SO(2,1)-invariant quantization of the Liouville theory

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The recently proposed SO(2,1)-invariant quantization of the Liouville theory is elaborated. We develop a renormalized perturbation expansion which preserves this symmetry to all orders, but spontaneously breaks Poincaré invariance. Some Green's functions and scattering amplitudes are calculated in low perturbative order, and it is established that the S matrix is trivial in the tree approximation. Whether this is also true of the complete S matrix remains an open question.

I. INTRODUCTION

The Liouville field theory is tantalizing because the classical version can be completely and explicitly integrated, but in quantized form the model resists conventional analysis.[1–3] In this paper we elaborate the recent unconventional suggestion that the quantum theory be defined by an expansion about a position-dependent classical solution.[2,3] The energy density, measured with respect to this background, is positive for fluctuations of arbitrary strength, and the solution is stable. The fluctuations are quantized and a consistent renormalized perturbation expansion is developed for which all amplitudes are infrared finite. Space-translation symmetry, which is broken by the classical solution, is not restored by quantum corrections since no Goldstone zero modes are present to produce infrared divergences. Though translation invariance is lost, the quantum theory preserves an SO(2,1) subgroup of the infinite-dimensional conformal-symmetry group of the Liouville action. Physical states are classified in unitary representations of SO(2,1), and the perturbation theory is SO(2,1) invariant. Two-point function spectral representations and scattering amplitudes are derived. Although nontrivial scattering processes are permitted by SO(2,1) invariance, the $2\longleftrightarrow 2$ scattering amplitude vanishes in the tree approximation, and the same can be established for general $n\longleftrightarrow m$ amplitudes by using the conformal symmetry of the Liouville action. Anomalies prevent extension of our proof to the full quantum theory. Thus, the question whether the S matrix is trivial or begins in one-loop order remains open.

This paper presents a systematic exploration of the space-translational-noninvariant quantization. The motivation and the underlying hypotheses are discussed and the results mentioned above are derived. This unconventional procedure is consistent and seems a natural approach to the quantized Liouville theory, in view of its geometrical properties.

In Sec. II, some aspects of the Liouville theory are reviewed with emphasis on the infinite-parameter conformal symmetry. Our quantization procedure is outlined in Sec. III, and the SO(2,1) invariance group is described. In Sec. IV, physical states, classified as representations of SO(2,1), and the propagator are discussed. Spectral representations for two-point functions are derived, and a manifest SO(2,1)-invariant formulation on a hyperbolic space is outlined. Section V is devoted to the general structure of renormalized perturbation theory as well as to the calculation of the propagator and of the three- and four-point functions. The canonical mapping and operator arguments which prove that the tree-level S matrix is trivial are given in Sec. VI, and a concluding discussion follows in Sec. VII. One form of the spectral representation is derived in Appendix A, while the renormalized quantum energy-momentum tensor is constructed in Appendix B.

II. REVIEW

A. Classical theory

The Liouville model is governed by the [Minkowski space: $x=(t,x^1)$] Lagrange density

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi-\frac{m^2}{\beta^2}e^{\beta\Phi}\ ,\quad \beta,m^2>0\ , \tag{2.1}$$

giving the following equation of motion:

$$\Box\Phi+\frac{m^2}{\beta}e^{\beta\Phi}=0\ , \tag{2.2}$$

whose most general solution is

$$\Phi(x)=\frac{1}{\beta}\ln\frac{F'(x^+)G'(x^-)}{\left[1+\frac{m^2}{4}F(x^+)G(x^-)\right]^2}\ ,\qquad x^\pm=\frac{1}{\sqrt{2}}(t\pm x^1)\ , \tag{2.3}$$

where F and G are arbitrary functions. While this equation has appeared in many problems of physics and mathematics,[1] here we stress only its geometrical significance. In two dimensions, any metric tensor $\gamma_{\mu\nu}$ can be made conformally flat by coordinate redefinition: $\gamma_{\mu\nu}=e^{\beta\Phi}g_{\mu\nu}$, where $g_{\mu\nu}$ is the Minkowski metric tensor. Also in two dimensions, the curvature tensor is entirely determined by the scalar curvature $R=\beta e^{-\beta\Phi}g^{\mu\nu}\partial_\mu\partial_\nu\Phi$. When Φ satisfies (2.2), R is a negative constant, and $\gamma_{\mu\nu}$ describes a two-dimensional surface of constant negative curvature. The isometry group of such surfaces is SO(2,1), and correspondingly, all solutions (2.3) of the Liouville theory are invariant under an SO(2,1) isotropy group.

The Liouville theory is conformally invariant: the action changes only by surface terms when fields are transformed according to the infinitesimal rule

$$\delta_f \Phi = f^\mu \partial_\mu \Phi + \frac{1}{\beta} \partial_\mu f^\mu , \tag{2.4}$$

where f^μ is an infinitesimal conformal transformation of coordinates

$$\delta x^\mu = -f^\mu , \tag{2.5}$$

satisfying the conformal Killing equation

$$\partial_\mu f_\nu + \partial_\nu f_\mu - g_{\mu\nu} \partial_\alpha f^\alpha = 0 . \tag{2.6}$$

In dimensions greater than two, solutions of the conformal Killing equation generate the finite-dimensional restricted conformal algebra with the following elements:

translations: $f^\mu = a^\mu$, (2.7a)

Lorentz transformations: $f^\mu = \omega^{\mu\nu} x_\nu$, (2.7b)

dilatations: $f^\mu = d x^\mu$, (2.7c)

special conformal transformations: $f^\mu = 2x^\mu c\cdot x - c^\mu x^2$. (2.7d)

Here a^μ, $\omega^{\mu\nu} = -\omega^{\nu\mu}$, d, and c^μ are constant transformation parameters. In two dimensions, however, (2.6) is similar to the Cauchy-Riemann equation and possesses an infinity of solutions, corresponding to the infinite-parameter two-dimensional conformal algebra:

$$f^+ = f^+(x^+) , \quad f^- = f^-(x^-) . \tag{2.8}$$

The restricted conformal transformations (2.7) form an $SO(2,2) = SO(2,1) \times SO(2,1)$ subalgebra of the full two-dimensional conformal algebra. The conformal transformations (2.4)—(2.6) satisfy a composition law

$$[\delta_f, \delta_g] = \delta_h \tag{2.9a}$$

with f, g, and h conformal Killing vectors, the last given by the Lie bracket of the first two:

$$h^\mu = f^\alpha \partial_\alpha g^\mu - g^\alpha \partial_\alpha f^\mu . \tag{2.9b}$$

The conserved conformal currents may be simply expressed in terms of a conserved, symmetric and traceless energy-momentum tensor[1]:

$$J_f^\mu = \theta^{\mu\nu} f_\nu , \tag{2.10a}$$

$$\partial_\mu J_f^\mu = 0 , \tag{2.10b}$$

$$\theta^{\mu\nu} = \partial^\mu \Phi\, \partial^\nu \Phi - g^{\mu\nu} \mathscr{L} + \frac{2}{\beta} (g^{\mu\nu} \Box - \partial^\mu \partial^\nu) \Phi$$
$$= \partial^\mu \Phi\, \partial^\nu \Phi - \tfrac{1}{2} g^{\mu\nu} \partial_\alpha \Phi\, \partial^\alpha \Phi + \frac{1}{\beta} (g^{\mu\nu} \Box - 2\partial^\mu \partial^\nu) \Phi \tag{2.10c}$$

When a canonical formulation is defined with equal-time Poisson brackets,

$$\{\Phi(t,x^1), \Phi(t,y^1)\} = \{\dot\Phi(t,x^1), \dot\Phi(t,y^1)\} = 0 , \tag{2.11a}$$

$$\{\dot\Phi(t,x^1), \Phi(t,y^1)\} = \delta(x^1 - y^1) , \tag{2.11b}$$

one finds that the charges

$$Q_f = \int dx^1 J_f^0(t,x^1) \tag{2.12}$$

generate the transformation (2.4):

$$\{Q_f, \Phi\} = \delta_f \Phi = f^\mu \partial_\mu \Phi + \frac{1}{\beta} \partial_\mu f^\mu . \tag{2.13}$$

However, the algebra (2.9) is realized only with a center:

$$\{Q_f, Q_g\} = -Q_h + \beta^{-2} \triangle(f,g) , \tag{2.14a}$$

$$\triangle(f,g) = \int dx^+ (f^+ \partial_+^3 g^+ - g^+ \partial_+^3 f^+)) + \int dx^- (f^- \partial_-^3 g^- - g^- \partial_-^3 f^-) . \tag{2.14b}$$

One may ask whether the conformal charges can be modified by adding terms independent of the canonical variables such that the resulting algebra is realized without center. This can be done for a finite-dimensional SO(2,2) subalgebra, with conformal Killing vectors

$$f^+ = \frac{m}{2} \frac{F^2}{F'} a_1 + \frac{F}{F'} a_2 + \frac{2}{m} \frac{1}{F'} a_3 , \tag{2.15a}$$

$$f^- = \frac{2}{m} \frac{1}{G'} b_1 - \frac{G}{G'} b_2 + \frac{m}{2} \frac{G^2}{G'} b_3 , \tag{2.15b}$$

where $F(x^+)$ and $G(x^-)$ are two arbitrary functions which specify the embedding of the subalgebra and the a's and b's are constants.[4] It is easy to show that for both f and g belonging to this SO(2,2) algebra, we have[5]

$$\triangle(f,g) = \int dx^1 \theta^{0\nu}(\Phi_0) h_\nu , \tag{2.16}$$

where Φ_0 is a solution of (2.2) constructed from F and G as in (2.3), and $\theta^{\mu\nu}(\Phi_0)$ is the value of the improved stress tensor (2.10c) on the solution Φ_0. Then the modified charges

$$Q_f' = \int dx^1 [\theta^{0\nu}(\Phi) - \theta^{0\nu}(\Phi_0)] f_\nu \tag{2.17}$$

do realize the SO(2,2) algebra without center:

$$\{Q_f', Q_g'\} = -Q_h' . \tag{2.18}$$

Although we do not have a complete proof for unbounded spaces, it appears to us that the only subalgebra of the full conformal group which can be realized without center must be finite dimensional. By a theorem of Lie,[6] the largest of these is the SO(2,2) just described. It is easy to see that the diagonal subalgebra of (2.15)—the SO(2,1) specified by $a_i = b_i$—leaves the classical solution Φ_0 invariant: $\delta_f \Phi_0 = 0$. This fact is related to the geometrical significance of SO(2,1) as the symmetry group of a two-dimensional surface of constant negative curvature.

B. Quantum theory

One may quantize the Liouville theory by postulating that equal-time commutators replace the Poisson brackets (2.11). Let us assume for the moment that there is a conformally invariant regularization procedure[7] in which all the charges Q_f, which implement conformal transformations by commutation, can be constructed:

$$\frac{i}{\hbar}[Q_f,\Phi]=\delta_f\Phi \ , \tag{2.19}$$

so that they satisfy the algebra

$$[Q_f,Q_g]=i\hbar Q_h+c\triangle(f,g) \ , \tag{2.20}$$

with possible quantum modification in the definition of Q_f and the necessary quantum correction to the coefficient of the center[8] included in c.

In the search for the ground state of the quantum theory, one wishes to consider states of maximal symmetry, and it is clear from (2.20) that such states can only be invariant under those subalgebras that can be realized without a center. Therefore, the largest symmetry that a ground state could possibly support is SO(2,2). The remainder of the conformal group must be spontaneously broken, or else beset by anomalies—hence, not preserved by the quantum theory.

