Quantization of gauge theories with linearly dependent generators

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The quantization rules for gauge theories with open algebras are generalized to the case of linearly dependent generators. The given zero-eigenvalue eigenvectors of the generators may also be linearly dependent and possess zero-eigenvalue eigenvectors which may also be linearly dependent and so on. We give the solution for the general case of such a hierarchy.

I. INTRODUCTION

In Ref. 1, we proposed a method which gave a closed formulation of the quantization rules for gauge theories with open algebras under the conditions of irreducibility and completeness. The condition of irreducibility requires the generators of gauge transformations to be linearly independent at the stationary part of the classical action (and consequently in a neighborhood of that point). The condition of completeness requires the degeneracy of the classical action to be totally due to its gauge invariance described by the given generators.

These conditions are fulfilled in a large class of gauge theories, but recently examples have appeared in the literature (Refs. 2–9) in which the condition of irreducibility does not hold. Of course, one can always single out a basis of linearly independent generators, but then one will generally lose either locality or relativistic covariance. The local and covariant generators are generally linearly dependent. In particular, this is the case when one describes spin-0 particles by an antisymmetric tensor or spin-$\frac{5}{2}$ particles by a symmetric spin tensor.

The purpose of the present work is the generalization of the method of Ref. 1 to reducible gauge theories. The condition of irreducibility will be replaced by a "hierarchy of reducibility" which may have any number of stages, in the following sense. Let the infinitesimal gauge transformations of a gauge field ϕ^i be represented in the conventional condensed form[10] as $\delta\phi^i = R^i_{\alpha_0}(\phi)\delta\theta^{\alpha_0}$, where $\delta\theta^{\alpha_0}$ are the parameters. If the vectors $R^i_{\alpha_0}$ enumerated by α_0 are linearly independent, by which we mean that the matrix $R^i_{\alpha_0}(\phi)$ at the stationary point has the maximal rank, the theory is irreducible and will be called a zero-stage theory. A first-stage theory corresponds to the case in which $R^i_{\alpha_0}(\phi)$ at the stationary point has nontrivial zero-eigenvalue eigenvectors $Z^{\alpha_0}_{1\alpha_1}$ with respect to the index α_0, but the matrix $Z^{\alpha_0}_{1\alpha_1}$ has the maximal rank. In a second-stage theory the matrix $Z^{\alpha_0}_{1\alpha_1}$ at the stationary point also has nontrivial zero-eigenvalue eigenvectors $Z^{\alpha_1}_{2\alpha_2}$ (with respect to the index α_1), but these $Z^{\alpha_1}_{2\alpha_2}$ have the maximal rank, and so on. The only limitation imposed is that the theory be of a finite stage. It is only important that the final Z shall have the maximal rank.

We shall suppose that the above zero-eigenvalue eigenvectors Z are all explicitly given, as well as the initial generators $R^i_{\alpha_0}$, and are local covariant operators. This is indeed the case in the known examples. The problem is to formulate the quantization rules in terms of these given quantities and avoid the introduction of other bases which may destroy locality and other indices which may destroy relativistic covariance.

The condition of completeness remains unchanged. Therefore the number of admissible gauge conditions needed to remove the degeneracy of the action must equal the true number of gauge invariances, i.e., the number of linearly independent generators. However, it may prove difficult to introduce such a set of gauge conditions in a local and covariant way. One would like the gauge conditions to carry the index α_0 of the generators $R^i_{\alpha_0}$, but then there will be too many gauge conditions. The problem arises of introducing gauge conditions using only indices matching those of R and the Z's, and at the same time guaranteeing the correct number of δ functions in the functional integral. For linear theories this problem has been solved in particular examples.[7,8] We shall give the general rules for gauge fixing and constructing ghost Lagrangians for theories with arbitrary reducible and open gauge algebras.

Our principal result is that the general formalism given in Ref. 1 is valid for gauge theories of any finite state of reducibility. A modification is needed only in two places: (1) in the determination of the minimal set of fields and antifields, on which the proper solution of the master equation $(S,S)=0$ can be realized, and (2) in the determination of the structure of the gauge fermion Ψ, securing the admissibility of gauge conditions.

The summary and the plan of the paper are as follows. Section II contains a precise formulation of our assumptions about reducible gauge theories. Section III contains the general construction of quantization rules and gauge algebra, unique for all reducible and irreducible theories. (This is a review of the method of Ref. 1.) In Sec. IV, we obtain the Feynman rules for first-stage theories. The solution of both aspects of the problem, i.e., the construction of reducible gauge algebras and the introduction of redundant gauge conditions, is discussed in detail. A new feature, which arises when passing from the first to higher stages of reducibility, is encountered in Sec. V, where we consider theories of the second stage. Section VI contains a complete formulation of the Feynman rules for theories of an arbitrary stage of reducibility. The reader who is interested only in the results without derivations and explanations need read only Secs. II and VI.

In Sec. VII, we consider two examples: the spin-0 antisymmetric tensor field[5-9] and the spin-$\frac{5}{2}$ symmetric spin-tensor field.[2-4] Both are linear theories, so the gauge algebra is trivial. However the problem of gauge fixing is nontrivial even at the linear level.[7,8] These examples are intended to illustrate the use of general formulas.

The Appendix contains the theory of the master equation. The properties of the master equation were briefly formulated in Ref. 1. Here they are considered in more detail.

Notation and conventions. We mainly use the notation of Ref. 1. In particular, $\epsilon_A=\epsilon(A)$ denotes the Grassmann parity of A; ∂_r and ∂_l are right and left derivatives. The ghost number of A will be denoted by $\mathrm{gh}(A)$.

The rank of a matrix is the maximal size of its invertible square minor. In the case of an even-parity matrix X one may speak about two ranks: those of the Bose-Bose and Fermi-Fermi blocks. These ranks will be denoted by $\mathrm{rank}_{\pm}X$. The rank of an even-parity matrix is $\mathrm{rank}X=\mathrm{rank}_{+}X+\mathrm{rank}_{-}X$. $\mathrm{Ber}X$ is the superdeterminant (Berezinian) of the even-parity square matrix X.

By linear independence of k vectors V^i_μ, $\mu=1,\ldots,k$, we always mean $\mathrm{rank}V^i_\mu=k$. In the presence of Grassmann numbers, this should be mentioned specially.

We use the condensed notation for the gauge field: ϕ^i, $i=1,2,\ldots,n=n_{+}+n_{-}$, where n_{+} (n_{-}) is the number of bosonic (fermionic) components. $\mathscr{S}(\phi)$ is the classical gauge action and is a bosonic quantity. ϕ_0 denotes everywhere a stationary point of $\mathscr{S}(\phi)$.

II. POSTULATES OF REDUCIBLE GAUGE THEORIES

Although the postulates of gauge theory, formulated in Ref. 1, must here be weakened, the following assumptions remain unchanged.

The classical gauge action $\mathscr{S}(\phi)$ is assumed to have at least one stationary point ϕ_0,

$$\left.\frac{\partial_r\mathscr{S}}{\partial\phi^i}\right|_{\phi_0}=0\ , \tag{2.1}$$

and to be regular (infinitely differentiable) in the neighborhood of ϕ_0. Further, m_{0+} bosonic and m_{0-} fermionic Noether identities are assumed to hold in a neighborhood of the stationary point:

$$\frac{\partial_r\mathscr{S}}{\partial\phi^i}R^i_{\alpha_0}=0\ ,\quad \alpha_0=1,\ldots,m_0=m_{0+}+m_{0-}\ , \tag{2.2}$$

where $R^i_{\alpha_0}(\phi)$ are regular functions, and $\epsilon(R^i_{\alpha_0})=\epsilon_i+\epsilon_{\alpha_0}$. Here $\epsilon_i=\epsilon(\phi^i)$, and ϵ_{α_0} is the Grassmann parity of the parameters of gauge transformations. The formulation of the remaining assumptions, namely, those of irreducibility and completeness, is different for theories having different stages of reducibility.

First-stage theories. For these theories we have

$$\mathrm{rank}_{\pm}R^i_{\alpha_0}\,|_{\phi_0}=(m_0-m_1)_{\pm},\quad m_{0\pm}>m_{1\pm}\ . \tag{2.3}$$

Then there exist $m_1=m_{1+}+m_{1-}$ zero-eigenvalue eigenvectors $Z^{\alpha_0}_{1\alpha_1}$:

$$R^i_{\alpha_0}Z^{\alpha_0}_{1\alpha_1}\,|_{\phi_0}=0\ ,$$
$$\epsilon(Z^{\alpha_0}_{1\alpha_1})=\epsilon_{\alpha_0}+\epsilon_{\alpha_1}\ ,\quad \alpha_1=1,\ldots,m_1\ . \tag{2.4}$$

These eigenvectors are assumed to be regular and linearly independent in a neighborhood of the stationary point:

$$\mathrm{rank}_{\pm}Z^{\alpha_0}_{1\alpha_1}\,|_{\phi_0}=m_{1\pm}\ . \tag{2.5}$$

The condition of completeness reads

$$\mathrm{rank}_{\pm}\left.\frac{\partial_l\partial_r\mathscr{S}}{\partial\phi^i\partial\phi^j}\right|_{\phi_0}=n_{\pm}-(m_0-m_1)_{\pm} \tag{2.6}$$

and $n_{\pm}>(m_0-m_1)_{\pm}$.

Second-stage theories. For these theories we have

$$\mathrm{rank}_{\pm}R^i_{\alpha_0}\,|_{\phi_0}=m_{0\pm}-(m_1-m_2)_{\pm}\ ,$$
$$m_{1\pm}>m_{2\pm},\quad m_{0\pm}>(m_1-m_2)_{\pm}\ . \tag{2.7}$$

There exist $m_1=m_{1+}+m_{1-}$ zero-eigenvalue eigenvectors $Z^{\alpha_0}_{1\alpha_1}$:

$$R^i_{\alpha_0}Z^{\alpha_0}_{1\alpha_1}\,|_{\phi_0}=0\ ,$$
$$\epsilon(Z^{\alpha_0}_{1\alpha_1})=\epsilon_{\alpha_0}+\epsilon_{\alpha_1}\ ,\quad \alpha_1=1,\ldots,m_1\ , \tag{2.8}$$

which are regular in a neighborhood of the stationary point, and

$$\mathrm{rank}_{\pm}Z^{\alpha_0}_{1\alpha_1}\,|_{\phi_0}=(m_1-m_2)_{\pm}\ . \tag{2.9}$$

Because of (2.9), there exist $m_2=m_{2+}+m_{2-}$ zero-eigenvalue eigenvectors $Z^{\alpha_1}_{2\alpha_2}$,

$$Z^{\alpha_0}_{1\alpha_1}Z^{\alpha_1}_{2\alpha_2}\,|_{\phi_0}=0\ ,$$
$$\epsilon(Z^{\alpha_1}_{2\alpha_2})=\epsilon_{\alpha_1}+\epsilon_{\alpha_2}\ ,\quad \alpha_2=1,\ldots,m_2\ , \tag{2.10}$$

which are assumed to be regular and linearly independent in a neighborhood of the stationary point:

$$\mathrm{rank}_{\pm}Z^{\alpha_1}_{2\alpha_2}\,|_{\phi_0}=m_{2\pm}\ . \tag{2.11}$$

The condition of completeness reads

$$\mathrm{rank}_{\pm}\left.\frac{\partial_l\partial_r\mathscr{S}}{\partial\phi^i\partial\phi^j}\right|_{\phi_0}=n_{\pm}-[m_0-(m_1-m_2)]_{\pm} \tag{2.12}$$

and $n_{\pm}>m_{0\pm}-(m_1-m_2)_{\pm}$.

Similarly, we define a theory having any finite stage of reducibility.

III. THE GENERAL CONSTRUCTION OF QUANTIZATION RULES AND GAUGE ALGEBRA

The general scheme for the construction of quantization rules and gauge algebra is still that of Ref. 1. Here we recall some definitions.

