hat{\tilde{L}}_{n+m}+\left[\frac{D}{12}n(n^2-1)-4\pi Cn^3+2n\beta\right]\delta_{n,-m} . \quad (6.5)$$

This expression is of the same form as (4.3) except for the term containing the constant C. The ghost part is identical here to that in Sec. IV. Thus we will only slightly modify (4.5), the nilpotency condition of Q

$$\hat{Q}^2=\tfrac{1}{2}\sum_{n=1}^{\infty}\left[\frac{D-26-48\pi C}{6}n^3+\frac{2-D+24\beta}{6}n\right]\hat{\eta}_n^\dagger\hat{\eta}_n \quad (6.6)$$

so that the nilpotency of $\hat{Q}$ is restored at the quantum level if

$$C=\frac{D-26}{48\pi},\quad \beta=\frac{D-2}{24} \quad (6.7)$$

and thus we find that the anomaly vanishes if and only if $D=26$. Thus the value of the anomaly constant C is identical to that of Polyakov.[1] Marnelius[2] obtained the same values for both C and β. But whereas this was done by checking the Lorentz algebra in a light-cone gauge, we achieve the same results in a truly covariant way.

VII. CONCLUSIONS

We have in this paper made a covariant quantization of the bosonic string, first without bothering about the trace or conformal anomaly and then taken it into account in the interpretation of Marnelius.[2] The BRS approach shows that, for the standard theory, it is only possible to maintain the nilpotency of Q, at the quantum level, if $D=26$ and with the intercept parameter $\alpha_0=1$. Thus a consistent quantum theory is only found for $D=26$. The result coincides with that of Kato and Ogawa,[4] but whereas their treatment starts in representation space, ours starts in phase space. Our Hamiltonian (3.12) corresponds to a BRS-invariant Lagrangian:

$$\mathscr{L}(\tau,\sigma)=\tfrac{1}{2}(\dot{y}^2-y'^2)+\bar{\eta}_1(\dot{\eta}^1-2\eta^{1'})-\bar{\eta}_1'\eta^1 +\bar{\eta}_2(\dot{\eta}^2+2\eta^{2'})+\bar{\eta}_2'\eta^2 , \tag{7.1}$$

where $\bar{\eta}_\alpha$ are auxiliary ghost variables. The Lagrangian (2.14) in Ref. 4 is essentially the same as (7.1) after the functional integration over $g^{\alpha\beta}$. Notice that the gauge fixing of $g^{\alpha\beta}$ is trivial, i.e., only represented by a δ function.

Taking into consideration the conformal anomaly, we find that a consistent quantization in the BRS formulation is possible if the anomaly constant is given by $C=(D-26)/48\pi$ and $\beta=(D-2)/24$. This holds under the assumption that the algebra of the Liouville Fourier coefficients M_n is unchanged at the quantum level. This value of C agrees with the result of the covariant quantization of Polyakov[1] in the path-integral formulation, namely, that it is only for $D=26$ the conformal anomaly vanishes and that an exact quantization seems plausible for $D<26$. The question of unitarity and the connection to the standard theory in the BRS formulation of Fradkin and Vilkovisky is established using the formal proof of Ref. 6.

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APPENDIX: CONNECTION BETWEEN THE BRS QUANTIZATION AND THE CONVENTIONAL COVARIANT QUANTIZATION

We shall here explicitly exhibit the connection between the BRS formulation and the conventional covariant quantization. The approach is basically that of Ref. 6. With the correspondence to the conventional treatment established, the unitarity is secured.

The physical space $\mathscr{S}$ of the whole space of states is projected out by the condition

$$\hat{Q}\,|\,\text{phys}\rangle=0 . \tag{A1}$$

The space $\mathscr{S}$ consists of two parts $\mathscr{S}_0$ and $\mathscr{S}_1$:

$$\begin{aligned} &|\,\text{phys}\rangle_0\equiv|\,\text{phys}\rangle\in\mathscr{S}_0\text{: } \langle\text{phys}\,|\,\text{phys}\rangle_0=0 ,\\ &|\,\text{phys}\rangle_1\equiv|\,\text{phys}\rangle\in\mathscr{S}_1\text{: } {}_1\langle\text{phys}\,|\,\text{phys}\rangle_1=C , \end{aligned} \tag{A2}$$

where C is some constant, which is positive if we have unitarity. From (A1) and (A2) we see that a zero-norm physical state $|\,\text{phys}\rangle_0$ may be found by

$$|\,\text{phys}\rangle_0=\hat{Q}\,|\,\text{state}\rangle \tag{A3}$$

if $\hat{Q}$ is Hermitian.

Define a physical operator F as an operator for which

$$\hat{F}\,|\,\text{phys}\rangle=|\,\text{phys}\rangle . \tag{A4}$$

The physical operators are of two types:

$$\begin{aligned} &\hat{A}\in\{\hat{F}\}\text{: } \hat{A}\,|\,\text{phys}\rangle_1=|\,\text{phys}\rangle_1 ,\\ &\hat{B}\in\{\hat{F}\}\text{: } \hat{B}\,|\,\text{phys}\rangle=|\,\text{phys}\rangle_0 , \end{aligned} \tag{A5}$$

(A1) and (A3) implies that the operator

$$\hat{F}\equiv[\hat{Q},\hat{C}]_\pm \tag{A6}$$

is a B operator. C may be any operator.