Conventionally, one expects the Poincaré group—a subgroup of SO(2,2)—to be the invariance group of the vacuum. However, there is evidence that a normalizable translation-invariant ground state does not exist.[1–3] Such a state would conflict with the positivity of the normal-ordered source $\beta^{-2}m^2e^{\beta\Phi}$. Further, the complete effective potential[9] is also an exponential, so that a translation-invariant vacuum cannot be found by minimizing it. Possibly, translation invariance is recovered if the expectation value of Φ in the ground state is $-\infty$. However, since this point is an essential singularity of the action, one cannot develop a perturbation theory about it.[10]

Even for a finite, translation-invariant $\langle 0|\Phi|0\rangle$, there are obstacles to conventional perturbation theory. Expansion in β produces a β^{-1} tadpole which cannot be shifted away. Perturbation theory in the entire interaction, i.e., in m^2 cannot be carried out either. By dimensional analysis it is infrared divergent, becoming more and more singluar with increasing order. Furthermore, m^2 is not a true parameter of the theory since it can be modified by a constant shift in Φ.

It may be that the quantized Liouville theory has no ground state: the energy spectrum is bounded below, but the bound is not attained.[1] (The one-dimensional, i.e., quantum-mechanical, Liouville model behaves in this way.[1]) Since we are unable to develop calculational procedures to test this possibility, we adopt an alternative: a ground state *does* exist, but it is not Poincaré invariant; rather its symmetry group is the SO(2,1) invariance group of classical solutions. In this paper we explore this assumption, and show that one can build a consistent perturbation theory for a ground state which is space-translation noninvariant.

III. SPACE-TRANSLATION-NONINVARIANT APPROACH

A. Classical static solutions

In the previous section we argued that the invariance group of the Liouville vacuum is a SO(2,1) subgroup of SO(2,2). In principle, a quantum field theory can be constructed for any of the subalgebras described in (2.15). To proceed, however, we shall assume that time translations are part of the unbroken subalgebra, and quantize the Liouville field Φ about a time-independent classical solution.

The most general static solution of (2.2),[11]

$$\Phi_s^\epsilon(x^1)=-\frac{1}{\beta}\ln\frac{m^2}{2\epsilon}\sinh^2\sqrt{\epsilon}(x^1-x_0^1) \ , \tag{3.1}$$

contains two constants of integration: x_0^1, associated with space translations, and ϵ, which can be positive, zero, or negative. For negative ϵ, there are periodically space singularities; otherwise, only the point $x^1=x_0^1$ is singular.

The conventional energy density

$$\mathscr{E}=\tfrac{1}{2}\dot{\Phi}^2+\tfrac{1}{2}\Phi'^2+\frac{m^2}{\beta^2}e^{\beta\Phi} \ , \tag{3.2a}$$

evaluated on the static solution $\Phi=\Phi_s^\epsilon$, is

$$\mathscr{E}=\frac{4}{\beta^2}\left[\frac{\epsilon}{\sinh^2\sqrt{\epsilon}(x^1-x_0^1)}+\frac{\epsilon}{2}\right] . \tag{3.2b}$$

The total energy is infinite owing to ultraviolet singularities and for $\epsilon\neq0$ to an infrared (volume) infinity. But it is also clear that the $\epsilon=0$ solution is of lower energy than those with $\epsilon\neq0$. Furthermore, the conformally improved energy density (2.10c) differs from $\mathscr{E}$ by a total spatial derivative

$$\theta^{00}=\mathscr{E}-\frac{2}{\beta}\Phi'' \ , \tag{3.3a}$$

and for static solutions, it is constant:

$$\theta_s^{00}=\frac{2\epsilon}{\beta^2} \ . \tag{3.3b}$$

This energy density, unlike $\mathscr{E}$, is not positive and vanishes for $\epsilon=0$. In principle, our quantization procedure could be developed for any of these classical static solutions along lines similar to those presented in the sequel, but henceforth we shall discuss only the $\epsilon=0$ solution, which is

$$\Phi_s(x^1)=-\frac{1}{\beta}\ln\frac{m^2(x^1-x_0^1)^2}{2} \ . \tag{3.4}$$

The SO(2,1) algebra that leaves (3.4) invariant is the timelike, special conformal algebra and its generators are obtained from (2.15):

$$f_H^\mu=g^{\mu0} \ , \tag{3.5a}$$

$$f_D^\mu=(x^\mu-x_0^\mu) \ , \tag{3.5b}$$

$$f_K^\mu=2(x^\mu-x_0^\mu)(t-t_0)-g^{\mu0}(x-x_0)^2 \ . \tag{3.5c}$$

The generators of time translation H, dilatation D, and special conformal transformation in the timelike direction K satisfy the following SO(2,1) commutation relations:

$$\frac{i}{\hbar}[D,H]=H \ , \tag{3.6a}$$

$$\frac{i}{\hbar}[D,K]=-K \ , \tag{3.6b}$$

$$\frac{i}{\hbar}[K,H]=2D \ . \tag{3.6c}$$

The fluctuation field

$$\hat{\Phi}=\Phi-\Phi_s \qquad (3.7)$$

responds homogeneously to the transformations (3.5),

$$\delta_f\hat{\Phi}=f^\mu\partial_\mu\hat{\Phi} , \qquad (3.8)$$

and the conserved currents are

$$\hat{J}^\mu_f=\hat{\theta}^{\mu\nu}f_\nu \qquad (3.9)$$

with

$$\hat{\theta}^{\mu\nu}=\partial^\mu\hat{\Phi}\,\partial^\nu\hat{\Phi}-g^{\mu\nu}\left[\tfrac{1}{2}\partial_\alpha\hat{\Phi}\,\partial^\alpha\hat{\Phi}-\frac{2}{\beta^2r^2}(e^{\beta\hat{\Phi}}-\beta\hat{\Phi}-1)\right] ,$$

$$r=x^1-x^1_0 . \qquad (3.10)$$

Similarly, the Lagrangian density and the equation of motion may be expressed in terms of $\hat{\Phi}$:

$$\hat{\mathscr{L}}=\tfrac{1}{2}\partial_\mu\hat{\Phi}\,\partial^\mu\hat{\Phi}-\frac{2}{\beta^2r^2}(e^{\beta\hat{\Phi}}-\beta\hat{\Phi}-1) , \qquad (3.11a)$$

$$\Box\hat{\Phi}+\frac{2}{\beta r^2}(e^{\beta\hat{\Phi}}-1)=0 . \qquad (3.11b)$$

It is remarkable that the mass has disappeared from (3.11). Also, $\hat{\theta}^{\mu\nu}$ is the canonical energy-momentum tensor derived from the Lagrangian $\hat{\mathscr{L}}$. The energy density of (3.10),

$$\hat{\theta}^{00}=\tfrac{1}{2}(\partial_0\hat{\Phi})^2+\tfrac{1}{2}(\partial_1\hat{\Phi})^2+\frac{2}{\beta^2r^2}(e^{\beta\hat{\Phi}}-\beta\hat{\Phi}-1) , \qquad (3.12)$$

is always non-negative and vanishes only when $\hat{\Phi}=0$. Thus, the solution $\hat{\Phi}=0$ is stable since variations in the field will always increase the energy.

Stability against infinitesimal fluctuations may also be examined by linearizing the equations of motion (2.2) around Φ_s:

$$\Phi(x)=\Phi_s(r)+e^{-i\omega t}\psi_\omega(r) . \qquad (3.13)$$

The functions ψ_ω satisfy

$$-\psi''_\omega+\frac{2}{r^2}\psi_\omega=\omega^2\psi_\omega . \qquad (3.14)$$

This equation admits regular solutions only for positive ω^2, so that ω is real and Φ_s is stable. The regular solutions to (3.14) are given by

$$\psi_\omega(r)=\left[\frac{2}{\pi}\right]^{1/2}\left[\frac{\sin\omega r}{\omega r}-\cos\omega r\right]$$

$$=(\omega r)^{1/2}J_{3/2}(\omega r) . \qquad (3.15)$$

The modes ψ_ω are of even parity and are complete only on the half line, and they have been so normalized in (3.15). It is important that there are no zero-frequency modes: $\psi_\omega|_{\omega=0}=0$. Of course, at $\omega=0$ (3.14) is solved by two linearly independent functions:

$$\frac{\partial}{\partial r}\Phi_s \text{ and } \frac{\partial}{\partial\epsilon}\Phi^\epsilon_s\bigg|_{\epsilon=0} .$$

However, neither is normalizable, not even with a continuum convention. The absence of zero modes is the key to the consistency of our procedure, which we shall now explain.[11]

B. Postulates for the space-translation-noninvariant quantization procedure

We postulate that space-translation invariance is spontaneously broken, and that there is a family of degenerate, normalizable ground states labeled by x^1_0, which are all physically equivalent. We also assume that Φ_s is the lowest-order, classical approximation to the expectation value of the Liouville field Φ in the ground state. We have

$$\langle x^1_0\,|\,\Phi(x)\,|\,x^1_0\rangle=-\frac{1}{\beta}\ln\frac{m^2(x^1-x^1_0)^2}{2}$$

$$+\text{quantum correction} ,$$

$$\langle x^1_0\,|\,x^1_0\rangle=1 . \qquad (3.16)$$

The fact that the conventional energy is infinite on this classical solution is physically inconsequential, since the vacuum energy is unobservable. This is equivalent to using the conformally improved energy (3.3).

Because the small fluctuations are complete only on the half line, we postulate that space spontaneously contracts to the half line (henceforth taken to be $x^1\geq x^1_0$). This is the phenomenon of spontaneous (semi)compactification.

Finally we assume that the states of the theory can be identified as representations of SO(2,1)—the invariance group of the classical solution Φ_s.

The consistency of our postulates is revealed by showing that higher-order quantum corrections as well as scattering amplitudes for physical states can be computed order by order in a well-defined perturbative expansion. Since the value of x^1_0 is immaterial, we set it to zero and rename $x^1=r\geq 0$.

IV. PHYSICAL STATES, THE PROPAGATOR, AND A MANIFESTLY SO(2,1)-COVARIANT FORMULATION

A. Physical states

The SO(2,1) group may be used to classify the states of our quantum field theory in unitary representations of SO(2,1); just as in ordinary theories, states appear in unitary representations of the Poincaré group. The latter group is in fact a contraction of the former. The infinite-dimensional unitary representations of SO(2,1) are conveniently labeled by the eigenvalue of some SO(2,1) generator, which we choose to be H, and by the eigenvalue of the quadratic Casimir operator:

$$C=\tfrac{1}{2}HK+\tfrac{1}{2}KH-D^2 . \qquad (4.1)$$

Using more conventional generators J^a, one can also write

$$C=J^aJ_a=(J^1)^2-(J^2)^2-(J^3)^2 ,$$

and the metric of the three-dimensional space ($a=1,2,3$) has signature $(1,-1,-1)$. The generators J^a, related to the generators H, D, and K, with the help of an arbitrary scale parameter Λ, are

$$J^1=\frac{\Lambda}{2}H+\frac{1}{2\Lambda}K ,$$
$$J^2=\frac{\Lambda}{2}H-\frac{1}{2\Lambda}K , \tag{4.2}$$
$$J^2=D ,$$

and obey the usual SO(2,1) commutation relations

$$[J^a,J^b]=i\hbar\epsilon^{abc}J_c , \quad \epsilon^{123}=1 . \tag{4.3}$$

The eigenvalues of the Casimir operator C can be written as $j(j+1)$, but j may be continuous.[12] The Casimir operator C is analogous to the mass-square operator in Poincaré-invariant field theory.