Let Φ^A be a set of bosonic fermionic fields which contains the initial gauge field:

$$\phi^i \subset \Phi^A . \tag{3.1}$$

To each Φ^Z adjoin an antifield Φ^*_A having opposite statistics:

$$\epsilon(\Phi^A)=\epsilon_A , \quad \epsilon(\Phi^*_A)=\epsilon_A+1 . \tag{3.2}$$

For any two functions X,Y on the phase space of Φ,Φ^*, define an operation called antibrackets:

$$(X,Y)=\frac{\partial_r X}{\partial\Phi^A}\frac{\partial_l Y}{\partial\Phi^*_A}-\frac{\partial_r X}{\partial\Phi^*_A}\frac{\partial_l Y}{\partial\Phi^A} . \tag{3.3}$$

Consider the following class of surfaces in the phase space

$$\Sigma:\ \Phi^*_A=\frac{\partial\psi(\Phi)}{\partial\Phi^A} , \tag{3.4}$$

where $\Psi(\Phi)$ is some fermionic function which will be called *the gauge fermion*. This class of surfaces can be equivalently defined in the canonically invariant form[11]:

$$\Sigma:\ \chi_A(\Phi,\Phi^*)=0 , \quad (\chi_A,\chi_B)=0 , \qquad \operatorname{Ber}\frac{\partial_r\chi_A}{\partial\Phi^*_B}\neq 0 . \tag{3.5}$$

Let the bosonic function $W(\Phi,\Phi^*)$ satisfy the equation

$$\Delta \exp\left[\frac{i}{\hbar}W\right]=0 , \quad \Delta=\frac{\partial_r}{\partial\Phi^A}\frac{\partial_l}{\partial\Phi^*_A} , \tag{3.6}$$

or, equivalently,

$$\tfrac{1}{2}(W,W)=i\hbar\Delta W . \tag{3.7}$$

Let $W_\Sigma(\Phi)$ be the restriction of $W(\Phi,\Phi^*)$ to the surface (3.4):

$$W_\Sigma(\Phi)=W(\Phi,\Phi^*)\,|_\Sigma\equiv W\left[\Phi,\Phi^*=\frac{\partial\Psi}{\partial\Phi}\right] . \tag{3.8}$$

Then the functional integral of the theory may be expressed in the form[1]

$$\mathscr{Z}_\Psi=\int \exp\left[\frac{i}{\hbar}W_\Sigma(\Phi)\right]\prod_A d\Phi^A , \tag{3.9}$$

where the content of the field Φ and boundary conditions to Eq. (3.7) are to be determined from two requirements: the nondegeneracy of the functional integral and the correctness of the classical limit.

The gauge fermion $\Psi(\Phi)$ is arbitrary modulo the restrictions imposed by the requirement of nondegeneracy. Making the transformation

$$\delta\Phi^A=(\Phi^A,W)\frac{i}{\hbar}\delta\psi(\Phi)\,|_\Sigma \tag{3.10}$$

in (3.9) and using (3.7), one can prove that the functional integral does not depend on the choice of Ψ: $\mathscr{Z}_\Psi=\mathscr{Z}_{\Psi+\delta\Psi}$. The arbitrariness of Ψ is the most general gauge-fixing arbitrariness existing in the theory. The transformation (3.10) is the most general version of the Becchi-Rouet-Stora (BRS) transformation.

The solution of Eq. (3.7) can be expanded in powers of $\hbar$:

$$W=S+\sum_{p=1}^{\infty}\hbar^p M_p . \tag{3.11}$$

This gives

$$p=0:\ (S,S)=0 , \tag{3.12}$$

$$p=1:\ (M_1,S)=i\Delta S , \tag{3.13}$$

$$p\geq 2:\ (M_p,S)=i\Delta M_{p-1}-\tfrac{1}{2}\sum_{q=1}^{p-1}(M_q,M_{p-q}) . \tag{3.14}$$

The classical part of W, denoted by S, satisfies *the master equation* (3.12) and, remaining restricted to Σ, represents the full action of a theory. The remaining terms of (3.11) give the quantum integration measure.

The requirement of correctness of the classical limit leads to (3.1) and the following boundary condition for S:

$$S(\Phi,\Phi^*)\,|_{\Phi^*=0}=\mathscr{S}(\phi) . \tag{3.15}$$

The nondegeneracy requires first of all that S be *the proper solution* of the master equation 1. The solution is called proper if the Hessian of $S(\Phi,\Phi^*)$ has the maximal possible rank at the stationary point of $S(\Phi,\Phi^*)$. It can be shown that this maximal possible rank equals the number of fields Φ^A (half of the total number of fields and antifields). This is because the Hessian of $S(\Phi,\Phi^*)$ at the stationary point is nilpotent. The solution is proper only if this Hessian has no other zero-eigenvalue eigenvectors except those contained in itself. For details see the Appendix.

The requirement that the solution be proper dictates the minimal content of Φ^A as well as the boundary conditions for S, supplementing (3.5). From this point on the construction of the proper solution crucially depends on the properties of reducibility of the theory. Let us recall how this question was solved for irreducible theories.[1]

Suppose that Φ^A contains only the initial gauge field ϕ^i. Then both the master equation and the boundary condition (3.15) will be satisfied if we simply put $S(\Phi,\Phi^*)=\mathscr{S}(\Phi)$ with no dependence on antifields. But if $\mathscr{S}(\Phi)$ is the gauge action the rank of its Hessian at the stationary point is

$$\operatorname{rank}_\pm \left.\frac{\partial_l\partial_r\mathscr{S}}{\partial\phi^i\partial\phi^j}\right|_{\phi_0}=n_\pm-\operatorname{rank}_\pm R^i_{\alpha_0}\,|_{\phi_0} \tag{3.16}$$

(in consequence of the completeness) and is smaller than the number of fields. Therefore such a solution is not proper. Owing to the boundary condition (3.15), there are initially m_0 zero-eigenvalue eigenvectors $R^i_{\alpha_0}$ that are not contained in the Hessian of S at the stationary point. In order to make the solution proper one must include these eigenvectors in the Hessian. The rows and columns in which they are to be included may be inferred from the nilpotency relations (see the Appendix):

$$\left.\frac{\partial_l\partial_r S(\Phi,\Phi^*)}{\partial\Phi^A\partial\Phi^B}\frac{\partial_l\partial_r S(\Phi,\Phi^*)}{\partial\Phi^*_B\partial\Phi^C}\right|_{\frac{\partial S}{\partial\Phi}=\frac{\partial S}{\partial\Phi^*}=0}+(-1)^{\epsilon_A\epsilon_C}(A\leftrightarrow C)=0 .$$

Now, in virtue of (3.15), one has

$$\frac{\partial_l \partial_r \mathscr{S}(\phi)}{\partial\phi^i \partial\phi^j} R^i_{\alpha_0} \Bigg|_{\frac{\partial\mathscr{S}}{\partial\phi}=0} = 0 \,.$$

Therefore $R^i_{\alpha_0}$ must be contained in a block of the form

$$R^i_{\alpha_0} = \frac{\partial_l \partial_r S(\Phi,\Phi^*)}{\partial\phi^*_i \partial(?)^{\alpha_0}} \,,$$

where the question mark stands for some new field. At this point one introduces m_0 auxiliary fields $C_0^{\alpha_0}$ having the statistics $\epsilon(C_0^{\alpha_0}) = \epsilon_{\alpha_0} + 1$ and requires that

$$C_0^{\alpha_0} \subset \Phi^A \,, \quad \frac{\partial_l \partial_r S(\Phi,\Phi^*)}{\partial\phi^*_i \partial C_0^{\alpha_0}} \Bigg|_{\Phi^*=0} = R^i_{\alpha_0}(\phi) \,. \qquad (3.17)$$

Then the number of fields Φ^A becomes $(n+m_0)_\pm$. The Hessian of S at the stationary point is shown in Fig. 1. Its rank equals

$$\operatorname{rank}_\pm \frac{\partial_l \partial_r \mathscr{S}(\phi)}{\partial\phi^i \partial\phi^j}\Bigg|_{\phi_0} + 2\operatorname{rank}_\pm R^i_{\alpha_0}\,|_{\phi_0} = n_\pm + \operatorname{rank}_\pm R^i_{\alpha_0}\,|_{\phi_0} \qquad (3.18)$$

by virtue of (3.6). *If the theory is irreducible,* then

$$\operatorname{rank}_\pm R^i_{\alpha_0}\,|_{\phi_0} = m_{0\pm} \,, \qquad (3.19)$$

and the rank of the Hessian equals the number of fields. In this way one is led to the introduction of ghosts $C_0^{\alpha_0}$.

For irreducible theories one sees that the proper solution of the master equation is realized with the minimal set of fields

$$\Phi^A_{\min} = \{\phi^i; C_0^{\alpha_0}\} \qquad (3.20)$$

and satisfies the boundary conditions (3.15) and (3.17).

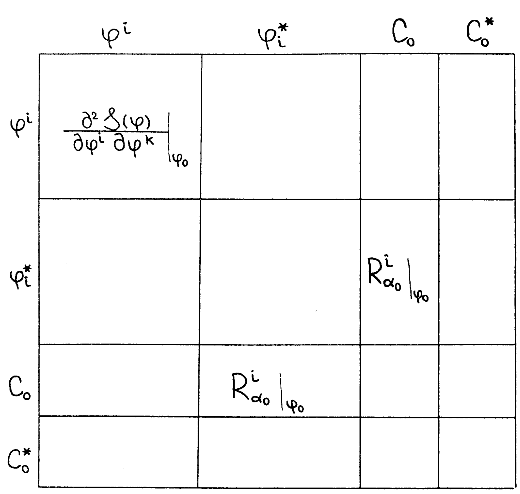

FIG. 1. The Hessian of $S(\Phi_{\min}\cdot\Phi^*_{\min})$ at the stationary point for irreducible theories. The stationary point is $\phi=\phi_0$, $\phi^*=C_0=C_0^*=0$. The empty blocks are zeros due to ghost-number conservation.

Here it is convenient to introduce the notion of *ghost number.* The ghost numbers of the fields ϕ^i, $C_0^{\alpha_0}$, ϕ^*_i, and $C^*_{0\alpha_0}$ are defined by

$$\mathrm{gh}(\phi^i)=0 \,, \quad \mathrm{gh}(\phi^*_i) = -1 \,, \qquad \mathrm{gh}(C_0^{\alpha_0}) = 1 \,, \quad \mathrm{gh}(C^*_{0\alpha_0}) = -2 \,. \qquad (3.21)$$

It can be proved that there exists a proper solution of the master equation with boundary conditions (3.15) and (3.17) and (conserved) ghost number equal to zero. This solution generates Feynman rules via expressions (3.9) and (3.11). Expanded in powers of antifields it yields the generating function (Ref. 1) of the open-gauge algebra.[12]

Conservation of ghost number is a property of the gauge algebra. In the usual derivation of the gauge algebra,[12] structure functions are not associated with every set of indices. There is a certain ratio between the number of group (upper and lower) and field indices. The structure functions of the algebra are the coefficients of the expansion of $S(\Phi_{\min},\Phi^*_{\min})$ in powers of ϕ^*_i, $C^*_{0\alpha_0}$, and $C_0^{\alpha_0}$, and conservation of ghost number is the selection rule for these structure functions. Note that ghost-number conservation is essential for the solution to be proper. It is the addition of this requirement to the boundary condition (3.17) that gives the Hessian of S at the stationary point the form of Fig. 1.

The requirement of nondegeneracy imposes also restrictions on the gauge fermion. *This is the second point at which the properties of reducibility of the theory play a crucial role.* The number of gauge conditions *in irreducible theories* must equal the number of generators $R^i_{\alpha_0}$. In Ref. 1 these gauge conditions were introduced in the following way. The space (3.20) was extended by $2m_0$ additional auxiliary fields:

$$\bar{C}_{0\alpha_0}, \ \pi_{0\alpha_0} \subset \Phi^A \,, \qquad \epsilon(\bar{C}_{0\alpha_0}) = \epsilon_{\alpha_0} + 1 \,, \quad \epsilon(\pi_{0\alpha_0}) = \epsilon_{\alpha_0} \,. \qquad (3.22)$$

The dependence of S on these new fields was chosen to be trivial:

$$S(\Phi,\Phi^*) = S(\Phi_{\min},\Phi^*_{\min}) + \bar{C}_0^{*\alpha_0} \pi_{0\alpha_0} \,. \qquad (3.23)$$

It is easy to see that (3.23) satisfies the master equation in the extended space if it is satisfied in the minimal space. It is also clear that the solution remains proper. One easily finds that gauge conditions are generated by the gauge fermion in the form

$$\frac{\partial\Psi}{\partial\bar{C}_{0\alpha_0}} = 0 \qquad (3.24)$$

with $\pi_{0\alpha_0}$ playing the role of Lagrange multipliers. Indeed, one has, from (3.8),

$$S(\Phi,\Phi^*)\,|_\Sigma = S\left[\Phi_{\min}, \frac{\partial\Psi}{\partial\Phi_{\min}}\right] + \frac{\partial\Psi}{\partial\bar{C}_{0\alpha_0}} \pi_{0\alpha_0} \,. \qquad (3.25)$$

The fields $C_0^{\alpha_0}$ and $\bar{C}_{0\alpha_0}$ are the Feynman-DeWitt–Faddeev-Popov ghosts, and the corresponding matrix (the inverse ghost propagator) stands on the lefthand side of the following equality:

$$\operatorname{rank}_{\pm}\left[\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{0\alpha_0}\partial\phi^i}R^i_{\beta_0}\right]\Bigg|_{\substack{\phi=\phi_0\\ C_0=\bar{C}_0=0}}=m_{0\pm}\ . \tag{3.26}$$

This equality is the only restriction on the gauge fermion *in irreducible theories.*

The only quantity that remains undefined in the functional integral (3.9) is the measure. As explained in Ref. 1, the boundary conditions to Eqs. (3.13) and (3.14) for the measure cannot be formulated without an appeal to the canonical formalism.[13–15] However, in a local basis of the gauge algebra the right-hand sides of Eqs. (3.13) and (3.14) are proportional to $\delta(0)$. In the framework of a regularization which annihilates such divergences one may put $M_p=0$, $p\geq 1$.