From the Jacobi identities, we find that the B operators defined by (A6) form a closed algebra

$$[\hat{B}_a,\hat{B}_b]_\pm=C^c_{ab}\hat{B}_c . \tag{A7}$$

For the ordinary string theory in $D=26$, we have a Q operator given by (4.4), from which we find the following B operators:

$$\begin{aligned} \hat{B}_k^{\text{I}}&\equiv[\hat{Q},\hat{\mathscr{P}}_k]_+\\ &=\hat{L}_k'+\sum_{n=-\infty}^{\infty}(2k-n):\hat{\mathscr{P}}_n\hat{\eta}_{k-n}: , \end{aligned} \tag{A8}$$

$$\hat{B}_k^{\text{II}}\equiv[\hat{Q},\hat{\chi}_k]_-=\sum_{n=-\infty}^{\infty}[\hat{L}_n',\hat{\chi}_k]_-\hat{\eta}_{-n} , \tag{A9}$$

$$\hat{B}_k^{\text{III}}\equiv[\hat{Q},\hat{\eta}_k]_+=\sum_{n=-\infty}^{\infty}n\hat{\eta}_n\hat{\eta}_{k-n} , \tag{A10}$$

where $\hat{\chi}_k$ are a complete set of gauge-fixing operators to $\hat{L}_n$. In the BRS formulation of the string, the B_n^{I} operators are the true gauge operators of the theory, which unlike the Virasoro operators satisfy a closed algebra, provided $\hat{Q}^2=0$.

The physical space of the ordinary string theory is spanned by the so-called DDF (Del Guidice-Di Vecchia-Fubini) operators.[10] Since all these operators commute with $\hat{L}_n$ they also commute with $\hat{B}_n^{\text{I}}$ and the string part of the physical space is unchanged. Our task is now to show that these states are coupled to a trivial ghost state.

Using (A8), we see that we may rewrite $\hat{Q}$ as

$$\hat{Q}=\sum_{n=-\infty}^{\infty}\hat{L}_n'\hat{\eta}_{-n}+\sum_{m=1}^{\infty}(\hat{\mathscr{P}}_m+\hat{\mathscr{P}}_{-m})(\hat{B}_m^{\text{III}}+\hat{B}_{-m}^{\text{III}}) +\hat{\mathscr{P}}_0\hat{B}_0^{\text{III}}+\sum_{m=1}^{\infty}m\hat{\eta}_0 . \tag{A11}$$

From (A9) we get

$$\sum_{n=-\infty}^{\infty} [\hat{L}_n',\hat{\chi}_k]_-\hat{\eta}_{-n}\,|\,\text{phys}\rangle_1 = |\,\text{phys}\rangle_0 \,.$$

Clearly since $\hat{\chi}_k$ fixes the gauge completely, we have

$$[\hat{L}_n',\hat{\chi}_k]_-\,|\,\text{phys}\rangle_1 = (\text{invertible matrix})\,|\,\text{phys}\rangle_1$$

which means that

$$\hat{\eta}_{-n}\,|\,\text{phys}\rangle_1 = |\,\text{phys}\rangle_0 \,. \tag{A12}$$

This implies that $\hat{B}_n^{\text{III}}\,|\,\text{phys}\rangle_1=0$ so that from (A11) we find

$$0=\hat{Q}\,|\,\text{phys}\rangle_1 = \left[\sum_{n=-\infty}^{\infty} \hat{L}_n'\hat{\eta}_{-n} + \sum_{m=1}^{\infty} m\hat{\eta}_0\right]|\,\text{phys}\rangle_1$$

and thus since $\sum_{m=1}^{\infty} m$ is an infinite constant

$$\hat{\eta}_0\,|\,\text{phys}\rangle_1=0 \,, \qquad \sum_{n=-\infty}^{\infty} \hat{L}_n'\hat{\eta}_{-n}\,|\,\text{phys}\rangle_1=0 \,. \tag{A13}$$

The operators $\hat{L}_n'$ form a complete set of independent operators so that

$$\hat{L}_n'\hat{\eta}_{-n}\,|\,\text{phys}\rangle_1=0 \;\; \forall n \,. \tag{A14}$$

This should be satisfied together with the equations

$$_1\langle \text{phys}\,|\,\hat{B}_k^{\text{I}}\,|\,\text{phys}\rangle_1 = {}_1\langle \text{phys}\,|\,\hat{B}_k^{\text{II}}\,|\,\text{phys}\rangle_1 = {}_1\langle \text{phys}\,|\,\hat{B}_k^{\text{III}}\,|\,\text{phys}\rangle_1 = 0 \,.$$

A solution to all these equations is

$$\hat{\eta}_n\,|\,\text{phys}\rangle_1=0, \;\; n\geq 0 \,, \qquad \hat{L}_n'\,|\,\text{phys}\rangle_1=0, \;\; n\geq 0 \,, \tag{A15}$$

which means that the physical space is separated into two parts, a string part and a ghost state:

$$|\,\text{phys}\rangle_1 = |\,\text{phys}\rangle_1^{\text{string}}\,|\,0\rangle^{\text{ghost}} \,, \tag{A16}$$

where $|\,0\rangle^{\text{ghost}}$ is just the ghost vacuum state. Notice that for the Virasoro operators, (A15) and (A8) imply

$$_1\langle \text{phys}\,|\,\hat{L}_n'\,|\,\text{phys}\rangle_1=0 \;\; \forall n \,. \tag{A17}$$

This establishes that the conventional treatment of the string is contained in our BRS formulation and thus unitarity is secured.

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