We shall now assume, following our postulates of the previous section, that physical states are labeled by j and ω, the eigenvalues of C and H.[13] We have

$$J^aJ_a\,|\,j,\omega\rangle=j(j+1)\hbar^2\,|\,j,\omega\rangle , \tag{4.4a}$$

$$H\,|\,j,\omega\rangle=\omega\hbar\,|\,j,\omega\rangle . \tag{4.4b}$$

By definition, the vacuum state is annihilated by all the generators:

$$J^a\,|\,0\rangle=0 . \tag{4.5}$$

In the next section we shall show that such a state may indeed by constructed in the full quantum theory, order by order in perturbation theory.

Upon linearizing (3.11b), we see that free or asymptotic states are governed by a field $\hat{\Phi}_0$, which satisfies

$$\left[\Box+\frac{2}{r^2}\right]\hat{\Phi}_0=0 . \tag{4.6}$$

This equation may be rewritten in a manifestly SO(2,1)-invariant form:

$$\frac{i}{\hbar}\left[J^a,\frac{i}{\hbar}[J_a,\hat{\Phi}_0]\right]=r^2\Box\hat{\Phi}_0=-2\hat{\Phi}_0 . \tag{4.7}$$

Comparison with (4.4a) shows that the "single-particle" state $\hat{\Phi}_0\,|\,0\rangle$ has $j=1$, so that the Liouville "particle" lives in a single specific irreducible unitary representation of SO(2,1), which lies in the discrete spectrum.[12] Consequently, if asymptotic fields for multiparticle states are constructed by applying polynomials in the fields $\hat{\Phi}_0$ to $|\,0\rangle$, only states in the discrete series of unitary representations will result.[12] Thus, it seems plausible to conjecture that the asymptotic states of the theory form the complete discrete series of nonspinor representations of SO(2,1). Still, other states may exist.

B. The propagator

With this group-theoretical machinery, one may obtain spectral representations for time-ordered products of local operators. As an example, we analyze the propagator

$$G(x,x')=G(tr,t'r')=\langle 0\,|\,T\hat{\Phi}(x)\hat{\Phi}(x')\,|\,0\rangle . \tag{4.8}$$

The free propagator $\mathscr{D}(x,x')$ is constructed from the quadratic terms in $\mathscr{L}$ and is given by the small-oscillation modes (3.15):

$$\mathscr{D}(x,x')=\mathscr{D}(tr,t'r')=\frac{\hbar}{2}\int_0^\infty\frac{d\omega}{\omega}e^{-i\omega|t-t'|}\psi_\omega(r)\psi_\omega(r')$$
$$=\frac{\hbar}{\pi}\int_0^\infty\frac{d\omega}{\omega}e^{-i\omega|t-t'|}\left[\frac{\sin\omega r}{\omega r}-\cos\omega r\right]\left[\frac{\sin\omega r'}{\omega r'}-\cos\omega r'\right] . \tag{4.9}$$

The absence of the zero modes guarantees that no infrared singularities occur in this integral, and it can be evaluated. We have

$$\mathscr{D}(x,x')=\frac{\hbar}{2\pi}Q_1\left[\frac{(t-t')^2-r^2-r'^2}{2rr'}-i0\right] . \tag{4.10}$$

Here Q_1 is a Legendre function of the second kind,

$$Q_1(z)=-1+\frac{z}{2}\ln\frac{z+1}{z-1} , \tag{4.11}$$

and it is easy to see that $\mathscr{D}$ also is the free-field Green's function:

$$\left[\Box_x+\frac{2}{r^2}\right]\mathscr{D}(x,x')=-i\hbar\delta^2(x-x') . \tag{4.12}$$

One may explicitly check that the infrared behavior as the points separate is regular, while the ultraviolet singularity of $\mathscr{D}$ at coincident points is the same as in a free two-dimensional scalar theory.

We now derive an analog of the Lehmann spectral representation in our SO(2,1)-invariant quantization scheme. The two-point function is presented with the help of a complete set of intermediate states:

$$\langle 0\,|\,\hat{\Phi}(x)\hat{\Phi}(x')\,|\,0\rangle$$
$$=\sum_j\int_0^\infty d\omega\langle 0\,|\,\hat{\Phi}(x)\,|\,j,\omega\rangle\langle j,\omega\,|\,\hat{\Phi}(x')\,|\,0\rangle . \tag{4.13}$$

(The summation on j may include an integration if states with continuous eigenvalues of the Casimir operator occur.) From the equality

$$\langle 0\,|\,[J^2,\hat{\Phi}(x)]\,|\,j,\omega\rangle=-j(j+1)\hbar^2\langle 0\,|\,\hat{\Phi}(x)\,|\,j,\omega\rangle , \tag{4.14a}$$

one obtains the Casimir equation for $\langle 0\,|\,\hat{\Phi}(x)\,|\,j,\omega\rangle$.

$$r^2\Box\langle 0\,|\,\hat{\Phi}(x)\,|\,j,\omega\rangle=-j(j+1)\langle 0\,|\,\hat{\Phi}(x)\,|\,j,\omega\rangle . \tag{4.14b}$$

The action of finite dilatations and time translations is

$$\hat{\Phi}(t,r)=e^{i(tH/\hbar)}e^{i[(\ln r)D/\hbar]}\hat{\Phi}(0,1)e^{-i[(\ln r)D/\hbar]}e^{-i(tH/\hbar)} , \tag{4.15}$$

so that the time dependence of $\langle 0 | \hat{\Phi}(x) | j,\omega \rangle$ is $e^{-i\omega t}$. The regular solutions of (4.14) with this time dependence are Bessel functions

$$\langle 0 | \hat{\Phi}(x) | j,\omega \rangle = \sqrt{\hbar} N(j) e^{-i\omega t} (\omega r)^{1/2} J_{j+1/2}(\omega r) , \quad (4.16)$$

where $N(j)$ gives the normalization. The full propagator can be represented as

$$\langle 0 | T\hat{\Phi}(x)\hat{\Phi}(x') | 0 \rangle = \frac{\hbar}{2} \sum_j \rho(j) \int_0^\infty d\omega\, e^{-i\omega|t-t'|} \sqrt{rr'} J_{j+1/2}(\omega r) J_{j+1/2}(\omega r')$$
$$= \frac{\hbar}{2\pi} \sum_j \rho(j) Q_j \left[\frac{(t-t')^2 - r^2 - r'^2}{2rr'} - i0 \right] , \quad \rho(j) = |N(j)|^2 . \quad (4.17)$$

This may also be presented in spectral form with respect to energy:

$$\langle 0 | T\hat{\Phi}(x)\hat{\Phi}(x') | 0 \rangle = \sum_j \rho(j) \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \int_0^\infty d\omega\, e^{-i\alpha(t-t')} \frac{i\hbar}{\alpha^2 - \omega^2 - i0} (\omega^2 rr')^{1/2} J_{j+1/2}(\omega r) J_{j+1/2}(\omega r'). \quad (4.18)$$

Equal-time commutators imply the Lehmann sum rule for $\rho(j)$:

$$\sum_j \rho(j) = 1 . \quad (4.19)$$

It should be pointed out that these spectral representations are valid not only for the canonical field $\hat{\Phi}$ but, with different weight $\rho(j)$, for two-point functions of any local operator of scale dimension zero, such as polynomials in $\hat{\Phi}$.

Another spectral representation uses a formula which is dispersive in the Casimir eigenvalue. This expression is conveniently given in Euclidean space; it is derived in Appendix A:

$$G_E(x,x') = \sum_j \rho(j) \sum_{m=-\infty}^{\infty} \int d\lambda\, Y^*_{\lambda,m}(\theta',\phi') \times \frac{i\hbar}{(j+\frac{1}{2})^2 + \lambda^2} Y_{\lambda,m}(\theta,\phi) \quad (4.20a)$$

with

$$Y_{\lambda,m}(\theta,\phi) = \frac{\Gamma(i\lambda + \frac{1}{2} - m)}{\Gamma(i\lambda)} \frac{e^{-im\phi}}{\sqrt{2\pi}} P^m_{-1/2+i\lambda}(\cosh\theta) . \quad (4.20b)$$

Here Γ is the gamma function, $P^m_{-1/2+i\lambda}$ the associated Legendre function (toroidal function), and θ and ϕ are coordinates parametrizing a hyperboloid. We have

$$\cosh\theta = \frac{\Lambda^2 + t^2 + r^2}{2\Lambda r} ,$$
$$\sinh\theta \cos\phi = \frac{\Lambda^2 - t^2 - r^2}{2\Lambda r} , \quad (4.21)$$
$$\sinh\theta \sin\phi = t/r .$$

The functions $Y_{\lambda,m}$ are harmonics on the hyperboloid. Just as in (4.2), Λ is an arbitrary scale. A Minkowski version of (4.20) and (4.21) can be obtained by analytic continuation.

C. Manifest SO(2,1)-covariant formalism

The coordinate system (4.21) may also be used to parametrize the hyperboloid in a manifestly SO(2,1)-covariant fashion. We define the coordinates

$$\xi^1 = \frac{\Lambda^2 + t^2 + r^2}{2\Lambda r} ,$$
$$\xi^2 = \frac{\Lambda^2 - t^2 - r^2}{2\Lambda r} ,$$
$$\xi^3 = \frac{t}{r} , \quad (4.22)$$
$$\xi^a \xi_a = 1 .$$

The contraction is formed with the three-dimensional metric of signature $(1,-1,-1)$. Under SO(2,1), the vector ξ transforms linearly by (pseudo)rotations. The argument of the Euclidean version of the propagators (4.10) and (4.17),

$$\frac{(t-t')^2 + r^2 + r'^2}{2rr'} ,$$

is just $\xi \cdot \xi'$, so that the free propagator is SO(2,1) invariant: $\mathscr{D}(\xi,\xi') = \mathscr{D}(\xi \cdot \xi')$.

In fact, the full theory can be conveniently reformulated in terms of these coordinates. The SO(2,1) invariance of Euclidean action is then manifest, since it may be expressed as

$$\int d^2x_E \hat{\mathscr{L}}_E = \int d^3\xi\, \delta(\xi\cdot\xi - 1) \times \left[\tfrac{1}{2} \hat{\Phi} \partial^a \partial_a \hat{\Phi} + \frac{2}{\beta^2} (e^{\beta\hat{\Phi}} - \beta\hat{\Phi} - 1) \right] . \quad (4.23)$$

The operators ∂_a are tangential derivatives

$$\partial_a \equiv \frac{\partial}{\partial \xi^a} - \xi_a \xi^b \frac{\partial}{\partial \xi^b} , \quad (4.24a)$$

satisfying

$$[\partial_a, \partial_b] = \xi_a \partial_b - \xi_b \partial a , \quad (4.24b)$$
$$\partial^a \partial_a \equiv \hat{\Box} = -r^2 \Box . \quad (4.24c)$$

Equation (4.23) is derived with the help of the following equivalence of measures:

$$\frac{d^2x_E}{r^2} \cong d^3\xi\,\delta(\xi\cdot\xi-1)\ . \tag{4.25}$$

The field equations for $\hat{\Phi}$ then reads

$$\hat{\Box}\hat{\Phi}+\frac{2}{\beta}(e^{\beta\hat{\Phi}}-1)=0\ . \tag{4.26}$$

V. PERTURBATION THEORY

To confirm the consistency of our space-translation-noninvariant approach to the Liouville theory, we show that both Green's functions and amplitudes for transitions between physical states can be computed order by order in a perturbative expansion in the dimensionless coupling constant β. First, the one- and two-point Green's functions are evaluated to one-loop order, and an SO(2,1)-invariant regularization scheme is exhibited. Then we give Feynman rules for a manifestly SO(2,1)-invariant perturbation theory that is renormalized by normal ordering the interaction. Finally, we consider scattering amplitudes in the tree approximation. Certain scattering amplitudes vanish for SO(2,1)-group-theoretic reasons: The decay of the one-particle state into any multiparticle state is prohibited. The amplitude for $2\longleftrightarrow 2$ particle states, however, is also found to vanish in an explicit tree-approximation calculation, even though it is group theoretically allowed. We then prove that all on-shell scattering amplitudes vanish in the tree approximation, and we discuss the prospects of extending the result to the exact S matrix.