The presence of the measure makes the functional integral independent of the choice of the basis of generators. In nonlocal bases the measure may generate all sorts of Feynman diagrams; only in a local basis is the measure inessential.

The status of the measure in reducible theories is the same. Since the generating function S of the gauge algebra in reducible theories will here be constructed in a local basis the measure may be ignored.

IV. QUANTIZATION RULES FOR FIRST-STAGE THEORIES

If the theory is reducible, then condition (3.17) does not guarantee that S is already the proper solution. Although the rank of the Hessian (Fig. 1) is given by Eq. (3.18), as before, we have

$$\operatorname{rank}_{\pm}R^i_{\alpha_0}\,|_{\phi_0}<m_{0\pm}\ ,$$

so that the rank (3.18) is smaller than the number of fields. The reason is that the block $R^i_{\alpha_0}$ of Fig. 1 now has zero-eigenvalue eigenvectors $Z^{\alpha_0}_{1\alpha_1}$ which are not included in the Hessian. In other words, the ghost $C_0^{\alpha_0}$ becomes a gauge field. The same considerations as before lead to the conclusion that we must introduce new auxiliary fields $C_1^{\alpha_1}$, $\alpha_1=1,\ldots,m_1$, $\epsilon(C_1^{\alpha_1})=\epsilon_{\alpha_1}$ and require that

$$C_1^{\alpha_1}\subset\Phi^A\ ,$$
$$\frac{\partial_l}{\partial C^*_{0\alpha_0}}\frac{\partial_r}{\partial C_1^{\alpha_1}}S(\Phi,\Phi^*)\Bigg|_{\Phi^*=0}=Z^{\alpha_0}_{1\alpha_1}\ , \tag{4.1}$$

in order to include new zero-eigenvalue eigenvectors in the Hessian.[16] The number of fields Φ^A is now $(n+m_0+m_1)_{\pm}$, and the Hessian is shown in Fig. 2. Its rank equals

$$\operatorname{rank}_{\pm}\frac{\partial_l\partial_r\mathcal{S}(\phi)}{\partial\phi^i\partial\phi^j}\Bigg|_{\phi_0}+2\operatorname{rank}_{\pm}R^i_{\alpha_0}\,|_{\phi_0}+2\operatorname{rank}_{\pm}Z^{\alpha_0}_{1\alpha_1}\,|_{\phi_0}\ . \tag{4.2}$$

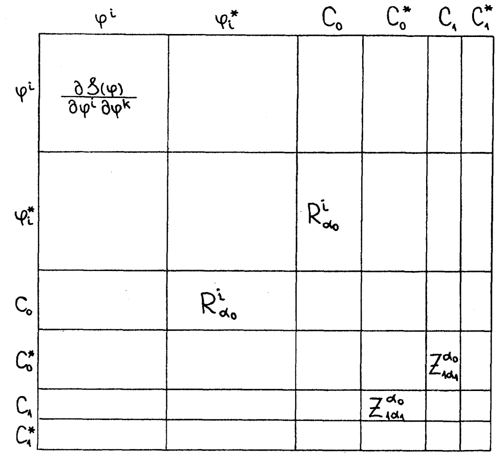

FIG. 2. The Hessian of $S(\Phi_{\min},\Phi^*_{\min})$ at the stationary point for first-stage theories. All quantities are restricted to $\phi=\phi_0$. The empty blocks are zeros due to ghost-number conservation.

If the theory is a first-stage theory, then Eqs. (2.3)—(2.6) hold, and the rank (4.2) exactly equals the number of fields.

Thus in first-stage theories the minimal content of Φ^A is

$$\Phi^A_{\min}=\{\phi^i,C_0^{\alpha_0},C_1^{\alpha_1}\} \tag{4.3}$$

and the boundary conditions for the master equation are (3.15), (3.17), and (4.1). We assign ghost numbers

$$\operatorname{gh}(C_1^{\alpha_1})=+2\ ,\quad \operatorname{gh}(C^*_{1\alpha_1})=-3 \tag{4.4}$$

to the new ghost and its antifield. The ghost numbers of the old fields appearing in (4.3), and of their antifields, remain the same as before. Again we require that S be bosonic with (conserved) ghost number equal to zero.

Under these conditions one can prove a theorem analogous to that formulated in Ref. 1 for irreducible theories, namely, that a solution of the master equation exists as a power series in antifields. The coefficients of this series are polynomials in C_0,C_1 and are infinitely differentiable in ϕ in a neighborhood of the stationary point. The proof is based on postulates (2.1)—(2.6).

The generic monomial in the expansion of $S(\Phi_{\min},\Phi^*_{\min})$ is proportional to

$$(\phi^*)^l(C_0^*)^p(C_1^*)^t(C_0)^r(C_1)^s\ , \tag{4.5}$$

where

$$r+2s=l+2p+3t \tag{4.6}$$

in consequence of the requirement that the ghost number vanish. The lowest-order terms in the expansion of $S(\Phi_{\min},\Phi^*_{\min})$ are of the form

$$S(\Phi_{\min},\Phi^*_{\min})=\mathcal{S}+\phi_i^*R^i_\alpha C^\alpha+C^*_\alpha(Z^\alpha_a\eta^a+T^\alpha_{\beta\gamma}C^\gamma C^\beta)+\eta^*_a(A^a_{b\alpha}C^\alpha\eta^b+F^a_{\alpha\beta\gamma}C^\gamma C^\beta C^\alpha)$$
$$+\phi_i^*\phi_j^*(B^{ji}_a\eta^a+E^{ji}_{\alpha\beta}C^\beta C^\alpha)+2C^*_\alpha\phi_i^*(G^{i\alpha}_{a\mu}C^\mu\eta^a+D^{i\alpha}_{\beta\gamma\delta}C^\delta C^\gamma C^\beta)+\cdots\ , \tag{4.7}$$

where the new ghost $C_1^{\alpha_1}$ is here denoted by η^a for simplicity. In (4.7) we keep terms at most quadratic in the antifields, with $r+2s\le 3$. The coefficients, $R^i_\alpha, Z^\alpha_a, T^\alpha_{\beta\gamma},\ldots$ are functions of ϕ.

The general relations which the master equation imposes upon the coefficients of the expansion in terms of antifields are derived in the Appendix. In the present case the lowest-order relations imposed upon the coefficients of (4.7) are

$$\frac{\partial_r \mathscr{S}}{\partial\phi^i}R^i_\alpha C^\alpha=0\;, \tag{4.8}$$

$$R^i_\alpha Z^\alpha_a\eta^a-2\frac{\partial_r\mathscr{S}}{\partial\phi^j}B^{ji}_a\eta^a(-1)^{\epsilon_i}=0\;, \tag{4.9}$$

$$\frac{\partial_r R^i_\alpha C^\alpha}{\partial\phi^j}R^j_\beta C^\beta+R^i_\mu T^\mu_{\alpha\beta}C^\beta C^\alpha-2\frac{\partial_r\mathscr{S}}{\partial\phi^j}E^{ji}_{\alpha\beta}C^\beta C^\alpha(-1)^{\epsilon_i}=0\;, \tag{4.10}$$

$$\frac{\partial_r T^\alpha_{\beta\gamma}C^\gamma C^\beta}{\partial\phi^j}R^j_\delta C^\delta+2T^\alpha_{\mu\beta}C^\beta T^\mu_{\gamma\delta}C^\delta C^\gamma+Z^\alpha_a F^a_{\beta\gamma\delta}C^\delta C^\gamma C^\beta+2\frac{\partial_r\mathscr{S}}{\partial\phi^j}D^{j\alpha}_{\beta\gamma\delta}C^\delta C^\gamma C^\beta(-1)^{\epsilon_\alpha}=0\;, \tag{4.11}$$

$$\frac{\partial_r Z^\alpha_a\eta^a}{\partial\phi^j}R^j_\beta C^\beta+2T^\alpha_{\mu\beta}C^\beta Z^\mu_a\eta^a+Z^\alpha_b A^b_{a\beta}C^\beta\eta^a+2\frac{\partial_r\mathscr{S}}{\partial\phi^j}G^{j\alpha}_{a\beta}C^\beta\eta^a(-1)^{\epsilon_\alpha}=0\;. \tag{4.12}$$

Note that Noether identities (4.8) and commutation relations (4.10) are generated by the master equation in the same form as in irreducible theories, but the Jacobi identity (4.11) changes. Besides the usual open term proportional to $\partial\mathscr{S}/\partial\phi$, this identity acquires a new term proportional to zero-eigenvalue eigenvectors Z^α_a. In the usual derivation of the Jacobi identity this term would arise when dividing by R^i_α. Equations (4.9) and (4.12) are the new algebraic relations. The lowest-order new consequence of the master equation is Eq. (4.9). It shows how the reducibility condition (2.4) looks off the mass shell. The totality of relations generated by the master equation forms *a new algebra:* that of a first-stage theory.

Turn now to the question of gauge fixing. We must determine the restrictions that the gauge fermion $\Psi(\Phi)$ must satisfy in order that it generate admissible gauge conditions. First of all we require that

$$\mathrm{gh}(\Psi)=-1\;, \tag{4.13}$$

in agreement with (3.4) and the assigned ghost numbers. Secondly, as pointed out in Ref. 1, the minimal set of fields Φ^A may be extended by any number of pairs of fields of the type (3.22) appearing in S as in Eq. (3.23). This freedom proves to be extremely useful in reducible theories. Instead of one pair (3.22) we shall introduce three pairs of auxiliary fields,

$$\bar C_{0\alpha_0}\;,\;\pi_{0\alpha_0}\;;\;\bar C_{1\alpha_1}\;,\;\pi_{1\alpha_1}\;;\;C_1'^{\alpha_1}\;,\;\pi_1'^{\alpha_1}\subset\Phi^A\;, \tag{4.14}$$

$$\begin{aligned}&\epsilon(\bar C_{0\alpha_0})=\epsilon_{\alpha_0}+1\;,\;\;\epsilon(\pi_{0\alpha_0})=\epsilon_{\alpha_0}\;,\\&\epsilon(\bar C_{1\alpha_1})=\epsilon_{\alpha_1}\;,\;\;\epsilon(\pi_{1\alpha_1})=\epsilon_{\alpha_1}+1\;,\\&\epsilon(C_1'^{\alpha_1})=\epsilon_{\alpha_1}\;,\;\;\epsilon(\pi_1'^{\alpha_1})=\epsilon_{\alpha_1}+1\;,\end{aligned} \tag{4.15}$$

and define

$$S(\Phi,\Phi^*)=S(\Phi_{\min},\Phi^*_{\min})+\bar C_0^{*\alpha_0}\pi_{0\alpha_0}+\bar C_1^{*\alpha_1}\pi_{1\alpha_1}+C'^*_{1\alpha_1}+C'^*_{1\alpha_1}\pi_1'^{\alpha_1}\;, \tag{4.16}$$

where $\Phi_{\min}$ is the minimal sector (4.3). The following ghost numbers must be assigned to these fields:

$$\begin{aligned}&\mathrm{gh}(\bar C_{0\alpha_0})=-1\;,\;\;\mathrm{gh}(\bar C_0^{*\alpha_0})=0\;,\\&\mathrm{gh}(\pi_{0\alpha_0})=0\;,\;\;\mathrm{gh}(\pi_0^{*\alpha_0})=-1\;,\\&\mathrm{gh}(\bar C_{1\alpha_1})=-2\;,\;\;\mathrm{gh}(\bar C_1^{*\alpha_1})=+1\;,\\&\mathrm{gh}(\pi_{1\alpha_1})=-1\;,\;\;\mathrm{gh}(\pi_1^{*\alpha_1})=0\;,\\&\mathrm{gh}(C_1'^{\alpha_1})=0\;,\;\;\mathrm{gh}(C'^*_{1\alpha_1})=-1\;,\\&\mathrm{gh}(\pi_1'^{\alpha_1})=+1\;,\;\;\mathrm{gh}(\pi'^*_{1\alpha_1})=-2\;.\end{aligned} \tag{4.17}$$

Then the following restrictions on the gauge fermion guarantee the correctness of the gauge fixing (Φ_0 is the stationary point of the action $S\,|_\Sigma$):

$$\mathrm{rank}_\pm\left.\frac{\partial_l\partial_r\Psi}{\partial\bar C_{0\alpha_0}\partial\phi^i}\right|_{\Phi_0}=(m_0-m_1)_\pm\;, \tag{4.18a}$$

$$\mathrm{rank}_\pm\left.\frac{\partial_l\partial_r\Psi}{\partial\bar C_{0\alpha_0}\partial\phi^i}R^i_{\beta_0}\right|_{\Phi_0}=(m_0-m_1)_\pm\;, \tag{4.18b}$$

$$\mathrm{rank}_\pm\left.\frac{\partial_l\partial_r\Psi}{\partial\bar C_{1\alpha_1}\partial\bar C_0^{\alpha_0}}Z^{\alpha_0}_{1\beta_1}\right|_{\Phi_0}=m_{1\pm}\;, \tag{4.18c}$$

$$\mathrm{rank}_\pm\left.\bar Z^{\alpha_1}_{1\alpha_0}\frac{\partial_l\partial_r\Psi}{\partial\bar C_{0\alpha_0}\partial C_1'^{\beta_1}}\right|_{\Phi_0}=m_{1\pm}\;. \tag{4.18d}$$