A. One-loop perturbation theory

In tree approximation, we expand the theory about the classical solutions Φ_s of (3.4), as required by our postulates. This background field will acquire further quantum corrections $\delta\Phi$, which may be calculated by minimizing the effective action to the order of perturbation theory under consideration. In this new background field

$$\Phi_s+\delta\Phi=\langle 0\,|\,\Phi\,|\,0\rangle\ ,$$

we can calculate Green's functions for the field fluctuation $\hat{\Phi}=\Phi-\langle 0\,|\,\Phi\,|\,0\rangle$.

The effective action to order $\hbar$ is given by

$$\Gamma(\Phi)=\int d^2x\left[\tfrac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi-\frac{m^2+\hbar\delta m^2}{\beta^2}e^{\beta\Phi}\right] +\frac{i\hbar}{2}\mathrm{Tr}\ln(-\Box-m^2e^{\beta\Phi})\ , \tag{5.1}$$

where $\hbar\delta m^2$ is a mass renormalization of order $\hbar$. We now search for a minimum of $\Gamma(\Phi)$:

$$0=\frac{\delta\Gamma(\Phi)}{\delta\Phi(x)}=-\Box\Phi(x)-\frac{m^2+\hbar\delta m^2}{\beta}e^{\beta\Phi(x)} -i\frac{\hbar\beta m^2}{2}e^{\beta\Phi(x)}(-\Box-m^2e^{\beta\Phi})^{-1}(x,x)\ . \tag{5.2}$$

Assuming that $\delta\Phi$ is of order $\hbar$, it is easy to see that $\delta\Phi$ must satisfy the equation

$$\left[\Box+\frac{2}{r^2}\right]\delta\Phi=\frac{-1}{r^2}\left[\beta\mathscr{D}(x,x)+2\frac{\hbar\delta m^2}{\beta m^2}\right]\ , \tag{5.3}$$

where $\mathscr{D}$ is the propagator (4.10), which has an ultraviolet divergence at coincident points. In conventional scalar field theories in two-dimensional space-time, this is the only divergence; a fact which remains true in our approach, since (4.10) possesses an ordinary short-distance limit. Multiplication of two propagators is always regular. Consequently, regularizing the coincident-point singularity will remove all ultraviolet divergences.

In a Poincaré-invariant theory, a Poincaré-invariant regulator would be chosen; for example, one would set

$$\mathscr{D}_{\mathrm{reg}}(x,x)=\mathscr{D}(x-\epsilon,x+\epsilon)$$

with ϵ an infinitesimal vector. In our problem, Poincaré-invariant regularization is not called for; rather, we seek a regularization which preserves the SO(2,1) symmetry of the theory.

An SO(2,1)-invariant regularization of the singularity at coincident points of the Euclidean space propagator is achieved by passing to the coordinates (4.22), which linearize the SO(2,1) transformation, and by recalling that the propagator—a function of ξ and ξ'—depends only on $\xi\cdot\xi'$. The limit of coincident points is therefore $\xi\cdot\xi'\to\xi\cdot\xi=1$, and the regularized propagator at coincidence points is obtained by setting $\xi\cdot\xi'$ equal to $1+\eta$ for small η (Ref. 14):

$$\mathscr{D}_{\mathrm{reg}}(x,x)=\mathscr{D}_E(1+\eta)=\frac{\hbar}{2\pi}\left[-1+\tfrac{1}{2}\ln\frac{\eta}{2}\right]\ . \tag{5.4}$$

This expression is constant and thus SO(2,1) invariant.

With the help of the hyperbolic coordinates (4.22), it is easy to solve equation (5.3):

$$\delta\Phi(\xi)=\frac{-1}{\hbar}\int d^3\xi'\delta(\xi'\cdot\xi'-1)\mathscr{D}_E(\xi\cdot\xi') \times\left[\beta\mathscr{D}_E(1+\eta)+2\frac{\hbar\delta m^2}{\beta m^2}\right]\ . \tag{5.5}$$

Graphically, this equation is represented in Fig. 1(a). The manifest SO(2,1) invariance of the integral implies that $\delta\Phi$ can only depend on $\xi\cdot\xi=1$, so that $\delta\Phi$ is actually constant. In this case, (5.3) is trivially solved and we obtain

$$\delta\Phi=-\frac{\beta}{2}\mathscr{D}_E(1+\eta)-\frac{\hbar\delta m^2}{\beta m^2}\ . \tag{5.6}$$

This one-loop correction to $\langle 0\,|\,\Phi\,|\,0\rangle$ is absorbed in the mass renormalization by defining

$$\hbar\delta m^2=-\frac{m^2\beta^2}{2}\mathscr{D}_E(1+\eta)\ , \tag{5.7}$$

and we have $\langle 0\,|\,\hat{\Phi}\,|\,0\rangle=O(\hbar^2)$.

Higher Green's functions are evaluated by expanding around the minimum of $\Gamma(\Phi)$. For instance, the inverse propagator, to one-loop order, becomes

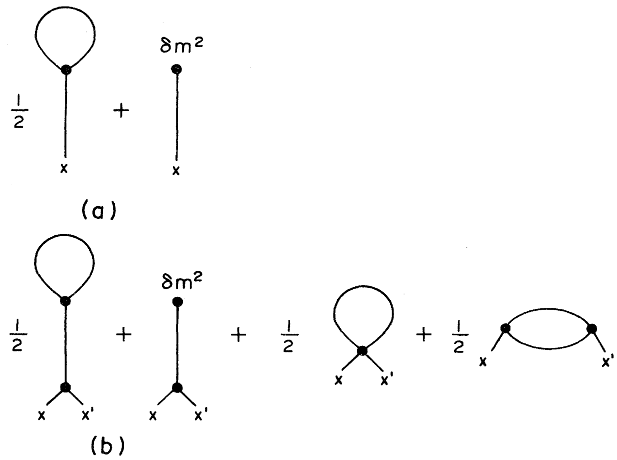

FIG. 1. The one-loop contributions and mass renormalization for the (a) one- and (b) two-point functions.

$$
\begin{aligned}
i\hbar G^{-1}(x,x') &= \left.\frac{\delta^2\Gamma(\Phi)}{\delta\Phi(x)\delta\Phi(x')}\right|_{\Phi=\langle 0|\Phi|0\rangle} \\
&= -\left[\Box_x + \frac{2}{r^2}\right]\delta(x-x') \\
&\quad + 2i\hbar^{-1}\beta^2\frac{\mathcal{D}^2(x,x')}{r^2r'^2} + O(\hbar^2)\,.
\end{aligned} \tag{5.8}
$$

The Feynman graphs contributing to this quantity are depicted in Fig. 1(b). Note that the mass does not occur in (5.8), so that G^{-1} does not depend on the mass renormalization. Also, G^{-1} is SO(2,1) invariant and infrared finite.

Finally, since

$$\Sigma(x,x') = -2\hbar^{-1}\beta^2\mathcal{D}^2(x,x')$$

is the lowest-order contribution to the time-ordered product of two interaction Lagrangian densities, it should have an SO(2,1)-invariant spectral representation like (4.17). This is indeed so, and in Appendix A we compute the appropriate spectral function $\sigma(n)$,

$$
\sigma(n) = \begin{cases} 0, & n \text{ even}\,, \\ \dfrac{(n+2)(n+\frac{1}{2})(n-1)}{(n+3)(n+1)n(n-2)}\,, & n \text{ odd}\,. \end{cases} \tag{5.9}
$$

B. General structure of perturbation theory

In higher orders of perturbation theory, the only source of ultraviolet infinities continues to be the propagator at coincident points, and only the mass need be renormalized. In particular, all tadpole contractions on the exponential interaction may be performed and the result is

$$e^{\beta\hat\Phi} = :e^{\beta\hat\Phi}:\exp\left[\frac{\beta^2}{2}\mathcal{D}(1+\eta)\right]\,, \tag{5.10}$$

where : : means that no contractions of two $\hat\Phi$'s should be made on the interaction vertex. To define the bare Lagrangian we replace the classical mass m by the bare mass m_0 in (2.1), and then shift Φ by the classical solution (3.4). Neglecting total derivative terms, we arrive at the Lagrangian for $\hat\Phi$

$$\hat{\mathcal{L}} = \tfrac{1}{2}\partial_\mu\hat\Phi\,\partial^\mu\hat\Phi - \frac{2m_0^2}{m^2\beta^2r^2}e^{\beta\hat\Phi} + \frac{2}{r^2\beta}\hat\Phi\,, \tag{5.11a}$$

which may also be rewritten by using (5.10):

$$
\begin{aligned}
\hat{\mathcal{L}} &= \tfrac{1}{2}\partial_\mu\hat\Phi\,\partial^\mu\hat\Phi - \frac{1}{r^2}\hat\Phi^2 \\
&\quad - \frac{2}{r^2}\left[\frac{m_0^2}{m^2\beta^2}e^{(\beta^2/2)\mathcal{D}(1+\eta)}:e^{\beta\hat\Phi}: - \tfrac{1}{2}\hat\Phi^2 - \frac{1}{\beta}\hat\Phi\right]\,.
\end{aligned} \tag{5.11b}
$$

With the help of a convenient renormalization of the mass

$$m_0^2 = m^2\exp\left[-\frac{\beta^2}{2}\mathcal{D}(1+\eta)\right]\,, \tag{5.12}$$

both the terms linear and quadratic in $\hat\Phi$ are completely cancelled from the interaction Lagrangian and we find

$$\hat{\mathcal{L}} = \tfrac{1}{2}\partial_\mu\hat\Phi\,\partial^\mu\hat\Phi - \frac{1}{r^2}\hat\Phi^2 + \hat{\mathcal{L}}_I\,, \tag{5.13a}$$

$$\hat{\mathcal{L}}_I = -\frac{2}{r^2}:\left[\frac{1}{\beta^2}e^{\beta\hat\Phi} - \tfrac{1}{2}\hat\Phi^2 - \frac{1}{\beta}\hat\Phi - \frac{1}{\beta^2}\right]:\,. \tag{5.13b}$$

It follows, therefore, that a renormalized perturbation theory can be based on $\hat{\mathcal{L}}_I$, provided the instruction is appended that two $\hat\Phi$'s of the same interaction vertex should not be contracted. Equivalently, $e^{\beta\hat\Phi}$ is normal ordered.

The perturbation theory derived from (5.13) is now ultraviolet finite, diagram by diagram. By inspection, one also sees that the theory is infrared finite. Furthermore, propagators and vertices are obtained in a manifestly SO(2,1)-invariant fashion. Thus, the full perturbative expansion is SO(2,1) invariant order by order. As a consequence, the vacuum expectation value of $\hat\Phi(\xi)$, which can depend only on $\xi\cdot\xi=1$ by SO(2,1) invariance, must be a

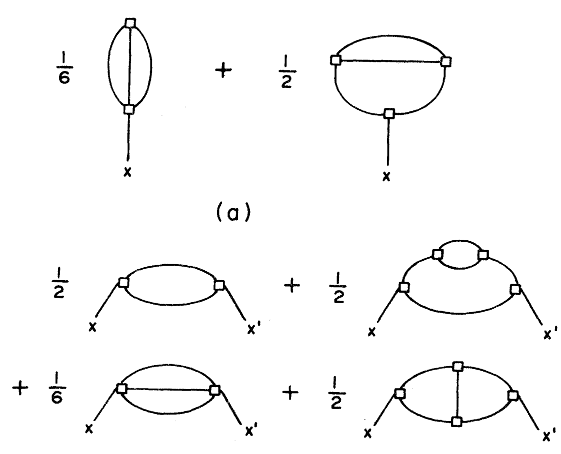

FIG. 2. The two-loop contributions for the (a) one- and (b) two-point functions. Squares represent the normal-ordered interaction vertices of (5.13b).

finite constant. To two-loop order, the graphs contributing to $\langle 0|\hat{\Phi}|0\rangle$ are shown in Fig. 2(a). The value of $\langle 0|\hat{\Phi}|0\rangle$ may of course also be absorbed into a finite mass renormalization of m. Thus, $\langle 0|\hat{\Phi}|0\rangle$ actually depends on the relation between the renormalized mass and m. Higher Green's functions, on the other hand, do not depend on the renormalized mass. The two-loop contributions to the inverse propagator are given in Fig. 2(b).