Here the $\bar Z^{\alpha_1}_{1\alpha_0}$ are zero-eigenvalue left eigenvectors of the matrix (4.18a):

$$\begin{aligned}&\left.\bar Z^{\alpha_1}_{1\alpha_0}\frac{\partial_l\partial_r\Psi}{\partial\bar C_{0\alpha_0}\partial\phi^i}\right|_{\Phi_0}=0\;,\\&\mathrm{rank}_\pm\bar Z^{\alpha_1}_{1\alpha_0}\,|_{\Phi_0}=m_{1\pm}\;,\;\;\epsilon(\bar Z^{\alpha_1}_{1\alpha_0})=\epsilon_{\alpha_0}+\epsilon_{\alpha_1}\;.\end{aligned} \tag{4.19}$$

From (3.8) and (4.16) one obtains

$$S(\Phi,\Phi^*)\,|_\Sigma=S\left[\Phi_{\min},\frac{\partial\Psi}{\partial\Phi_{\min}}\right]+\frac{\partial\Psi}{\partial\bar{C}_{0\alpha_0}}\pi_{0\alpha_0}+\frac{\partial\Psi}{\partial\bar{C}_{1\alpha_1}}\pi_{1\alpha_1}+\frac{\partial\Psi}{\partial C_1'^{\alpha_1}}\pi_1'^{\alpha_1}\,. \tag{4.20}$$

Therefore the integration over all π's will give three sets of gauge conditions:

$$\text{(a)}\ \frac{\partial\Psi}{\partial\bar{C}_{0\alpha_0}}=0\,,\quad\text{(b)}\ \frac{\partial\Psi}{\partial\bar{C}_{1\alpha_1}}=0\,,\quad\text{(c)}\ \frac{\partial\Psi}{\partial C_1'^{\alpha_1}}=0\,. \tag{4.21}$$

In irreducible theories we had only m_0 gauge conditions (4.21a), which removed m_0 components of the initial gauge field ϕ^i. In first-stage theories only (m_0-m_1) components of ϕ^i should be removed. This is guaranteed by (4.18a). The requirement (4.18b) replaces (3.26) and guarantees the admissibility of the gauge conditions imposed on ϕ^i.

Further the quadratic part of the action of the Feynman-DeWitt–Faddeev-Popov ghosts is of the form

$$\bar{C}_{0\alpha_0}\left[\left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{0\alpha_0}\partial\phi^i}R^i_{\beta_0}\right|_{\Phi_0}\right]C_0^{\beta_0}\,. \tag{4.22}$$

The matrix in (4.22) (the inverse ghost propagator) is degenerate according to (4.18b). Its zero-eigenvalue right eigenvectors $Z^{\beta_0}_{1\beta_1}$ coincide with those of $R^i_{\beta_0}$. Therefore C_0 is a gauge field, and the Z_1 play the role of the corresponding generators. This is the reason why we introduced the second pair in (4.14). The second set of gauge conditions, Eqs. (4.21b), removes m_1 components of $C_0^{\beta_0}$, and (4.18c) is the standard requirement of admissibility of gauge conditions imposed on the $C_0^{\beta_0}$.

The matrix in (4.22) has also zero-eigenvalue *left* eigenvectors, which coincide with the $\bar{Z}_1$ of Eq. (4.19). This means that $\bar{C}_0$ is *also* a gauge field; the corresponding generators are the $\bar{Z}_1$. For this reason we introduced the third pair in (4.14). The third set of gauge conditions, Eqs. (4.21c), removes m_1 components of $\bar{C}_{0\beta_0}$, and (4.18d) is the standard requirement of admissibility for the gauge conditions imposed on $\bar{C}_{0\beta_0}$.

Thus the conditions (4.18) secure the removal of the degeneracy connected with the gauge fields $\phi, C_0, \bar{C}_0$. The new ghost C_1 and its $\bar{C}_1$ [introduced in the second pair of Eqs. (4.14)] are not gauge fields. The quadratic part of their action is of the form

$$\bar{C}_{1\alpha_1}\left[\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{1\alpha_1}\partial C_0^{\beta_0}}Z^{\beta_0}_{1\beta_1}\,|_{\Phi_0}\right]C_1^{\beta_1}\,, \tag{4.23}$$

and, for first-stage theories, is nondegenerate due to (4.18c). There remains only one redundant field: the "extraghost" C_1' introduced in the third pair of Eqs. (4.14). Note that it has no partner in the minimal sector (4.3). However there also remain m_1 redundant gauge conditions in (4.21a). Since

$$\operatorname{rank}_\pm\left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{0\alpha_0}\partial C_1'^{\beta_1}}\right|_{\Phi_0}=m_{1\pm} \tag{4.24}$$

in virtue of (4.18d), the redundant gauge conditions in (4.21a) exactly remove the extraghost C_1'.

Thus the m_0 primary gauge conditions (4.21a) play a double role: (m_0-m_1) of them remove the degeneracy of the initial gauge field, and the remaining m_1 conditions eliminate the extraghost. This is the general solution of the problem of redundant gauge conditions in reducible theories.

Note that the zero-eigenvalue left eigenvectors $\bar{Z}_1$ (in contrast to the right ones) do not enter the gauge algebra (4.7) and the functional integral. They are defined only by the choice of the gauge fermion [Eq. (4.19)] and are needed only to verify the correctness of gauge fixing [Eq. (4.18d)].

The simplest way to construct a gauge fermion satisfying the above requirements is to put

$$\Psi=\bar{C}_{0\alpha_0}\chi^{\alpha_0}(\phi)+\bar{C}_{1\alpha_1}\omega^{\alpha_1}_{\alpha_0}C_0^{\alpha_0}+\bar{C}_{0\alpha_0}\sigma^{\alpha_0}_{\alpha_1}C_1'^{\alpha_1}\,, \tag{4.25}$$

where $\omega^{\alpha_1}_{\alpha_0}$ and $\sigma^{\alpha_0}_{\alpha_1}$ are some convenient matrices having maximal rank and satisfying Eqs. (4.18c) and (4.18d), and the $\chi^{\alpha_0}(\phi)$ are some redundant gauge conditions satisfying Eqs. (4.18a) and (4.18b). Then Eqs. (4.21) take the form

$$\begin{aligned}&\text{(a)}\ \chi^{\alpha_0}(\phi)+\sigma^{\alpha_0}_{\alpha_1}C_1'^{\alpha_1}=0\,,\\&\text{(b)}\ \omega^{\alpha_1}_{\alpha_0}C_0^{\alpha_0}=0\,,\\&\text{(c)}\ \bar{C}_{0\alpha_0}\sigma^{\alpha_0}_{\alpha_1}=0\,,\end{aligned} \tag{4.26}$$

and allow the elimination of the redundant components of ϕ^i, $C_0^{\alpha_0}, \bar{C}_{0\alpha_0}$, and the extraghost $C_1'^{\alpha_1}$.

The gauge conditions (4.21) or (4.26) arise as δ functions in the functional integral only if the gauge fermion $\Psi(\Phi)$ does not depend on Lagrange multipliers π. Such a dependence is possible, however, since $\pi\subset\Phi$. In order to have the usual quadratic gauge-breaking Lagrangian, one must introduce a linear dependence of Ψ on the π's. Then the integration over the π's will give Gaussian averages of gauge conditions instead of δ functions. In the simplest case one must add the following terms to (4.25):

$$\tfrac{1}{2}(\bar{C}_{0\alpha_0}\kappa^{\alpha_0\beta_0}\pi_{0\beta_0}+\bar{C}_{1\alpha_1}\rho^{\alpha_1}_{\beta_1}\pi_1'^{\beta_1}-\pi_{1\alpha_1}\rho^{\alpha_1}_{\beta_1}(-1)^{\epsilon_{\alpha_1}}C_1'^{\beta_1})\,, \tag{4.27}$$

where $\kappa^{\alpha_0\beta_0}$ and $\rho^{\alpha_1}_{\beta_1}$ are some invertible matrices.

The integration over the π's is useful if the matrices $\kappa^{\alpha_0\beta_0}$ and $\rho^{\alpha_1}_{\beta_1}$ do not contain derivatives, but if

$$\kappa^{-1}{}_{\alpha_0\beta_0}\,,\quad \rho^{-1\alpha_1}{}_{\beta_1}$$

are required to be differential operators, then integration over the π's is inconvenient because it yields nonlocal determinants. In this case the following device is useful. Make the replacement

$$\pi_{0\alpha_0}\to\kappa^{-1}{}_{\alpha_0\beta_0}[A^{\beta_0\gamma_0}\pi_{0\gamma_0}-(\chi^{\beta_0}+\sigma^{\beta_0}_{\alpha_1}C'^{\alpha_1}_1)] ,$$
$$\bar C_{0\alpha_0}\to\bar C_{0\gamma_0}A^{\gamma_0\beta_0}\kappa^{-1}_{\beta_0\alpha_0} , \tag{4.28}$$

as well as

$$\pi_{1\alpha_1}\to[\pi_{1\gamma_1}B^{\gamma_1}_{\beta_1}-\bar C_{0\sigma_0}A^{\sigma_0\gamma_0}\kappa^{-1}{}_{\gamma_0\beta_0}\sigma^{\beta_0}_{\beta_1}]\times\rho^{-1}{}^{\beta_1}_{\alpha_1}(-1)^{\epsilon_{\alpha_1}+1} , \tag{4.29}$$

$$\bar C_{1\alpha_1}\to\bar C_{1\gamma_1}B^{\gamma_1}_{\beta_1}\rho^{-1}{}^{\beta_1}_{\alpha_1}(-1)^{\epsilon_{\alpha_1}+1} ,$$
$$\pi'^{\alpha_1}_1\to\pi'^{\alpha_1}_1-\rho^{-1}{}^{\alpha_1}_{\beta_1}\omega^{\beta_1}_{\gamma_0}C^{\gamma_0}_0 , \tag{4.30}$$

where $A^{\alpha_0\beta_0}$ and $\beta^{\alpha_1}_{\beta_1}$ are arbitrary invertible matrices. The Jacobian of this replacement equals unity, because π_0 and $\bar C_0$, as well as π_1 and $\bar C_1$, necessarily have opposite statistics, while (4.30) is simply a shift of origin.

As a result of the above replacement the gauge-breaking terms of the action (4.20) take the form

$$\left[\frac{\partial\Psi}{\partial\bar C_{0\alpha_0}}\pi_{0\alpha_0}+\frac{\partial\Psi}{\partial\bar C_{1\alpha_1}}\pi_{1\alpha_1}+\frac{\partial\Psi}{\partial C'^{\alpha_1}_1}\pi'^{\alpha_1}_1\right]$$
$$\to[\tfrac12(-1)^{\epsilon_{\alpha_0}}\pi_{0\alpha_0}\Gamma^{\alpha_0\beta_0}\pi_{0\beta_0}+\pi_{1\beta_1}B^{\beta_1}_{\alpha_1}\pi'^{\alpha_1}_1$$
$$-\tfrac12(\chi^{\alpha_0}+\sigma^{\alpha_0}_{\alpha_1}C'^{\alpha_1}_1)\kappa^{-1}{}_{\alpha_0\beta_0}(\chi^{\beta_0}+\sigma^{\beta_0}_{\beta_1}C'^{\beta_1}_1)$$
$$-\bar C_{0\alpha_0}A^{\alpha_0\beta_0}\kappa^{-1}{}_{\beta_0\gamma_0}\sigma^{\gamma_0}_{\beta_1}\rho^{-1}{}^{\beta_1}_{\sigma_0}C^{\sigma_0}_0] , \tag{4.31}$$

where

$$\Gamma\equiv A^T\kappa^{-1}A , \quad A^{T\alpha_0\beta_0}\equiv(-1)^{\epsilon_{\alpha_0}\epsilon_{\beta_0}}A^{\beta_0\alpha_0} . \tag{4.32}$$

One must also remember to make the appropriate replacement in the argument $\partial\Psi/\partial\Phi_{\min}$ of the action (4.20).

The term $\chi\kappa^{-1}\chi$ of (4.31) removes the degeneracy of the initial gauge action, while the last term of (4.31) removes the degeneracy of the action of the Faddeev-Popov ghosts $\bar C_0, C_0$. If κ^{-1} contains derivatives, then, as seen from (4.31), the Lagrange multiplier π_0 becomes a propagating field. A propagating π_0 arises already in irreducible theories and is the so-called Nielsen-Kallosh ghost.[17–19] Whether the new Lagrange multipliers π_1 and π'_1 will propagate or not depends on the form of the matrix B. The extraghost C'_1, however, will always be a propagating field, because in specific examples either σ or κ^{-1} always contain derivatives. (See examples considered in Sec. VIII of this paper.)