Another important consequence is that the eigenvalue of the Casimir operator C for the elementary "particles" of the theory is unchanged by quantum correction, as can already be recognized in the one-loop evaluation of G^{-1} in (5.8).

C. Transition amplitudes

An $n\longleftrightarrow m$ particle transition amplitude is calculated by first determining the connected $(n+m)$-point function, truncating the external lines by factoring away the complete propagator, and then passing on-shell by integrating with the appropriate "single-particle" wave functions. The relevant wave function for an incoming particle of energy $\hbar\omega$ is

$$(2\pi)^{-1/2}e^{-i\omega t}(\omega r)^{1/2}J_{3/2}(\omega r)\ ,$$

while the outgoing particle's wave function is the complex conjugate of the above. One may also define an off-shell transition amplitude by using off-shell wave functions

$$(2\pi)^{-1/2}e^{-i\alpha t}(\omega r)^{1/2}J_{3/2}(\omega r)\ \text{ with } \alpha^2\neq\omega^2\ .$$

Time-translation invariance ensures energy conservation, while the further SO(2,1) symmetry puts additional constraints on the amplitudes.

We now compute the tree approximation to three- and four-particle processes. The amputated three-point function is especially simple:

$$\Gamma_3(x_1,x_2,x_3)=\frac{-2i\beta}{r_1^2}\delta^2(x_1-x_2)\delta^2(x_1-x_3)\ . \quad (5.14)$$

Γ_3 is represented in Fig. 3(a). Integrating this with off-shell wave functions, we get

$$T_3(\alpha_1\omega_1,\alpha_2\omega_2,\alpha_3\omega_3)$$
$$=-i\beta\left[\frac{2}{\pi}\right]^{1/2}\delta(\alpha_1+\alpha_2+\alpha_3)V_3(\omega_1,\omega_2,\omega_3)\ , \quad (5.15)$$

where the three-vertex is

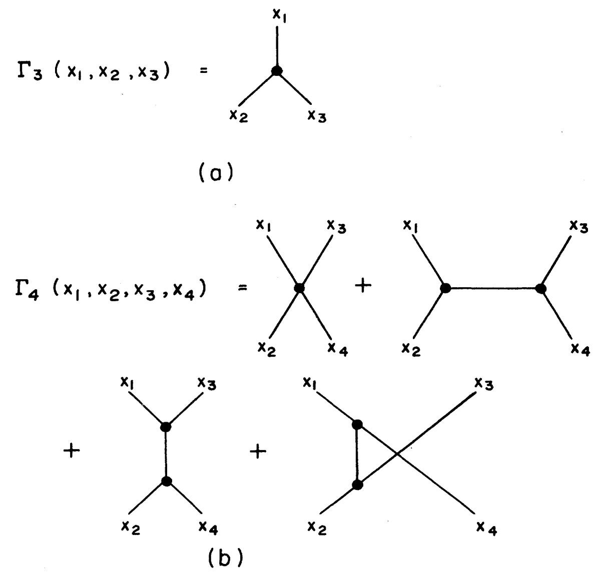

FIG. 3. The (a) three- and (b) four-point functions to the tree approximation in position space.

$$V_3(\omega_1,\omega_2,\omega_3)=\int_0^\infty \frac{dr}{r^2}\psi_{\omega_1}(r)\psi_{\omega_2}(r)\psi_{\omega_3}(r)$$
$$=\int_0^\infty \frac{dr}{\sqrt{r}}J_{3/2}(\omega_1 r)J_{3/2}(\omega_2 r)J_{3/2}(\omega_3 r)\ . \quad (5.16)$$

The integral over the Bessel functions is equal to

$$\left[\frac{2}{\pi}\right]^{1/2}\frac{\Delta^2}{(\omega_1\omega_2\omega_3)^{1/2}}$$

when ω_1, ω_2, and ω_3 form the sides of a triangle of area Δ; otherwise the integral vanishes.

To go on-shell, we set $\alpha_1=\omega_1$, $\alpha_2=-\omega_2$, $\alpha_3=-\omega_3$, and $\omega_1=\omega_2+\omega_3$; then T_3 describes the disintegration of 1 into 2 and 3. However, in that case the triangle formed from the three energies has zero area and the amplitude vanishes. In fact, using the SO(2,1) group theory, one may show that any process where a one-particle state goes to any multiparticle state is forbidden: the tensor product of SO(2,1) states $|j,\omega_2\rangle\otimes|j,\omega_3\rangle$ contains only states in the discrete series with $j'\geq 2j$, and these states are orthogonal to $|j,\omega_1\rangle$.[12] By extension of the argument to multiple tensor products, one sees that group theory predicts that a single particle is absolutely stable. Perturbation theory better agree.

The amputated four-point function is

$$\Gamma_4(x_1,x_2,x_3,x_4)=\frac{-2i\beta^2}{r_1^2}\delta^2(x_1-x_4)\delta^2(x_2-x_4)\delta^2(x_3-x_4)$$
$$-4\beta^2\left[\delta^2(x_1-x_2)\delta^2(x_3-x_4)\frac{1}{r_1^2r_3^2}\mathscr{D}(x_1,x_3)\right.$$
$$\left.+\delta^2(x_1-x_3)\delta^2(x_2-x_4)\frac{1}{r_1^2r_2^2}\mathscr{D}(x_1,x_2)+\delta^2(x_1-x_4)\delta^2(x_2-x_3)\frac{1}{r_1^2r_2^2}\mathscr{D}(x_1,x_2)\right]\ . \quad (5.17)$$

Γ_4 is represented in Fig. 3(b). With off-shell wave functions, T_4 becomes

$$T_4(\alpha_1\omega_1,\alpha_2\omega_2,\alpha_3\omega_3,\alpha_4\omega_4)$$
$$= -i\frac{\beta^2}{\pi}\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\left\{V_4(\omega_1,\omega_2,\omega_3,\omega_4) + 2\int_0^\infty d\omega \left[V_3(\omega_1,\omega_2,\omega)\frac{1}{(\alpha_1+\alpha_2)^2-\omega^2+i0}V_3(\omega,\omega_3,\omega_4) + V_3(\omega_1,\omega_3,\omega)\frac{1}{(\alpha_1+\alpha_3)^2-\omega^2+i0}V_3(\omega,\omega_2,\omega_4) + V_3(\omega_1,\omega_4,\omega)\frac{1}{(\alpha_1+\alpha_4)^2-\omega^2+i0}V_3(\omega,\omega_2,\omega_3)\right]\right\}, \tag{5.18}$$

where the four-vertex is

$$V_4(\omega_1,\omega_2,\omega_3,\omega_4) = \int_0^\infty \frac{dr}{r^2}\psi_{\omega_1}(r)\psi_{\omega_2}(r)\psi_{\omega_3}(r)\psi_{\omega_4}(r) = (\omega_1\omega_2\omega_3\omega_4)^{1/2}\int_0^\infty dr\, J_{3/2}(\omega_1 r)J_{3/2}(\omega_2 r)J_{3/2}(\omega_3 r)J_{3/2}(\omega_4 r)\,. \tag{5.19}$$

There are two on-shell configurations: $\alpha_1=\omega_1$, $\alpha_2=-\omega_2$, $\alpha_3=-\omega_3$, $\alpha_4=-\omega_4$, $\omega_1=\omega_2+\omega_3+\omega_4$, describing the disintegration of one particle into three, and $\alpha_1=\omega_1$, $\alpha_2=\omega_2$, $\alpha_3=-\omega_3$, $\alpha_4=-\omega_4$, $\omega_1+\omega_2=\omega_3+\omega_4$, describing two-particle scattering. The former should vanish; the single particle is stable. Indeed it does, since the three-point vertices in (5.18) cannot meet the triangle restriction on their arguments and the four-point vertex is zero in this configuration.

Remarkably, the particle-particle scattering amplitude also vanishes. Each of the four terms in (5.18) involves $(\omega_1\omega_2\omega_3\omega_4)^{-1}$ times a fifth-degree polynomial in the energies, and these highly nontrivial expressions finally sum to zero. (There are no significant partial cancellations.)

It is important to appreciate that this vanishing has nothing to do with SO(2,1) invariance, which in fact allows nontrivial scattering in which total energy is conserved but relative energy can change. One can obtain an SO(2,1)-invariant field theory with arbitrary potential $V(\hat{\Phi})$ simply by inserting that potential in (3.11a) or (5.13) in place of $e^{\beta\hat{\Phi}}-\beta\hat{\Phi}-1$. The effect in the $2\longleftrightarrow 2$ scattering calculation is simply to change the relative strengths of the four- and three-vertex contributions. The scattering amplitude would no longer vanish, although the decay amplitude still vanishes.

VI. ABSENCE OF CLASSICAL SCATTERING

The absence of two-body scattering in the tree approximation is not unexpected, since there exist classical canonical transformations which map the Liouville theory into a free massless theory.[1,2] The further shift of the Liouville field Φ to the fluctuating field $\hat{\Phi}$ is obviously canonical. Moreover, a harmonic field ϕ, obeying the free massless equation $\Box\phi=0$, may be converted to a free fluctuating Liouville field $\hat{\Phi}_0$, obeying the Casimir equation $(r^2\Box+2)\hat{\Phi}_0=0$, by the fixed-time operator $\partial/\partial r-1/r$, which satisfies the identity

$$(r^2\Box+2)\left[\frac{\partial}{\partial r}-\frac{1}{r}\right]=r^2\left[\frac{\partial}{\partial r}-\frac{1}{r}\right]\Box\,.$$

We have

$$\hat{\Phi}_0=\left[\frac{\partial}{\partial r}-\frac{1}{r}\right]\phi\,. \tag{6.1}$$

This too is a canonical transformation, and we see that the Liouville dynamics governing $\hat{\Phi}$ may be canonically transformed into free dynamics for $\hat{\Phi}_0$. While these transformations are only classical, they do ensure that the classical S matrix is trivial, i.e., all scattering amplitudes vanish in the tree approximation.[15] A possible approach to the general proof is the following.

In the Lehmann-Symanzik-Zimmermann formalism, transition amplitudes are obtained from truncated Green's functions in which free-particle wave operators are applied to extract external-line poles. Green's function of the canonical field are usually taken, and were used in the calculations of Sec. V. However, any local operator with nonvanishing matrix elements between the vacuum and the one-particle states can be used as an interpolating field. Thus, if there is a local composite operator $A(x)$ which satisfies the free-field equation $(r^2\Box+2)A=0$, but still connects the vacuum to one-particle states, the S matrix of the theory is trivial since the residues at external-line poles vanish.