V. QUANTIZATION RULES FOR SECOND-STAGE THEORIES

The postulates of second-stage theories are (2.1), (2.2), and (2.7)–(2.12). For these theories a qualitatively new feature arises in the construction of the gauge fermion.

No qualitatively new features arise in the construction of the gauge algebra. Note that the block $Z^{\alpha_0}_{1\alpha_1}$ of Fig. 2 now acquires zero-eigenvalue eigenvectors $Z^{\alpha_1}_{2\alpha_2}$ which are not included in the Hessian. Therefore one introduces a new ghost $C^{\alpha_2}_2$,

$$\epsilon(C^{\alpha_2}_2)=\epsilon_{\alpha_2}+1 , \quad \mathrm{gh}(C^{\alpha_2}_2)=+3 ,$$
$$\mathrm{gh}(C^*_{2\alpha_2})=-4 , \tag{5.1}$$

and a corresponding boundary condition [Eq. (5.6) below] which includes Z_2 in the Hessian and makes the solution of the master equation proper.

The minimal sector of the second stage, namely,

$$\Phi^A_{\min}=\{\phi^i,C^{\alpha_0}_0,C^{\alpha_1}_1,C^{\alpha_2}_2\} , \tag{5.2}$$

now contains a zero-stage ghost C_0, a first-stage ghost C_1, and a second-stage ghost C_2. The complete set of boundary conditions for the master equation (3.12) reads

$$S(\Phi,\Phi^*)\,|_{\Phi^*=0}=\mathscr{S}(\Phi) , \tag{5.3}$$

$$\left.\frac{\partial_l}{\partial\phi^*_i}\frac{\partial_r}{\partial C^{\alpha_0}_0}S(\Phi,\Phi^*)\right|_{\Phi^*=0}=R^i_{\alpha_0}(\phi) , \tag{5.4}$$

$$\left.\frac{\partial_l}{\partial C^*_{0\alpha_0}}\frac{\partial_r}{\partial C^{\alpha_1}_1}S(\Phi,\Phi^*)\right|_{\Phi^*=0}=Z^{\alpha_0}_{1\alpha_1}(\phi) , \tag{5.5}$$

$$\left.\frac{\partial_l}{\partial C^*_{1\alpha_1}}\frac{\partial_r}{\partial C^{\alpha_2}_2}S(\Phi,\Phi^*)\right|_{\Phi^*=0}=Z^{\alpha_1}_{2\alpha_2}(\phi) . \tag{5.6}$$

The bosonic proper solution of the master equation in the minimal sector (5.2), which satisfies the above boundary conditions and has (conserved) ghost number equal to zero, generates the gauge algebra.

The new feature that arises in the construction of the gauge fermion is the appearance of "extraghosts for the extraghosts." It is clear that since the first-stage ghost C_1 and its $\bar C_1$ are now gauge fields, two new sets of gauge conditions will be needed, and hence two new pairs of auxiliary fields in addition to (4.14). It is also clear that one of these two pairs will contain a new extraghost. It is less evident, however, that the old extraghost C'_1 is also now a gauge field, requiring a *third* pair containing an extraghost for the extraghost.

For second-stage theories the space (5.2) must be extended by six pairs of auxiliary fields:

$$\bar C_{0\alpha_0} , \ \pi_{0\alpha_0} ; \ \bar C_{1\alpha_1} , \ \pi_{1\alpha_1} ; \ C'^{\alpha_1}_1 , \ \pi'^{\alpha_1}_1 ;$$
$$\bar C_{2\alpha_2} , \ \pi_{2\alpha_2} ; \ C'^{\alpha_2}_2 , \ \pi'^{\alpha_2}_2 ; \ \bar C''_{2\alpha_2} , \ \pi''_{2\alpha_2}\subset\Phi^a . \tag{5.7}$$

The Grassmann parities and ghost numbers of new fields in (5.7) are

$$\epsilon(\bar C_{2\alpha_2})=\epsilon_{\alpha_2}+1 , \quad \epsilon(\pi_{2\alpha_2})=\epsilon_{\alpha_2} ,$$
$$\epsilon(C'^{\alpha_2}_2)=\epsilon_{\alpha_2}+1 , \quad \epsilon(\pi'^{\alpha_2}_2)=\epsilon_{\alpha_2} ,$$
$$\epsilon(\bar C''_{2\alpha_2})=\epsilon_{\alpha_2}+1 , \quad \epsilon(\pi''_{2\alpha_2})=\epsilon_{\alpha_2} ;$$
$$\mathrm{gh}(\bar C_{2\alpha_2})=-3 , \quad \mathrm{gh}(\bar C^{*\alpha_2}_2)=+2 ,$$
$$\mathrm{gh}(\pi_{2\alpha_2})=-2 , \quad \mathrm{gh}(\pi^{*\alpha_2}_2)=+1 ,$$
$$\mathrm{gh}(C'^{\alpha_2}_2)=+1 , \quad \mathrm{gh}(C'^*_{2\alpha_2})=-2 , \tag{5.8}$$

$$\mathrm{gh}(\pi_2^{\prime\,\alpha_2})=+2\ ,\ \ \mathrm{gh}(\pi_{2\alpha_2}^{\prime *})=-3\ ,$$
$$\mathrm{gh}(\bar{C}_{2\alpha_2}'')=-1\ ,\ \ \mathrm{gh}(\bar{C}_2^{\prime\prime *\alpha_2})=0\ , \tag{5.9}$$
$$\mathrm{gh}(\pi_{2\alpha_2}'')=0\ ,\ \ \mathrm{gh}(\pi_2^{\prime\prime *\alpha_2})=-1\ .$$

The solution of the master equation in the extended phase space is of the form

$$S(\Phi,\Phi^*)=S(\Phi_{\min},\Phi_{\min}^*)+\bar{C}_0^{*\alpha_0}\pi_{0\alpha_0}+\bar{C}_1^{*\alpha_1}\pi_{1\alpha_1}+\bar{C}_{1\alpha_1}^{\prime *}\pi_1^{\prime\,\alpha_1}+\bar{C}_2^{*\alpha_2}\pi_{2\alpha_2}+C_{2\alpha_2}^{\prime *}\pi_2^{\prime\,\alpha_2}+\bar{C}_2^{\prime\prime *\alpha_2}\pi_{2\alpha_2}''\ . \tag{5.10}$$

The classification of fields (5.7) and the corresponding conditions on the gauge fermion are the following. The first three pairs in (5.7) are the fields (4.14) of first-stage theories. Therefore the conditions

$$\mathrm{rank}_\pm \left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{0\alpha_0}\partial\phi^i}\right|_{\Phi_0}=\mathrm{rank}_\pm \left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{0\alpha_0}\partial\phi^i}R^i_{\beta_0}\right|_{\Phi_0}=[m_0-(m_1-m_2)]_\pm \tag{5.11}$$

replace (4.18a) and (4.18b), and the conditions

$$\begin{aligned}\mathrm{rank}_\pm \left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{1\alpha_1}\partial C_0^{\alpha_0}}\right|_{\Phi_0}&=\mathrm{rank}_\pm \left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{1\alpha_1}\partial C_0^{\alpha_0}}Z^{\alpha_0}_{1\beta_1}\right|_{\Phi_0}\\&=\mathrm{rank}_\pm \left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{0\alpha_0}\partial C_1^{\prime\,\alpha_1}}\right|_{\Phi_0}\\&=\mathrm{rank}_\pm \bar{Z}^{\alpha_1}_{1\alpha_0}\left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{0\alpha_0}\partial C_1^{\prime\,\beta_1}}\right|_{\Phi_0}\\&=\mathrm{rank}_\pm \bar{Z}^{\alpha_1}_{1\alpha_0}\,|_{\Phi_0}=(m_1-m_2)_\pm\end{aligned} \tag{5.12}$$

replace (4.18c) and (4.18d). The fourth pair in (5.7) is introduced in order to remove the redundant components of C_1, which is now a gauge field. This pair plays the same role as the second pair in (4.14). Therefore the requirement of admissibility of gauge conditions imposed on C_1 parallels (4.18c):

$$\mathrm{rank}_\pm \left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{2\alpha_2}\partial C_1^{\alpha_1}}Z^{\alpha_1}_{2\beta_2}\right|_{\Phi_0}=m_{2\pm}\ . \tag{5.13}$$

The fifth pair in (5.7) is introduced in order to remove the redundant components of $\bar{C}_1$ and plays the same role as the third pair in (4.14). Correspondingly, C_2' is the second-stage extraghost. The condition of admissibility parallels (4.18d):

$$\mathrm{rank}_\pm \bar{Z}^{\alpha_2}_{2\alpha_1}\left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{1\alpha_1}\partial C_2^{\prime\,\beta_2}}\right|_{\Phi_0}=m_{2\pm}\ , \tag{5.14}$$

where

$$\begin{gathered}\bar{Z}^{\alpha_2}_{2\alpha_1}\left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{1\alpha_1}\partial C_0^{\alpha_0}}\right|_{\Phi_0}=0\ ,\\ \epsilon(\bar{Z}^{\alpha_2}_{2\alpha_1})=\epsilon_{\alpha_1}+\epsilon_{\alpha_2}\ ,\\ \mathrm{rank}_\pm\bar{Z}^{\alpha_2}_{2\alpha_1}\,|_{\Phi_0}=m_{2\pm}\ .\end{gathered} \tag{5.15}$$

In the present case the first three pairs generate redundant gauge conditions. The redundant gauge conditions from the second pair remove the new extraghost C_2' the same way as C_1' was removed in first-stage theories. The redundant gauge conditions from the first pair must remove the old extraghost C_1' as before. However this time they remove only (m_1-m_2) components of C_1'. This is seen from the third rank in (5.12). The corresponding matrix has m_2 right zero-eigenvalue eigenvectors:

$$\begin{gathered}\left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{0\alpha_0}\partial C_1^{\prime\,\alpha_1}}Z^{\prime\,\alpha_1}_{2\beta_2}\right|_{\Phi_0}=0\ ,\\ \epsilon(Z^{\prime\,\alpha_1}_{2\alpha_2})=\epsilon_{\alpha_1}+\epsilon_{\alpha_2}\ ,\\ \mathrm{rank}_\pm Z^{\prime\,\alpha_1}_{2\beta_2}\,|_{\Phi_0}=m_{2\pm}\ ,\end{gathered} \tag{5.16}$$

which play the role of gauge generators for C_1'. The gauge components of C_1' survive. To remove them we introduced the sixth pair in (5.7), which contains the extraghost for the extraghost: $\bar{C}_2''$. The condition of admissibility is standard,

$$\mathrm{rank}_\pm \left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{2\alpha_2}''\partial C_1^{\prime\,\alpha_1}}Z^{\prime\,\alpha_1}_{2\beta_2}\right|_{\Phi_0}=m_{2\pm}\ , \tag{5.17}$$

where Z_2' are eigenvectors from (5.16). There still remain the redundant gauge conditions from the third pair. They exactly remove the extraghost for the extraghost. This is guaranteed by the condition

$$\mathrm{rank}_\pm \left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{2\alpha_2}''\partial C_1^{\prime\,\alpha_1}}\right|_{\Phi_0}=m_{2\pm}\ , \tag{5.18}$$

following from (5.17).

The simplest form for the gauge fermion, which satisfies the above conditions, is

$$\Psi=\bar{C}_{0\alpha_0}\chi^{\alpha_0}(\phi)+\bar{C}_{1\alpha_1}\omega^{\alpha_1}_{1\alpha_0}C_0^{\alpha_0}+\bar{C}_{0\alpha_0}\sigma^{\alpha_0}_{1\alpha_1}C_1^{\prime\,\alpha_1}+\bar{C}_{2\alpha_2}\omega^{\alpha_2}_{2\alpha_1}C_1^{\alpha_1}+\bar{C}_{1\alpha_1}\sigma^{\prime\,\alpha_1}_{2\alpha_2}C_2^{\prime\,\alpha_2}+\bar{C}_{2\alpha_2}''\sigma^{\prime\prime\alpha_2}_{2\alpha_1}C_1^{\prime\,\alpha_1}\ . \tag{5.19}$$

Here the functions $\chi^{\alpha_0}(\phi)$ are restricted by (5.11), the matrices ω_1 and σ_1 are degenerate and are restricted by (5.12), and the matrices ω_2, σ_2', and σ_2'' have the maximal rank and are restricted by (5.13), (5.14), and (5.17), respectively. The six sets of gauge conditions generated by six pairs of fields (5.7) take the form

$$\begin{aligned}
&\frac{\partial\Psi}{\partial\bar{C}_{0\alpha_0}}=\chi^{\alpha_0}+\sigma^{\alpha_0}_{1\alpha_1}C_1'^{\,\alpha_1}=0\ ,\\
&\frac{\partial\Psi}{\partial\bar{C}_{1\alpha_1}}=\omega^{\alpha_1}_{1\alpha_0}C_0^{\alpha_0}+\sigma'^{\,\alpha_1}_{2\alpha_2}C_2'^{\,\alpha_2}=0\ ,\\
&\frac{\partial\Psi}{\partial C_1'^{\,\alpha_1}}=\bar{C}_{0\alpha_0}\sigma^{\alpha_0}_{1\alpha_1}+\bar{C}''_{2\alpha_2}\sigma''^{\alpha_2}_{2\alpha_1}=0\ ,\\
&\frac{\partial\Psi}{\partial\bar{C}_{2\alpha_2}}=\omega^{\alpha_2}_{2\alpha_1}C_1^{\alpha_1}=0\ ,\\
&\frac{\partial\Psi}{\partial C_2'^{\,\alpha_2}}=\bar{C}_{1\alpha_1}\sigma'^{\,\alpha_1}_{2\alpha_2}=0\ ,\\
&\frac{\partial\Psi}{\partial\bar{C}''_{2\alpha_2}}=\sigma''^{\alpha_2}_{2\alpha_1}C_1'^{\,\alpha_1}=0\ .
\end{aligned}\tag{5.20}$$

VI. QUANTIZATION RULES FOR Lth- STAGE THEORIES

In Lth-stage theories we have gauge generators

$$\begin{aligned}
&R^i_{\alpha_0}\ ,\ i=1,\ldots,n=n_++n_-\ ,\\
&\qquad\alpha_0=1,\ldots,m_0=m_{0+}+m_{0-}\ ,\\
&\epsilon(R^i_{\alpha_0})=\epsilon_i+\epsilon_{\alpha_0}\ ,
\end{aligned}\tag{6.1}$$

and zero-eigenvalue eigenvectors

$$\begin{aligned}
&Z^{\alpha_{s-1}}_{s\alpha_s}\ ,\ \alpha_s=1,\ldots,m_s=m_{s+}+m_{s-}\ ,\\
&\qquad s=1,\ldots,L\ ,\\
&\epsilon(Z^{\alpha_{s-1}}_{s\alpha_s})=\epsilon_{\alpha_{s-1}}+\epsilon_{\alpha_s}\ ,
\end{aligned}\tag{6.2}$$

satisfying the Noether identities

$$\begin{aligned}
&\frac{\partial_r\mathscr{S}}{\partial\phi^i}R^i_{\alpha_0}=0\ ,\ \ R^i_{\alpha_0}Z^{\alpha_0}_{1\alpha_1}\,|_{\phi_0}=0\ ,\\
&Z^{\alpha_{s-2}}_{s-1\alpha_{s-1}}Z^{\alpha_{s-1}}_{s\alpha_s}\,|_{\phi_0}=0\ ,\ \ s=2,\ldots,L
\end{aligned}\tag{6.3}$$

and the conditions

$$\begin{aligned}
&\mathrm{rank}_\pm R^i_{\alpha_0}\,|_{\phi_0}=\left[m_0-\sum_{s=1}^{L}m_s(-1)^{s-1}\right]_\pm\ ,\\
&\mathrm{rank}_\pm Z^{\alpha_{s-1}}_{s\alpha_s}\,|_{\phi=\phi_0}=\left[\sum_{s'=s}^{L}m_{s'}(-1)^{s'-s}\right]_\pm\ .
\end{aligned}\tag{6.4}$$

Additionally, the postulate of completeness holds in the form

$$\left[\mathrm{rank}_\pm\frac{\partial_l\partial_r\mathscr{S}}{\partial\phi^i\partial\phi^j}+\mathrm{rank}_\pm R^i_{\alpha_0}\right]\Bigg|_{\phi_0}=n_\pm\ .\tag{6.5}$$

The minimal sector of fields Φ^A consists of

$$\begin{aligned}
&\Phi_{\min}=\{\phi^i;C_s^{\alpha_s}\,|\,\alpha_s=1,\ldots,m_s\ ;\ s=0,\ldots L\}\ ,\\
&\epsilon(C_s^{\alpha_s})=\epsilon_{\alpha_s}+s+1\ ,\\
&\mathrm{gh}(C_s^{\alpha_s})=s+1\ ,
\end{aligned}\tag{6.6}$$

where C_s is the ghost which arises at the sth stage to make the solution of the master equation proper. The boundary conditions for the master equation (3.12) are of the form

$$\begin{aligned}
&S(\Phi,\Phi^*)\,|_{\Phi^*=0}=\mathscr{S}(\phi)\ ,\\
&\frac{\partial_l}{\partial\phi_i^*}\frac{\partial_r}{\partial C_0^{\alpha_0}}S(\Phi,\Phi^*)\Bigg|_{\Phi^*=0}=R^i_{\alpha_0}(\phi)\ ,\\
&\frac{\partial_l}{\partial C^*_{s-1\alpha_{s-1}}}\frac{\partial_r}{\partial C_s^{\alpha_s}}S(\Phi,\Phi^*)\Bigg|_{\Phi^*=0}=Z^{\alpha_{s-1}}_{s\alpha_s}(\phi)\ .
\end{aligned}\tag{6.7}$$

The solution of the master equation in the minimal sector (6.6), which has vanishing (conserved) ghost number and satisfies the boundary conditions (6.7), generates the open Lth-stage gauge algebra.

The auxiliary sector of fields Φ^A is constructed according to the following rules. First to each ghost $C_s^{\alpha_s}$ of the minimal sector (6.6), one adjoins $(s+1)$ fields having the same statistics:

$$\begin{aligned}
&C_0^{\alpha_0}\to\bar{C}_{0\alpha_0}\ ,\\
&C_1^{\alpha_1}\to\bar{C}_{1\alpha_1}\ ,\ \ C_1'^{\,\alpha_1}\ ,\\
&C_2^{\alpha_2}\to\bar{C}_{2\alpha_2}\ ,\ \ C_2'^{\,\alpha_2}\ ,\ \ \bar{C}''_{2\alpha_2}\ ,\\
&C_3^{\alpha_3}\to\bar{C}_{3\alpha_3}\ ,\ \ C_3'^{\,\alpha_3}\ ,\ \ \bar{C}''_{3\alpha_3}\ ,\ \ C_3'''^{\,\alpha_3}\ ,\\
&\ldots\ .
\end{aligned}\tag{6.8}$$

Here the $(s+1)$th line from the top $(s=0,1,\ldots,L)$ contains a complete set of new ghosts arising at the sth stage. All ghosts on a given line have the same ghost number: the number of the stage at which they first appear. The C and $\bar{C}$ fields on each line alternate. The primed fields are extraghosts for extraghosts for ... or extraghosts of a certain generation. The number of primes increases monotonically along a line and indicates the number of a generation.

Next, to each field on the right of (6.8) one assigns a Lagrange multiplier of the opposite statistics:

$$\begin{aligned}
&\bar{C}_{0\alpha_0}\to\pi_{0\alpha_0}\ ,\\
&\bar{C}_{1\alpha_1}\to\pi_{1\alpha_1}\ ,\ \ C_1'^{\,\alpha_1}\to\pi_1'^{\,\alpha_1}\ ,\\
&\bar{C}_{2\alpha_1}\to\pi_{2\alpha_2}\ ,\ \ C_2'^{\,\alpha_2}\to\pi_2'^{\,\alpha_2}\ ,\ \ \bar{C}''_{2\alpha_2}\to\pi''_{2\alpha_2}\ ,\\
&\bar{C}_{3\alpha_3}\to\pi_{3\alpha_3}\ ,\ \ C_3'^{\,\alpha_3}\to\pi_3'^{\,\alpha_3}\ ,\\
&\bar{C}''_{3\alpha_3}\to\pi''_{3\alpha_3}\ ,C_3'''^{\,\alpha_3}\to\pi_3'''^{\,\alpha_3}\ ,\\
&\ldots\ .
\end{aligned}\tag{6.9}$$

All the C's in the $(L+1)$ lines of (6.8) and all the π's in the $(L+1)$ lines of (6.9) form, together with the initial gauge field ϕ, the complete set of fields Φ^A for the Lth-stage theory. To each Φ^A one then adjoins an antifield Φ^*_A. The solution of the master equation in the complete phase space has the form

$$\begin{aligned}S(\Phi,\Phi^*)=S(\Phi_{\min},\Phi^*_{\min})&\\ &+\bar{C}_0^{*\alpha_0}\pi_{0\alpha_0}\\ &+\bar{C}_1^{*\alpha_1}\pi_{1\alpha_1}+C'^{*}_{1\alpha_1}\pi_1'^{\alpha_1}\\ &+\bar{C}_2^{*\alpha_2}\pi_{2\alpha_2}+C'^{*}_{2\alpha_2}\pi_2'^{\alpha_2}+\bar{C}_2^{''\alpha_2}\pi''_{2\alpha_1}\\ &+\bar{C}_3^{*\alpha_3}\pi_{3\alpha_3}+C'^{*}_{3\alpha_3}\pi_3'^{\alpha_3}+\bar{C}_3^{''*\alpha_3}\pi''_{3\alpha_3}+C'''^{*}_{3\alpha_3}\pi_3'''^{\alpha_3}\\ &+\cdots\ . \end{aligned}\tag{6.10}$$

It remains to indicate the ghost numbers of the auxiliary fields and the conditions of the gauge fermion. For this purpose it is convenient to arrange the fields (6.8) in a triangle as shown in Fig. 3. The vertex of the triangle is the initial gauge field ϕ^i. The right side of the triangle, formed by bold bars, is the "algebraic branch." It contains the ghosts of the minimal sector (6.6), generating the gauge algebra. Each field of the algebraic branch gives rise to a left branch formed by arrows. Each horizontal dashed line contains all ghosts first arising at the given stage of reducibility. The horizontal lines are the lines of the table (6.8), written from right to left.

We now introduce the rule: the sum of the ghost numbers of any two fields connected by an arrow equals minus one. Since the ghost numbers of fields from the algebraic branch are known [Eq. (6.6)], the above rule determines the ghost numbers of all C's. The sum of ghost numbers of any field and its antifield also equals minus one. This determines the ghost numbers of all antifields. Finally, the ghost numbers of all the π's can be determined from Eq. (6.10) and the condition that the ghost number of S equals zero.

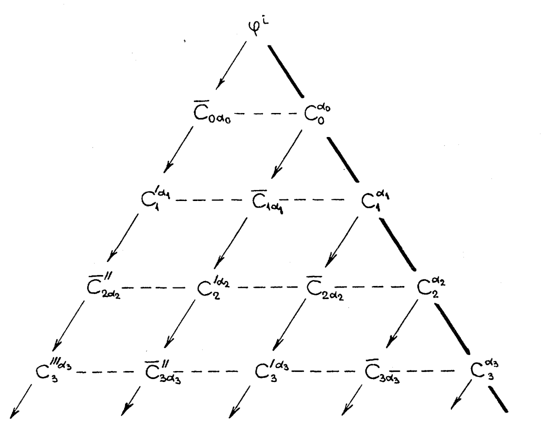

FIG. 3. The diagram of proliferation of ghosts. Each horizontal dashed line contains all ghosts which first arise at the given stage. The right bold line contains ghosts of the minimal (algebraic) sector. To each arrow there corresponds a nondegeneracy condition on the gauge fermion.

Each arrow in Fig. 3 connects two fields either as $C\to\bar{C}$ or as $\bar{C}\to C$. The connections $C\to\bar{C}$ and $\bar{C}\to C$ alternate along each left branch. The connections adjoining the algebraic branch are always the $C\to\bar{C}$ ones.

To each connection $C\to\bar{C}$ there corresponds a zero-eigenvalue right eigenvector Z and a condition on the gauge fermion. To each connection $\bar{C}\to C$ there corresponds a zero-eigenvalue left eigenvector $\bar{Z}$ and a condition on the gauge fermion. There are as many conditions on the gauge fermion as there are arrows in Fig. 3.