A candidate for such an interpolating field is the energy-momentum tensor $\theta^{\mu\nu}$ of (2.10c) which classically is conserved, symmetric, and traceless owing to the conformal invariance of the Liouville theory.[16] In two dimensions, any tensor with these properties satisfies[8] $\Box\theta^{\mu\nu}=0$. [Note that $\theta^{\mu\nu}$ is distinct from the energy-momentum tensor $\hat{\theta}^{\mu\nu}$ (3.10) of the shifted theory.] Although $\theta^{\mu\nu}$ is not directly suitable in the present SO(2,1)-invariant theory, it is easily converted to the interpolating field we need by shifting $\Phi=\Phi_s+\hat{\Phi}$ [with Φ_s given in

(3.4)], subtracting the value of $\theta^{\mu\nu}$ on Φ_s, and applying $\partial/\partial r - 1/r$. The resulting tensor

$$T^{\mu\nu}=\left[\frac{\partial}{\partial r}-\frac{1}{r}\right]\left[\partial^\mu\hat\Phi\,\partial^\nu\hat\Phi-\tfrac{1}{2}g^{\mu\nu}\partial_\alpha\hat\Phi\,\partial^\alpha\hat\Phi+\frac{1}{\beta}(g^{\mu\nu}\Box-2\partial^\mu\partial^\nu)\hat\Phi+\partial^\mu\hat\Phi\,\partial^\nu\Phi_s+\partial^\mu\Phi_s\,\partial^\nu\hat\Phi-g^{\mu\nu}\partial_\alpha\hat\Phi\,\partial^\alpha\Phi_s\right] \quad (6.2)$$

satisfies

$$(r^2\Box+2)T^{\mu\nu}=0\ . \quad (6.3)$$

Any of the two independent components of $T^{\mu\nu}$ can be used as the interpolating field. For example, the operator

$$T^{01}=\frac{2}{\beta}\left[\partial_0^3\hat\Phi-\left[\Box+\frac{2}{r^2}\right]\partial_0\hat\Phi\right]-\left[\frac{\partial}{\partial r}-\frac{1}{r}\right]\partial_0\hat\Phi\,\partial_1\hat\Phi \quad (6.4)$$

clearly connects the vacuum to one-particle states.

The complete S matrix of the SO(2,1) quantized Liouville model will be trivial if the required properties of $T^{\mu\nu}$ can be maintained after renormalization. However, when proper SO(2,1)-invariant regularization is used to define the energy-momentum tensor, anomalies prevent $T^{\mu\nu}$ from satisfying (6.3). (Details of the construction are given in Appendix B.) Thus, the question about the triviality of the full S matrix remains open, although at the tree level anomalies are absent and once again we can conclude that the classical S matrix is trivial.

VII. CONCLUSION

The breaking of translation invariance in the perturbative expansion, which has been developed for the Liouville field theory does not follow in every detail conventional examples of spontaneous symmetry breaking. The principal difference is that our static background solution has infinite energy. Obviously there exist static configurations, though not solutions, with finite energy. These configurations are also initial value data for finite-energy, time-dependent solutions. Thus, the Liouville equation certainly possesses solutions of lower energy than the one we have used; however, they are time dependent. Nevertheless, our perturbation theory is stable. Whether the finite-energy configurations and time-dependent solutions are relevant to some other sector of the quantum field theory is unclear. Also, the definition of energy is not without ambiguity: the conformally improved version with indefinite sign differs from the conventional, positive expression.

There are no Goldstone zero modes in the small-oscillation spectrum. This is gratifying since we need not concern ourselves whether translation symmetry is restored by infrared-singular fluctuations, as it is for the conventional soliton phenomenon. We recognize that the absence of a Goldstone translation mode is related to the infinite energy of the static solution: the divergent quantity $\int dr(\Phi_s')^2$ is the normalization integral for the translation mode and also a contribution to the energy.

It is again the singularity of the background field that makes the space semicompactified. In this respect our perturbation theory is similar to analyses of the Liouville theory with boundary conditions.[17] Indeed, at the origin our field satisfies

$$\Phi'=-\frac{m\sqrt{2}}{\beta}e^{(\beta/2)\Phi}\ .$$

However, it is to be emphasized that we do not impose such a boundary condition *a priori* and we do not restrict the Minkowski-space Liouville theory.

The background field can be viewed as a singular structure at $r=0$.[18] In this sense, what we are doing is analogous to inserting a singular Dirac monopole into quantum electrodynamics or a singular Wu-Yang monopole into Yang-Mills theory and developing the quantum field theory in the presence of that singular configuration. However, the difference is that in the Liouville model we *must* have the singular background field to define perturbation theory, while in the preceding two examples the theory is well defined in the absence of the singular background field. Moreover, the singularity does not describe a localized source at $r=0$; i.e., $(d^2/dr^2)\ln r$ does not give a δ function.

One may inquire how our procedure would operate in a model which possesses both a conventional perturbation expansion as well as singular, static solutions like, for example, two-dimensional Φ^4 theory with a positive quadratic mass term.[19] Of course, since translation symmetry-preserving procedures exist for this model, our method is not called for. However, if one does insist on using it, and expands around a singular background, the "theory" that would emerge is stable by the energy criterion, but a particle interpretation cannot be given. The reason is that there is no symmetry, like the Poincaré symmetry of conventional dynamics or the SO(2,1) symmetry of our Liouville model, that prevents the single particle from decaying. The three-point function, for instance, does not vanish on-shell, as we have verified by calculation. It is not clear whether such a theory can be well defined; of course, for the Φ^4 model there is no good reason to define it in the first place.

One may think of our background field as an SO(2,1)-invariant regularization of the infrared-divergent Liouville theory.[20] Indeed, with x_0^1 large and negative, the restricted space approximates the full space and processes taking place far from x_0^1 should be insensitive to the singular "source" at x_0^1. In this way we are approximating a free massless theory: Φ_s approaches negative infinity as x_0^1 becomes negatively large; the two-point function approaches that of the free massless theory, which ordinarily is infrared singular, but here is regulated by x_0^1; the tree S matrix is found to be trivial at finite x_0^1 and it remains trivial at $x_0^1\to-\infty$, etc. However, it must be stressed that the limit cannot ultimately be taken; the regulator cannot be removed: a free, massless, two-dimensional spinless theory does not exist; the free Liouville propagator has no limit as $x_0^1\to-\infty$, etc. Most explicitly we see this in the single-particle wave functions, which for large x_0^1 become the sum of two exponentials:

$$\left[\frac{2}{\pi}\right]^{1/2}\cos\omega(r-x_0^1)=\frac{1}{\sqrt{2\pi}}e^{i\omega(r-x_0^1)}+\frac{1}{\sqrt{2\pi}}e^{-i\omega(r-x_0^1)}$$

If one multiplies by $e^{i\omega x_0^1}$ or $e^{-i\omega r_0^1}$, one could say that one arrives in the limit at $(2\pi)^{-1/2}e^{i\omega r}$ or $(2\pi)^{-1/2}e^{-i\omega r}$, but this is only a formal statement and no true definition of the limit may be given. In the Liouville theory there is only one state per energy, and in the free theory there are two. The group-theoretical distinction between the Liouville and free theories is that while both give a realization of SO(2,1)-invariant dynamics, the former is characterized by the Casimir $j=1$, while the latter has $j=0$.

Thus, we suggest that in spite of its peculiarities, our procedure exemplifies spontaneous breaking of translational symmetry. Indeed, the full conformal group is also broken, and only the SO(2,1) group is realized on the states. Let us recall that in conventional examples of spontaneous symmetry breaking, the stability group is determined by the isotropy group of the lowest-energy solution. Here the concept of energy is ambiguous, but *all* solutions possess an SO(2,1) isotropy group; hence, it is not surprising that this should also be the group realized in the quantum theory. Indeed, other solutions can be used as a background field for the quantum theory; they would break the full conformal group into different SO(2,1) subgroups.

What is surprising is that the S matrix may be unity. Thus, the Liouville model may be similar to the Schwinger model: a symmetry is broken but there are no on-shell interactions, even though off-shell Green's functions are nontrivial.

We have no proof that the S matrix is trivial, but we can offer a suggestion why this may be so. We have placed the underlying geometrical significance of the Liouville equation in the background, but let us now recall that any classical solution Φ defines a space of constant negative curvature with metric tensor $e^{\beta\Phi}g_{\mu\nu}$. There is only one such space, and any variation of Φ can be compensated by coordinate redefinition. We did not emphasize a coordinate-free description, and an interaction appeared to act. However, it is possible that the apparent dynamics is merely the action of diffeomorphisms on the manifold. This geometrical interpretation may be linked to the triviality of the S matrix.

Finally, let us observe that the consistency of our procedure is guaranteed by the consistency of the SO(2,1)-invariant Lagrangian (5.13) with positive interaction. The SO(2,1)-invariant quantization may be extended to the supersymmetric Liouville theory.[5] The fact that an SO(2,1)-de Sitter—type geometry has emerged dynamically is unexpected, but reminiscent of similar effects in four-dimensional supergravities.[21]

Note added in proof. We have been informed that computer simulation of the Liouville quantum field theory by Monte Carlo methods support our claim that translation symmetry is broken [C. Bernard, B. Lautrup, and E. Rabinovici, CERN Report No. TH 3671 (unpublished)].

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APPENDIX A: SPECTRAL FORMS

In this appendix, we derive the spectral representation (4.20), show how it can be utilized to compute the effective potential for the fluctuation field $\hat{\Phi}$, and derive the spectral function (5.9).

1. Spectral representation (4.20)

To establish (4.20), we use (4.17) and derive the appropriate spectral representation for Q_j, which satisfies

$$[C+j(j+1)]Q_j(\xi\cdot\xi')=2\pi\delta^3(\xi-\xi') . \qquad (A1)$$

Since the Casimir operator $C=-r^2\Box$ commutes with the compact generator J^1, they can be simultaneously diagonalized:

$$CY_{\lambda,m}=(\lambda^2+\tfrac{1}{4})Y_{\lambda,m} , \qquad (A2a)$$

$$J^1Y_{\lambda,m}=mY_{\lambda,m} . \qquad (A2b)$$

Here, λ is a continuous eigenvalue and its range will be fixed later, while m is an integer (see below). To solve (A2), it is convenient to express C and J^1 in terms of the coordinates parametrizing the hyperboloid, which we have introduced in (4.21) and (4.22):

$$C=-\frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}(\sinh\theta)\frac{\partial}{\partial\theta}-\frac{1}{\sinh^2\theta}\frac{\partial^2}{\partial\phi^2} , \qquad (A3a)$$

$$J^1=i\frac{\partial}{\partial\phi} . \qquad (A3b)$$

From (A2b) and (A3b), we see that

$$Y_{\lambda,m}(\theta,\phi)=e^{-im\phi}p_\lambda^m(\theta) , \qquad (A4)$$

so that m must be an integer to assure that $Y_{\lambda,m}$ is single-valued, while (A2a) and (A3a) require that $p_\lambda^m(\theta)$ satisfy

$$-\frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}(\sinh\theta)\frac{\partial}{\partial\theta}p_\lambda^m(\theta)+\frac{m^2}{\sinh^2\theta}p_\lambda^m(\theta)=(\lambda^2+\tfrac{1}{4})p_\lambda^m(\theta) . \qquad (A5)$$

The solutions to Eq. (A5) are associate Legendre functions $P^m_{-1/2+i\lambda}$ (toroidal functions) which are normalizable provided λ is real. The functions $Y_{\lambda,m}$ are normalized by

$$Y_{\lambda,m}(\theta,\phi)=\frac{e^{-im\phi}}{\sqrt{2\pi}}\frac{\Gamma(i\lambda+\frac{1}{2}-m)}{\Gamma(i\lambda)}P^m_{-1/2+i\lambda}(\cosh\theta) . \qquad (A6)$$

The desired expression for Q_j is thus

$$Q_j(\cosh\bar{\theta})=\sum_{m=-\infty}^{\infty}\int d\lambda\, Y^*_{\lambda,m}(\theta',\phi')\times\frac{1}{(j+\frac{1}{2})^2+\lambda^2}Y_{\lambda,m}(\theta,\phi) , \qquad (A7)$$

where $\cosh\bar{\theta}=\cosh\theta\cosh\theta'-\sinh\theta\sinh\theta'\cos(\phi-\phi')$.