Consider a left branch and a connection

$$C^{\alpha_{s-1}}\to\bar{C}_{\alpha_s}\tag{6.11}$$

belonging to this branch (we omit primes). The condition corresponding to this connection is of the form

$$\begin{aligned}\text{rank}_\pm\left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{\alpha_s}\partial C^{\alpha_{s-1}}}\right|_{\Phi_0}&=\text{rank}_\pm\left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{\alpha_s}\partial C^{\alpha_{s-1}}}Z^{\alpha_{s-1}}_{\beta_s}\right|_{\Phi_0}\\ &=\left[\sum_{s'=s}^{L}m_{s'}(-1)^{s'-s}\right]_\pm\ .\end{aligned}\tag{6.12}$$

If the connection (6.11) does not adjoin the algebraic branch, then the right eigenvector $Z^{\alpha_{s-1}}_{\alpha_s}$ in (6.12) is defined by the previous connection

$$\bar{C}_{\alpha_{s-2}}\to C^{\alpha_{s-1}}$$

of the same left branch, namely,

$$\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{\alpha_{s-2}}\partial C^{\alpha_{s-1}}}Z^{\alpha_{s-1}}_{\alpha_s}\,|_{\Phi_0}=0\ ,\quad \epsilon(Z^{\alpha_{s-1}}_{\alpha_s})=\epsilon_{\alpha_{s-1}}+\epsilon_{\alpha_s}\ ,$$

$$\text{rank}_\pm Z^{\alpha_{s-1}}_{\alpha_s}\,|_{\Phi_0}=\left[\sum_{s'=s}^{L}m_{s'}(-1)^{s'-s}\right]_\pm\ .\tag{6.13}$$

If the connection (6.11) adjoins the algebraic branch (i.e., if the field $C^{\alpha_{s-1}}$ belongs to the minimal sector), then $Z^{\alpha_{s-1}}_{\alpha_s}$ is the initially given eigenvector (6.2) of the gauge algebra. This will be true also when $s=0$ if we identify

$$C^{\alpha_{-1}}=\phi^i\ ,\quad Z^{\alpha_{-1}}_{\alpha_0}=R^i_{\alpha_0}\ .\tag{6.14}$$

The condition corresponding to a connection

$$\bar{C}_{\alpha_{s-1}}\to C^{\alpha_s}\tag{6.15}$$

of a left branch is of the form

$$\begin{aligned}\text{rank}_\pm\left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{\alpha_{s-1}}\partial C^{\alpha_s}}\right|_{\Phi_0}&=\text{rank}_\pm\bar{Z}^{\alpha_s}_{\alpha_{s-1}}\left.\frac{\partial_l\partial_r\Psi}{\partial\bar{C}_{\alpha_{s-1}}\partial C^{\beta_s}}\right|_{\Phi_0}\\ &=\left[\sum_{s'=s}^{L}m_{s'}(-1)^{s'-s}\right]_\pm\ .\end{aligned}\tag{6.16}$$

The left eigenvector $\bar{Z}^{\alpha_s}_{\alpha_{s-1}}$ in (6.16) is defined by the previous connection

$$C^{\alpha_{s-2}}\to\bar{C}_{\alpha_{s-1}}$$

of the same left branch, namely,

$$\bar{Z}^{\alpha_s}_{\alpha_{s-1}} \frac{\partial_l \partial_r \Psi}{\partial \bar{C}_{\alpha_{s-1}} \partial C^{\alpha_s - 2}} \Bigg|_{\Phi_0} = 0 \ , \quad \epsilon(\bar{Z}^{\alpha_s}_{\alpha_{s-1}}) = \epsilon_{\alpha_{s-1}} + \epsilon_{\alpha_s} \ ,$$

$$\mathrm{rank}_{\pm} \bar{Z}^{\alpha_s}_{\alpha_{s-1}} \mid_{\Phi_0} = \left[\sum_{s'=s}^{L} m_{s'} (-1)^{s'-s} \right]_{\pm} \ . \tag{6.17}$$

In Eqs. (6.12)–(6.17), Φ_0 is the stationary point of the full action $S \mid_\Sigma$.

The conditions on the gauge fermion are thus formulated for each left branch separately. The simplest structure for the gauge fermion is a sum of products of each pair of C's that is connected by an arrow.

VII. EXAMPLES

To apply the methods of this paper to the simple linear theories mentioned in the Introduction is like cracking nuts with a sledgehammer. (The structure of the quantum action for these theories was already deduced earlier by other methods.[7,8]) Nevertheless, for the sake of illustrating the above general results and comparing them with other methods, we shall consider two examples: the spin-0 antisymmetric tensor field and the spin-$\frac{5}{2}$ symmetric spin-tensor field. Both examples are first-stage theories, so we may use the formulas of Sec. IV.

Since the gauge algebra in both examples is Abelian, only the first three terms of the expression (4.7) survive:

$$S(\Phi_{\min}, \Phi^*_{\min}) = \mathscr{S}(\phi) + \phi^*_i R^i_{\alpha_0} C^{\alpha_0}_0 + C^*_{0\alpha_0} Z^{\alpha_0}_{1\alpha_1} C^{\alpha_1}_1 \ . \tag{7.1}$$

The final action in the functional integral [Eq. (4.20)] takes the form

$$S \mid_\Sigma = \mathscr{S}(\phi) + \mathscr{S}_{\text{ghost}} + \mathscr{S}_{\text{gauge}} \ , \tag{7.2}$$

where

$$\mathscr{S}_{\text{ghost}} = \frac{\partial \Psi}{\partial \phi^i} R^i_{\alpha_0} C^{\alpha_0}_0 + \frac{\partial \Psi}{\partial C^{\alpha_0}_0} Z^{\alpha_0}_{1\alpha_1} C^{\alpha_1}_1 \ , \tag{7.3}$$

$$\mathscr{S}_{\text{gauge}} = \frac{\partial \Psi}{\partial \bar{C}_{0\alpha_0}} \pi_{0\alpha_0} + \frac{\partial \Psi}{\partial \bar{C}_{1\alpha_1}} \pi_{1\alpha_1} + \frac{\partial \Psi}{\partial C'^{\alpha_1}_1} \pi'^{\alpha_1}_1 \ . \tag{7.4}$$

A. The antisymmetric tensor field in an external metric

The action of the antisymmetric tensor field $A_{\mu\nu} = -A_{\nu\mu}$ is of the form[5]

$$\mathscr{S}(A_{\mu\nu}) = -\tfrac{1}{12} \int F_{\mu\nu\rho} F^{\mu\nu\rho} \sqrt{g}\, dx \ ,$$
$$F_{\mu\nu\rho} \equiv \partial_\mu A_{\nu\rho} + \partial_\rho A_{\mu\nu} + \partial_\nu A_{\rho\mu} \ , \tag{7.5}$$

and is invariant under the following transformations:

$$\delta A_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu$$

and vectorial bosonic parameters θ_μ.

Thus we have

$$\phi^i = A_{\mu\nu}(x) \ , \quad R^i_{\alpha_0} = (\partial_\nu \delta^\alpha_\mu) \delta(x - x_0) \ , \tag{7.6}$$

where $i = (\mu\nu, x)$ and $\alpha_0 = (\alpha, x_0)$. The generators are evidently linearly dependent and have the zero-eigenvalue eigenvectors

$$Z^{\alpha_0}_{1\alpha_1} = \partial_\alpha \delta(x_0 - x_1) \ , \quad R^i_{\alpha_0} Z^{\alpha_0}_{1\alpha_1} \equiv 0 \ , \tag{7.7}$$

where $\alpha_1 = (x_1)$. The eigenvectors Z_1 are evidently linearly independent, so the theory is first stage.

According to Sec. IV (or Fig. 3), the ghost content of this theory is the following. The zero-stage ghosts are vectors $C_{0\alpha}$ and $\bar{C}_{0\alpha}$ and are fermions. The normal first-stage ghosts are scalars C_1 and $\bar{C}_1$ and are bosons. The extraghost C'_1 is also a scalar and a boson. The zero-stage Lagrange multiplier $\pi_{0\alpha}$ is a vector and a boson. The first-stage Lagrange multipliers π_1, π'_1 are scalars and fermions.

Using the above information in Eqs. (7.3) and (7.4) we find, for the general form of ghost and gauge-breaking contributions to the total action in the functional integral,

$$\mathscr{S}_{\text{ghost}} = \int \left[\frac{\delta \Psi}{\delta A_{\mu\nu}} (\partial_\mu C_{0\nu} - \partial_\nu C_{0\mu}) + \frac{\delta \Psi}{\delta C_{0\alpha}} \partial_\alpha C_1 \right] dx \ , \tag{7.8}$$

$$\mathscr{S}_{\text{gauge}} = \int \left[\frac{\delta \Psi}{\delta \bar{C}_{0\alpha}} \pi_{0\alpha} + \frac{\delta \Psi}{\delta \bar{C}_1} \pi_1 + \frac{\delta \Psi}{\delta C'_1} \pi'_1 \right] dx \ . \tag{7.9}$$

It remains to construct the gauge fermion Ψ.

The following redundant set of gauge conditions suggests itself:

$$\chi^\alpha(A) = \sqrt{g}\, g^{\alpha\beta} \nabla^\mu A_{\mu\beta} \ , \quad \nabla_\alpha \frac{\chi^\alpha(A)}{\sqrt{g}} \equiv 0 \ . \tag{7.10}$$

This is the $\chi^{\alpha_0}(\phi)$ of Eq. (4.25). The simplest form for the gauge fermion is

$$\Psi = \int [\bar{C}_{0\alpha} g^{\alpha\beta} \nabla^\mu A_{\mu\beta} + \bar{C}_1 \nabla^\mu C_{0\mu} + C_{0\alpha} \nabla^\alpha C'_1] \sqrt{g}\, dx \tag{7.11}$$

with an obvious choice for the matrices $\omega^{\alpha_1}_{\alpha_0}$ and $\sigma^{\alpha_0}_{\alpha_1}$ of Eq. (4.25). The additional terms (4.27) take the form

$$\tfrac{1}{2} \int \left[\frac{1}{a} \bar{C}_{0\alpha} g^{\alpha\beta} \pi_{0\beta} + \frac{1}{b} \bar{C}_1 \pi'_1 - \frac{1}{b} \pi_1 C'_1 \right] \sqrt{g}\, dx \tag{7.12}$$

with an obvious choice for the matrices $\kappa^{\alpha_0\beta_0}$ and $\rho^{\alpha_1}_{\beta_1}$. The constants a and b in (7.12) are free parameters.

The ghost action (7.8) now takes the form

$$\mathscr{S}_{\text{ghost}} = \int [-\tfrac{1}{2} g^{\mu\alpha} g^{\nu\beta} (\nabla_\mu \bar{C}_{0\nu} - \nabla_\nu \bar{C}_{0\mu})(\nabla_\alpha C_{0\beta} - \nabla_\beta C_{0\alpha}) - (\nabla^\alpha \bar{C}_1)(\nabla_\alpha C_1)] \sqrt{g}\, dx \ . \tag{7.13}$$

The inverse propagator of the ghosts $\bar{C}_0$, C_0 is degenerate, as it should be, with right eigenvectors (7.7). The zero-eigenvalue left eigenvectors (the gauge generators for $\bar{C}_0$) coincide with the right ones:

$$\bar{Z}^{\alpha_1}_{1\alpha_0} = \partial_\alpha \delta(x_0 - x_1) \tag{7.14}$$

[this follows already from (7.10)]. This is the $\bar{Z}_1$ of Eq. (4.19) [the notation in (7.14) is the same as in (7.6) and

(7.7)]. The action for the ghosts $\bar{C}_0, C_0$ is of Maxwell's type. It contains no other gauge arbitrariness except that of the longitudinal parts of $\bar{C}_{0\mu}, C_{0\mu}$. The latter arbitrariness is described by the eigenvectors (7.7) and (7.14). This guarantees the fulfillment of the rank conditions (4.18a) and (4.18b). The action for the ghosts $\bar{C}_1, C_1$ in (7.13) is nondegenerate. This is equivalent to the rank condition (4.18c). Finally, the rank condition (4.18d) can be verified directly from (7.11) and (7.14).

If we do not add the terms (7.12) to (7.11), then the gauge-breaking action (7.9) will produce three sets of gauge conditions (4.26) or (4.21) which, in the present case, are of the form:

$$\nabla^\mu A_{\mu\alpha} + \nabla_\alpha C_1' = 0 \; , \quad \nabla^\mu C_{0\mu} = 0 \; , \quad \nabla^\mu \bar{C}_{0\mu} = 0 \; . \tag{7.15}$$

These conditions eliminate the longitudinal parts of $C_{0\mu}, \bar{C}_{0\mu}$. The application of ∇_α to the first equality of (7.15) gives the conditon $\Box C_1' = 0$ eliminating the extraghost C_1'. The remaining conditions eliminate the transverse part of $\nabla^\mu A_{\mu\alpha}$ and are the genuine gauge conditions on the antisymmetric tensor field.

If we add the terms (7.12) to (7.11), then the gauge-breaking action (7.9) will take the following final form after integration over the π's:

$$\mathscr{S}_{\text{gauge}} = \int \left[-\frac{a}{2} (\nabla^\mu A_{\mu\alpha})(\nabla^\nu A_{\nu\beta}) g^{\alpha\beta} + b(\nabla^\mu \bar{C}_{0\mu})^2 - \frac{a}{2} (\nabla^\alpha C_1')(\nabla_\alpha C_1') \right] \sqrt{g}\, dx \; . \tag{7.16}$$

The sum of expressions (7.5), (7.13), and (7.16) is the total action in the functional integral. The first term on the right-hand side of (7.16) is the usual quadratic gauge-breaking term, which removes the degeneracy of the initial action (7.5). The second term removes the degeneracy of the ghost action (7.13). The propagating fields C_1, $\bar{C}_1$ in (7.13) are the so-called ghosts for ghosts. The presence of the propagating extraghost C_1' in (7.16) is the price that one must pay for squaring redundant gauge conditions in the Lagrangian. The final result agrees with Refs. 7–9.