We may check formula (A7) by first performing the summation over m, remarking that

$$\Gamma(-i\lambda+\tfrac{1}{2}-m)\Gamma(i\lambda+\tfrac{1}{2}+m)=\frac{\pi(-1)^m}{\cosh\lambda\pi}\ , \qquad (A8)$$

and that[22]

$$\sum_m (-1)^m e^{-im(\phi-\phi')}\frac{\Gamma(i\lambda+\frac{1}{2}-m)}{\Gamma(i\lambda+\frac{1}{2}+m)}P^m_{-1/2+i\lambda}(\cosh\theta) \times P^m_{-1/2+i\lambda}(\cosh\theta')=P_{-1/2+i\lambda}(\cosh\bar{\theta})\ , \qquad (A9)$$

where $P_\mu \equiv P^0_\mu$. Thus, we are left with the integral over λ alone:

$$\int d\lambda \frac{\lambda \sinh\pi\lambda}{\cosh\pi\lambda}P_{-1/2+i\lambda}(\cosh\bar{\theta})\frac{1}{(j+\frac{1}{2})^2+\lambda^2}\ . \qquad (A10)$$

For λ ranging from 0 to ∞, this integral appears in the literature[23] and equals Q_j:

$$Q_j(\cosh\bar{\theta})=\int_0^\infty d\lambda \frac{\lambda\sinh\pi\lambda}{\cosh\pi\lambda} \times P_{-1/2+i\lambda}(\cosh\bar{\theta})\frac{1}{(j+\frac{1}{2})^2+\lambda^2}\ . \qquad (A11)$$

The group-theoretical meaning of (4.20) is that it corresponds to taking in the quantum Hilbert space a basis which diagonalizes the compact generator J^1.

2. Effective potential

We compute the effective potential for the fluctuation field $\hat{\Phi}$ to one-loop order. The one-loop Euclidean-space effective action for a constant field $\hat{\Phi}$ is

$$\Gamma(\hat{\Phi})=\int \frac{d^2x}{r^2}\left[2\frac{m^2+\hbar\delta m^2}{m^2\beta^2}e^{\beta\hat{\Phi}}-\frac{2}{\beta}\hat{\Phi}\right]+\Gamma_1(\hat{\Phi})\ ,$$

$$\Gamma_1(\hat{\Phi})=\frac{\hbar}{2}\ln \mathrm{Det}(-r^2\Box+2e^{\beta\hat{\Phi}})\ . \qquad (A12)$$

Since the functions $Y_{\lambda,m}$ form a complete orthonormal basis that diagonalizes $C=-r^2\Box$, $\Gamma_1(\hat{\Phi})$ is also equal to

$$\Gamma_1(\hat{\Phi})=\frac{\hbar}{2}\int d^3\xi\,\delta(\xi\cdot\xi-1)\sum_{m=-\infty}^{\infty}\int_0^\infty d\lambda\, K(\lambda,\hat{\Phi},m)\ ,$$

$$K(\lambda,\hat{\Phi},m)=Y^*_{\lambda,m}(\theta,\phi)\ln(\lambda^2+\tfrac{1}{4}+2e^{\beta\hat{\Phi}})Y_{\lambda,m}(\theta,\phi)\ . \qquad (A13)$$

The summation over m can be performed in the same way as it was for Q_j, and we obtain

$$\Gamma_1(\hat{\Phi})=\frac{\hbar}{2}\int d^3\xi\,\delta(\xi\cdot\xi-1) \times\int_0^\infty d\lambda\frac{\lambda\sinh\pi\lambda}{\cosh\pi\lambda}\ln(\lambda^2+\tfrac{1}{4}+2e^{\beta\hat{\Phi}})\ . \qquad (A14)$$

The volume element now clearly appears, and the effective potential to one-loop order is given by

$$V_{\mathrm{eff}}(\hat{\Phi})=2\frac{m^2+\hbar\delta m^2}{m^2\beta^2}e^{\beta\hat{\Phi}}-\frac{2}{\beta}\hat{\Phi}+\frac{\hbar}{2} \times\int_0^\infty d\lambda\frac{\lambda\sinh\pi\lambda}{\cosh\pi\lambda}\ln(\lambda^2+\tfrac{1}{4}+2e^{\beta\hat{\Phi}})\ . \qquad (A15)$$

The renormalization used in (5.7) is equivalent to demanding that the minimum of the effective action, or in this case the effective potential, be unchanged to one-loop order. Hence,

$$V_{\mathrm{eff}}(\hat{\Phi})=\frac{2}{\beta^2}(e^{\beta\hat{\Phi}}-\beta\hat{\Phi})+\frac{\hbar}{2}\int_0^\infty d\lambda\frac{\lambda\sinh\pi\lambda}{\cosh\pi\lambda}\left[\ln\frac{\lambda^2+\frac{1}{4}+2e^{\beta\hat{\Phi}}}{\lambda^2+(9/4)}-\frac{2(e^{\beta\hat{\Phi}}-1)}{\lambda^2+(9/4)}\right]. \qquad (A16)$$

3. Spectral function (5.9)

To derive a spectral representation for

$$\Sigma(x,x')=-2\hbar^{-1}\beta^2\mathscr{D}^2(x,x')\ ,$$

we proceed as follows. Since

$$\mathscr{D}^2(x,x')=\frac{\hbar^2}{4\pi^2}Q_1{}^2(z),\quad z=\frac{(t-t')^2-r^2-r'^2}{2rr'}\ , \qquad (A17)$$

we seek the coefficients of the expansion

$$Q_1{}^2(z)=\sum_n a_n Q_n(z)\ . \qquad (A18)$$

Use of the dispersive representation for Q_n shows that the above is equivalent to

$$Q_1{}^2(z)=\frac{1}{2}\sum_n a_n\int_{-1}^1 dz'\frac{P_n(z')}{z-z'}\ , \qquad (A18a)$$

where P_n is a Legendre polynomial. Taking discontinuities of both sides gives

$$2zQ_1(z)=-\sum_n a_n P_n(z)\ . \qquad (A19a)$$

Therefore, the coefficients are

$$a_n=-\int_{-1}^1 dz(2n+1)zP_n(z)Q_1(z)\ . \qquad (A19b)$$

This integral may be evaluated as follows:

$$a_n=-2\frac{(n-1)(2n+1)(n+2)}{(n-2)(n+1)n(n+3)}\ , \qquad (A20)$$

which then implies Eq. (5.9).

APPENDIX B: CONSTRUCTION OF THE ENERGY-MOMENTUM TENSOR

In order to construct the properly SO(2,1)-regulated energy-momentum tensor for the shifted Liouville theory, it is convenient to use the SO(2,1)-covariant formalism; we therefore work in Euclidean space. The formal stress tensor which follows from (4.23) is

$$\hat{\theta}_{ab}=\partial_a\hat{\Phi}\,\partial_b\hat{\Phi}-\frac{P_{ab}}{2}(\partial_c\hat{\Phi}\,\partial^c\hat{\Phi})+\frac{2g_{ab}}{\beta^2}(e^{\beta\hat{\Phi}}-\beta\hat{\Phi}-1)\ . \tag{B1}$$

Here P_{ab} is a tangential projection tensor,

$$P_{ab}(\xi)=g_{ab}-\xi_a\xi_b\ ,\quad \partial^a P_{ab}=-2\xi_b\ ,\quad P^a{}_a=2\ , \tag{B2}$$

and one readily verifies by virtue of the field equation (4.26) that $\hat{\theta}_{ab}$ is conserved, which is a statement of the theory's SO(2,1) invariance. The relation between $\hat{\theta}_{ab}$ and the (Euclidean-space) tensor $\hat{\theta}_{\mu\nu}$,

$$\hat{\theta}_{\mu\nu}=\partial_\mu\hat{\Phi}\,\partial_\nu\hat{\Phi}-\frac{g_{\mu\nu}}{2}\partial_\alpha\hat{\Phi}\,\partial^\alpha\hat{\Phi}-\frac{2g_{\mu\nu}}{\beta^2}(e^{\beta\hat{\Phi}}-\beta\hat{\Phi}-1)\ , \tag{B3}$$

may be presented in the following way. We consider only the bilinear parts of the respective tensors—the interaction terms are obviously identical:

$$\hat{\theta}^0_{ab}=\partial_a\hat{\Phi}\,\partial_b\hat{\Phi}-\tfrac{1}{2}P_{ab}(\partial_c\hat{\Phi}\,\partial^c\hat{\Phi})\ , \tag{B4a}$$

$$\hat{\theta}^0_{\mu\nu}=\partial_\mu\hat{\Phi}\,\partial_\nu\hat{\Phi}-\frac{g_{\mu\nu}}{2}\partial_\alpha\hat{\Phi}\,\partial^\alpha\hat{\Phi}\ . \tag{B4b}$$

Both tensors are traceless and symmetric; moreover, $\hat{\theta}^0_{ab}$ is orthogonal to ξ^a since $\xi^a\partial_a=0$. Hence, (B4a) has two independent components just as (B4b). Calling the diagonal components of $\hat{\theta}^0_{\mu\nu}$, $\mathscr{E}^0$, and $-\mathscr{E}^0$, and the off-diagonal one $\mathscr{P}^0$, we can establish the formulas

$$\hat{\theta}^0_{11}+\hat{\theta}^0_{22}-2\hat{\theta}^0_{12}=-\Lambda^2\mathscr{E}^0\ , \tag{B5a}$$

$$\hat{\theta}^0_{31}-\hat{\theta}^0_{32}=\Lambda t\mathscr{E}^0+\Lambda r\mathscr{P}^0\ . \tag{B5b}$$

Here Λ is the usual arbitrary scale, and the components on the left-hand side refer to the 3×3 SO(2,1) tensor given in (B1).

To regulate $\hat{\theta}_{ab}$, we first use the equation of motion (4.26) to present (B1) in equivalent form,

$$\hat{\theta}_{ab}=\partial_a\hat{\Phi}\,\partial_b\hat{\Phi}-\tfrac{1}{2}P_{ab}(\partial_c\hat{\Phi}\,\partial^c\hat{\Phi})-\frac{g_{ab}}{\beta}(\hat{\Box}+2)\hat{\Phi}\ , \tag{B6}$$

so that only the quadratic portion $\hat{\theta}^0_{ab}$ need be regulated. Since $\hat{\theta}^0_{ab}$ is manifestly traceless, the only question concerns its divergence, which formally is $\beta^{-1}\partial_b(\hat{\Box}+2)\hat{\Phi}$.

We define the normal-ordered, split-point bilinear by

$$:\hat{\Phi}(\xi)\hat{\Phi}(\xi'):=\hat{\Phi}(\xi)\hat{\Phi}(\xi')-\mathscr{D}(\xi\cdot\xi')\ , \tag{B7a}$$

$$:\partial_a\hat{\Phi}(\xi)\partial'_b\hat{\Phi}(\xi'):=\partial_a\hat{\Phi}(\xi)\partial'_b\hat{\Phi}(\xi')-\partial_a\partial'_b\mathscr{D}(\xi\cdot\xi')\ . \tag{B7b}$$

To pass to coincident points, ξ' is taken to be a function of ξ, with $\xi'-\xi$ of order $\sqrt{\eta}$, and $\xi'\cdot\xi=1+\eta$, as η goes to zero.

The regulated expression for $\hat{\theta}^0_{ab}$ is the normal-ordered version of the quadratic part of (B6), which can be written in terms of unordered products with the help of (B7b):

$$\begin{aligned}:\theta^0_{ab}:=\lim_{\eta\to 0}\{&\tfrac{1}{4}[\partial_a\hat{\Phi}(\xi)\partial'_b\hat{\Phi}(\xi')+\partial'_a\hat{\Phi}(\xi')\partial_b\hat{\Phi}(\xi)+\partial'_b\hat{\Phi}(\xi')\partial_a\hat{\Phi}(\xi)+\partial_b\hat{\Phi}(\xi)\partial'_a\Phi(\xi')]\\&-\tfrac{1}{4}P_{ab}(\xi)[\partial_c\hat{\Phi}(\xi)\partial'^c\Phi(\xi')+\partial'_c\hat{\Phi}(\xi')\partial^c\Phi(\xi)]+C_{ab}(\eta)\}\ .\end{aligned} \tag{B8}$$

Here C_{ab} is an η-dependent, singular c number, and all derivatives are with respect to argument. Formula (B8) is manifestly traceless, but the divergence must be recomputed. When (B8) is differentiated with respect to ξ, we encounter derivatives of ξ'_a with respect to ξ_b, and these differ from P_{ab} by terms of $O(\sqrt{\eta})$. It is therefore clear that one gets

$$\begin{aligned}\partial^a:\theta^0_{ab}:=\lim_{\eta\to 0}\{&\tfrac{1}{4}[\hat{\Box}\hat{\Phi}(\xi)\partial'_b\hat{\Phi}(\xi')+\hat{\Box}'\hat{\Phi}(\xi')\partial_b\hat{\Phi}(\xi)+\partial'_b\hat{\Phi}(\xi')\hat{\Box}\hat{\Phi}(\xi)+\partial_b\hat{\Phi}(\xi)\hat{\Box}'\hat{\Phi}(\xi')]\\&+\sqrt{\eta}(\text{terms bilinear in }\hat{\Phi})+c\text{ number}\}\ .\end{aligned} \tag{B9}$$

Again, all differentiation is with respect to the argument. The "(terms bilinear in $\hat{\Phi}$)" may be written in normal order plus an additional c number; after normal ordering, the factor $\sqrt{\eta}$ ensures that these operator bilinears vanish as $\eta\to 0$, so apart from a c number only the term in square brackets survives. We transform it as follows.

The operator equation of motion reads

$$\hat{\Box}\hat{\Phi}=-\frac{2}{\beta}(:e^{\beta\hat{\Phi}}:-1)\ , \tag{B10}$$

hence,

$$\begin{aligned}\hat{\Box}\hat{\Phi}(\xi)\partial'_b\hat{\Phi}(\xi')&=-\frac{2}{\beta}:e^{\beta\hat{\Phi}(\xi)}:\partial'_b\hat{\Phi}(\xi')+\frac{2}{\beta}\partial'_b\hat{\Phi}(\xi')\\&=-\frac{2}{\beta}:e^{\beta\hat{\Phi}(\xi)}\partial'_b\hat{\Phi}(\xi'):+\frac{2}{\beta}\partial'_b\hat{\Phi}(\xi')-2:e^{\beta\hat{\Phi}(\xi)}:\partial'_b\mathscr{D}(\xi\cdot\xi')\ .\end{aligned} \tag{B11}$$

In the first term the coincident-point limit may be taken to give

$$-\frac{2}{\beta}:e^{\beta\hat{\Phi}}\partial_b\hat{\Phi}:=-\frac{2}{\beta^2}\partial_b:e^{\beta\hat{\Phi}}:=\frac{1}{\beta}\partial_b\hat{\Box}\hat{\Phi}\ ,$$

while the second term leaves, apart from a c number,

$$\beta\hat{\Box}\hat{\Phi}(\xi)\partial'_b\mathscr{D}(\xi\cdot\xi')=\beta\hat{\Box}\hat{\Phi}(\xi)\xi^cP_{cb}(\xi')\mathscr{D}'(\xi\cdot\xi)\ .$$

In this way we find, in the limit $\xi'\to\xi$,

$$\partial^a:\theta^0_{ab}:=\frac{1}{\beta}\partial_b(\Box+2)\hat{\Phi}+\lim_{\eta\to 0}\left\{\frac{\beta}{2}[\Box\hat{\Phi}(\xi)\xi^cP_{cb}(\xi')+\Box'\hat{\Phi}(\xi')\xi'^cP_{cb}(\xi)]\mathscr{D}'(\xi\cdot\xi')\right\}\ . \quad \text{(B12)}$$

We have dropped irrelevant c numbers. The second term is the anomaly, which survives as $\eta\to 0$:

$$\frac{\beta}{2}[\hat{\Box}\hat{\Phi}(\xi)\xi^cP_{cb}(\xi')+\hat{\Box}'\hat{\Phi}(\xi')\xi'^cP_{cb}(\xi)]\mathscr{D}'(\xi\cdot\xi')\underset{\eta\to 0}{\to}-\frac{\beta\hbar}{4\pi}\xi_b\hat{\Box}\hat{\Phi}(\xi)+\frac{\beta\hbar}{8\pi\eta}(\xi'_b-\xi_b)[\hat{\Box}'\hat{\Phi}(\xi')-\hat{\Box}\hat{\Phi}(\xi)]\ , \quad \text{(B13)}$$

since

$$\mathscr{D}'(\xi'\cdot\xi)\to\frac{\hbar}{4\pi\eta}\left[1+\eta\ln\frac{\eta}{2}-\frac{1}{2}\right]\ .$$

Next, in the last term, $\hat{\Box}'\Phi(\xi')$ is expanded in powers of $\xi'-\xi$, to give

$$\frac{\beta\hbar}{8\pi\eta}(\xi'_a-\xi_a)(\xi'_b-\xi_b)\partial^a\hat{\Box}\hat{\Phi}(\xi)\ .$$

Because $\xi'-\xi$ is $O(\sqrt{\eta})$, this expression contributes an amount which depends on the precise regulation method. However, if we demand that the limiting result be manifestly SO(2,1) invariant, the only possible limiting form for $(1/\eta)(\xi'_a-\xi_a)(\xi'_b-\xi_b)$ is $-P_{ab}$, since the former has trace -2 and is transverse to ξ_a in the limit. Thus, we find

$$\partial^a:\theta^0_{ab}:=\frac{1}{\beta}\partial_b(\hat{\Box}+2)\hat{\Phi}-\frac{\beta\hbar}{8\pi}\xi_b\hat{\Box}\hat{\Phi}-\frac{\beta\hbar}{4\pi}\partial_b\hat{\Box}\hat{\Phi}\ . \quad \text{(B14)}$$

The conserved, normalized, energy-momentum tensor is

$$:\theta_{ab}:=:\theta^0_{ab}:-\frac{g_{ab}}{\beta}(\hat{\Box}+2)\hat{\Phi}+\frac{\beta\hbar}{16\pi}\xi_a\xi_b\hat{\Box}\hat{\Phi}+\frac{\beta\hbar}{4\pi}g_{ab}\hat{\Box}\hat{\Phi}\ . \quad \text{(B15)}$$

The last two terms are the anomaly. Note that the term proportional to g_{ab} can be absorbed in a renormalization of the corresponding classical term, however, the contribution proportional to $\xi_a\xi_b$ is new. Since the three conserved SO(2,1) currents are

$$J^a_b=:\theta^{ac}:\epsilon_{cdb}\xi^d\ ,$$
$$\partial_aJ^a_b=0\ , \quad \text{(B16)}$$

contributions to $:\theta^{ac}:$ proportional to ξ^c are irrelevant. Hence, an equivalent formula for energy-momentum tensor is

$$:\theta_{ab}:=:\partial_a\hat{\Phi}\,\partial_b\hat{\Phi}:-\tfrac{1}{2}P_{ab}:\partial_c\hat{\Phi}\,\partial^c\hat{\Phi}:-\frac{1}{\beta}P_{ab}\left[\left[1-\frac{\hbar\beta^2}{4\pi}\right]\hat{\Box}+2\right]\hat{\Phi}\ . \quad \text{(B17)}$$

This is not conserved, but the divergence is proportional to ξ.

One may now also try to construct a traceless, symmetric, and conserved tensor $:T_{\mu\nu}:$, which would generalize the classical formula (6.2). Clearly one contribution must involve $:\theta^0_{\mu\nu}:$. Since (B14) determines the divergence of $:\theta^0_{\mu\nu}:$, one may inquire whether there exists a traceless, time-local addition $\Delta T_{\mu\nu}$, which would cancel the nonzero divergence of $:\theta^0_{\mu\nu}:$. The answer is no; the anomaly provides an obstruction.

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[11]Static solutions to the Liouville equation were previously discussed by B. Barbashov, V. Nesterenko, and A. Chervyakov,

Teor. Mat. Fiz. 40, 15 (1979) [Theor. Math. Phys. 40, 572 (1979)]; however, the quantum fluctuations about these solutions are erroneously described because it is asserted that zero modes exist. In fact, they do not; see our Eq. (3.15) and the subsequent discussion.

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[13]Instead of H, one could diagonalize any SO(2,1) generator. A particularly interesting choice is J^1, which generates compact rotations, i.e., the SO(2) subgroup of SO(2,1). We do not pursue this here, but see Appendix A. This approach, to different problems, is developed by V. de Alfaro, S. Fubini, and G. Furlan, Nuovo Cimento A34, 569 (1976); R. Jackiw, Ann. Phys. (N.Y.) 129, 183 (1980).

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[15]The classical S matrix for the Liouville theory is also discussed by G. Dzhordzhadze, Teor. Mat. Fiz. 41, 33 (1979) [Theor. Math. Phys. 41, 867 (1979)]; we disagree with his conclusion that it is nontrivial.

[16]We thank C. Thorn for suggesting use of the energy-momentum tensor in this context.

[17]We thank A. Guth for discussions. Boundary conditions on the Liouville theory have been considered by B. Durhuus, H. Nielsen, P. Olesen, and J. Petersen, Nucl. Phys. B196, 498 (1982); B. Durhuus, P. Olesen, and J. Petersen, *ibid.* B198, 157 (1982); B201, 176 (1982); J.-L. Gervais and A. Neveu, *ibid.* B199, 59 (1982); B209, 125 (1982); A. Kihlberg and R. Marnelius, Göteborg Report No. 82-2, 1982 (unpublished); H. Bohr and H. Nielsen, Report No. ICTP 81, 82-7, 1982 (unpublished); L. Johanson, A. Kihlberg, and R. Marnelius; Göteborg Report No. 83-8, 1983 (unpublished). In these papers the boundary condition is parametrized by ρ; our theory corresponds to $\rho=1$, a value particularly discussed by Gervais and Neveu.

[18]We thank D. Gross and A. Guth for discussions.

[19]We thank S. Coleman and A. Guth for discussions.

[20]We thank D. Gross for discussions.

[21]P. Breitenlohner and D. Z. Freedman, Phys. Lett. 115B, 197 (1982); Ann. Phys. (N.Y.) 144, 249 (1982).

[22]N. Vilenkin, *Special Functions and the Theory of Group Representations,* translated in American Mathematical Society Translations (American Mathematical Society, Providence, Rhode Island, 1968), Vol, 22, p. 324. Further information concerning special functions related to the SO(2,1) group is given by R. Raczka, N. Limić, and J. Niederle, J. Math. Phys. 7, 1861 (1966); 7, 2026 (1966); 8, 1079 (1967).

[23]H. Bateman, in *Tables of Integral Transforms* (McGraw-Hill, New York, 1954), Vol. II, p. 330.