B. The free spin-$\frac{5}{2}$ field

The action of the spin-$\frac{5}{2}$ field is of the form[2]

$$\mathscr{S}(\psi_{\mu\nu}) = \int \left(-\tfrac{1}{2}\bar{\psi}_{\mu\nu}\partial\!\!\!/\psi_{\mu\nu} - \bar{\psi}_{\mu\nu}\gamma_\nu \partial\!\!\!/ \gamma_\lambda \psi_{\lambda\mu} + 2\bar{\psi}_{\mu\nu}\gamma_\nu \partial_\lambda \psi_{\lambda\mu} + \tfrac{1}{4}\bar{\psi}_{\lambda\lambda}\partial\!\!\!/\psi_{\mu\mu} - \bar{\psi}_{\lambda\lambda}\partial_\mu \gamma_\nu \psi_{\mu\nu} \right) dx \; , \tag{7.17}$$

where $\psi_{\mu\nu}$ is a symmetric Majorana spin tensor. The action is invariant under the transformation

$$\delta\psi_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \; , \tag{7.18}$$

where the parameter ϵ_μ is a fermionic Majorana spin vector restricted by the condition[2]

$$\gamma_\mu \epsilon_\mu = 0 \; . \tag{7.19}$$

The restrictions on the parameters are equivalent to a linear dependence of generators. Indeed, we may represent ϵ_μ as

$$\epsilon_\mu = (\delta_{\mu\nu} - \tfrac{1}{4}\gamma_\mu \gamma_\nu)\theta^\nu$$

and rewrite Eq. (7.18) as

$$\delta\psi_{\mu\nu} = (\partial_\mu \delta_\nu^\beta + \partial_\nu \delta_\mu^\beta)(\delta_{\beta\alpha} - \tfrac{1}{4}\gamma_\beta \gamma_\alpha)\theta^\alpha \; , \tag{7.20}$$

where θ^α are the genuine independent parameters, but the generators defined by (7.20) are linearly dependent.

Thus we have

$$\phi^i = \psi_{\mu\nu}(x) \; , \qquad R^i_{\alpha_0} = (\partial_\mu \delta_\nu^\beta + \partial_\nu \delta_\mu^\beta)(\delta_{\beta\alpha} - \tfrac{1}{4}\gamma_\beta\gamma_\alpha)\delta(x - x_0) \tag{7.21}$$

with an obvious identification of condensed indices. The zero-eigenvalue eigenvectors of the generators are

$$Z^{\alpha_0}_{1\alpha_1} = \gamma_\sigma \delta(x_0 - x_1) \; , \tag{7.22}$$

where σ, x_0, and one matrix of γ_σ and α_0, while the other matrix index of γ_σ and x_1 are α_1. The eigenvectors (7.22) are linearly independent, so the theory is of the first stage.

According to Sec. IV, the ghost content of this theory is the following. There are two bosonic spin vectors: C_0^α and $\bar{C}_{0\alpha}$. There are three fermionic spinors: C_1, $\bar{C}_1$, and the extraghost C_1'. Finally, there is one fermionic spin-vectorial Lagrange multiplier $\pi_{0\alpha}$ and two bosonic spinorial Lagrange multipliers π_1 and π_1'. All spinors are Majorana. All fields will be considered as spinors or conjugate spinors according to whether they have a bar or not, with the exception of $\pi_{0\alpha}$ and π_1. It is convenient always to write $\pi_{0\alpha}$ and π_1 on the left and consider them as conjugate spinors.

Using the above information in Eqs. (7.3) and (7.4), we find the general form for the ghost and gauge-breaking actions:

$$\mathscr{S}_{\text{ghost}} = \int \left[\frac{\delta\Psi}{\delta\psi_{\mu\nu}} (\partial_\mu \delta_\nu^\beta + \partial_\nu \delta_\mu^\beta)(\delta_{\beta\alpha} - \tfrac{1}{4}\gamma_\beta\gamma_\alpha) C_0^\alpha + \frac{\delta\Psi}{\delta C_0^\alpha}\gamma_\alpha C_1 \right] dx \; , \tag{7.23}$$

$$\mathscr{S}_{\text{gauge}} = \int \left[-\pi_{0\alpha} \frac{\partial\Psi}{\delta\bar{C}_{0\alpha}} + \pi_1 \frac{\delta\Psi}{\delta\bar{C}_1} + \frac{\delta\Psi}{\delta C_1'}\pi_1' \right] dx \; . \tag{7.24}$$

The gauge fermion Ψ remains to be constructed.

The authors of Ref. 2 propose the following redundant set of gauge conditions:

$$\chi_\alpha(\psi) = \gamma_\nu \psi_{\alpha\nu} - \tfrac{1}{4}\gamma_\alpha \psi_{\nu\nu} \; , \quad \gamma_\alpha \chi_\alpha(\psi) \equiv 0 \; . \tag{7.25}$$

In this case the simplest gauge fermion (4.25) can be written as

$$\Psi = \int [\bar{C}_{0\alpha}\chi_\alpha(\psi) + \bar{C}_1 \gamma_\mu C_0^\mu + \bar{C}_{0\alpha}\gamma_\alpha C_1'] dx \tag{7.26}$$

with an obvious choice for the matrices $\omega^{\alpha_1}_{\alpha_0}$ and $\sigma^{\alpha_0}_{\alpha_1}$. The additional terms (4.27) take the form

$$\tfrac{1}{2}\int (\bar{C}_{0\alpha}\kappa^{\alpha\beta}\pi^T_{0\beta} + \bar{C}_1 \rho \pi_1' + \pi_1 \rho C_1') dx \; , \tag{7.27}$$

where $\pi^T_{0\beta}$ is the transposed spinor. In (7.27) we shall choose

$$\kappa^{\alpha\beta}=\frac{1}{a}\delta^{\alpha\beta}\frac{1}{\Box}\partial\gamma^{-1}\ ,\quad \kappa^{-1}_{\alpha\beta}=a\delta_{\alpha\beta}\gamma\partial\ ,$$
$$\rho=\frac{1}{b}\frac{1}{\Box}\partial\ ,\quad \rho^{-1}=b\partial\ , \tag{7.28}$$

where $a,b,=$const, $\gamma_\mu^T=-\gamma\gamma_\mu\gamma^{-1}$, $\gamma^T=-\gamma$.

With this choice of the gauge fermion the explicit form of the ghost action (7.23) is

$$\mathscr{S}_{\text{ghost}}=\int[\bar{C}_{0\beta}(\gamma_\nu\delta_{\beta\mu}+\gamma_\mu\delta_{\beta\nu}-\tfrac{1}{2}\gamma_\beta\delta_{\mu\nu})\times(\delta_{\nu\alpha}-\tfrac{1}{4}\gamma_\nu\gamma_\alpha)\partial_\mu C_0^\alpha+4\bar{C}_1C_1]dx \tag{7.29}$$

and the gauge conditions following from (7.24) and (7.26) read

$$\chi_\alpha(\psi)+\gamma_\alpha C_1'=0\ ,\quad \gamma_\alpha C_0^\alpha=0\ ,\quad \bar{C}_{0\alpha}\gamma_\alpha=0\ . \tag{7.30}$$

The inverse propagator of the ghosts $\bar{C}_0$, C_0 has the zero-eigenvalue right eigenvectors (7.22) and the zero-eigenvalue left eigenvectors [see Eq. (4.19)]:

$$\bar{Z}^{\alpha_1}_{1\alpha_0}=\gamma_\sigma\delta(x_0-x_1)\ , \tag{7.31}$$

which coincide with the right ones, as seen already from (7.25). [The notation in (7.31) is the same as in (7.22)]. The gauge arbitrariness in the action for the ghosts $\bar{C}_{0\alpha}$, C_0^α is that of their γ-trace parts. The gauge conditions (7.30) exactly remove them. Application of γ_α to the first of Eqs. (7.30) yields the condition $C_1'=0$ which removes the extraghost. The remaining conditions remove the γ-traceless part of $\chi_\alpha(\psi)$ and are genuine gauge conditions for $\psi_{\mu\nu}$. This verifies the correctness of the choice of the gauge fermion.

If we add the terms (7.27) to (7.26), then the gauge-breaking action (7.24) will no longer produce δ functions. Since our matrices κ^{-1} and ρ^{-1} contain derivatives, we must make the replacement (4.28)–(4.30) instead of integrating over the π's. In this replacement we choose

$$A^{\alpha\beta}=\delta^{\alpha\beta}\gamma^{-1}\ ,\quad B=I\ ,$$

where I is the unit matrix in the space of spinors. Then the replacement (4.28)–(4.30) takes the form (literally)

$$\pi_{0\alpha}^T\to a\gamma\partial\{\gamma^{-1}\pi_{0\alpha}^T-[\chi_\alpha(\psi)+\gamma_\alpha C_1']\}\ ,$$
$$\bar{C}_{0\alpha}\to\bar{C}_{0\alpha}\partial a\ ,$$
$$\pi_1\to(\pi_1-\bar{C}_{0\alpha}\partial a\gamma_\alpha)b\partial\ ,$$
$$\bar{C}_1\to\bar{C}_1b\partial\ ,$$
$$\pi_1'\to\pi_1'-b\partial\gamma_\mu C_0^\mu\ ,$$

where all derivatives act formally to the right. When making the replacement in the gauge-breaking action (7.24) one may use Eq. (4.31).

After the above replacement the ghost and gauge-breaking contributions to the total action in the functional integral take the form

$$\mathscr{S}_{\text{ghost}}=\int[a\bar{C}_{0\beta}\partial(\gamma_\nu\delta_{\beta\mu}+\gamma_\mu\delta_{\beta\nu}-\tfrac{1}{2}\gamma_\beta\delta_{\mu\nu})(\delta_{\nu\alpha}-\tfrac{1}{4}\gamma_\nu\gamma_\alpha)\partial_\mu C_0^\alpha+4b\bar{C}_1\partial C_1]dx\ , \tag{7.32}$$

$$\mathscr{S}_{\text{gauge}}=\int\left[\frac{a}{2}(\bar{\chi}_\alpha+\bar{C}_1'\gamma_\alpha)\partial(\chi_\alpha+\gamma_\alpha C_1')-ab\bar{C}_{0\alpha}\partial\gamma_\alpha\partial\gamma_\beta C_0^\beta+\frac{a}{2}\pi_{0\alpha}\partial\underline{\pi}_{0\alpha}+\pi_1\pi_1'\right]dx\ , \tag{7.33}$$

where

$$\bar{C}_1'\equiv C_1'^T\gamma\ ,\quad \underline{\pi}_{0\alpha}\equiv(\gamma^T)^{-1}\pi_{0\alpha}^T$$

are redefinitions, not new fields, and where

$$\bar{\chi}_\alpha\equiv\bar{\psi}_{\alpha\nu}\gamma_\nu-\tfrac{1}{4}\bar{\psi}_{\nu\nu}\gamma_\alpha\ .$$

The integration fields are

$$\psi_{\mu\nu}\ ,\ \bar{C}_{0\alpha}\ ,\ C_0^\alpha\ ,\ \bar{C}_1\ ,\ C_1\ ,\ C_1'\ ,$$
$$\pi_{0\alpha}\ ,\ \pi_1\ ,\ \pi_1'\ .$$

Expressions (7.32) and (7.33) constitute the final result. The main difference from the previous example is the propagating Lagrange multiplier $\pi_{0\alpha}$. The Lagrange multipliers π_1 and π_1' do not propagate and can be integrated away. The propagating $\pi_{0\alpha}$ is the "Nielsen-Kallosh ghost" for the spin-$\frac{5}{2}$ field. As in the previous example, the extraghost C_1' is a propagating field.

APPENDIX: PROPERTIES OF THE MASTER EQUATION

We introduce the following collective notation for fields and antifields:

$$z^a=(\Phi^A;\Phi_A^*)\ ;\ A=1,\ldots,N\ ;$$
$$a=1,\ldots,2N\ ;\ \epsilon(z^a)\equiv\epsilon_a\ . \tag{A1}$$

Introduce the matrix

$$\zeta^{ab}=\begin{vmatrix}0 & \delta_B^A\\ -\delta_B^A & 0\end{vmatrix}\ , \tag{A2}$$

which permits the antibrackets (3.3) to be written in the form

$$(X,Y)=\frac{\partial_rX}{\partial z^a}\zeta^{ab}\frac{\partial_lY}{\partial z^b}\ . \tag{A3}$$

Evidently

$$\zeta^{ab}=-\zeta^{ba}\ ,\quad \epsilon\left[\zeta^{ab}\frac{\partial}{\partial z^b}\right]=\epsilon_a+1\ .$$

The master equation (3.12) for a bosonic $S(z)$ now takes the form

$$(S,S)\equiv\frac{\partial_rS}{\partial z^a}\zeta^{ab}\frac{\partial_lS}{\partial z^b}=0\ . \tag{A4}$$

Define

$$\mathscr{R}^a_c=\zeta^{ab}\frac{\partial_l\partial_rS}{\partial z^b\partial z^c}\ , \tag{A5}$$

which is the Hessian of $S(z)$ multiplied by the nonsingular matrix ζ. Let z_0 denote the stationary point of $S(z)$, which we suppose to exist.

Differentiation of (A4) yields

$$\frac{\partial_r S}{\partial z^a}\mathscr{R}^a_c=0\ . \tag{A6}$$

Hence any solution of the master equation is a gauge-invariant action. Equations (A6) are Noether identities, and the columns of the Hessian are generators of gauge transformations. The generators in (A6) are linearly dependent. Indeed, differentiation of (A6) yields

$$\mathscr{R}^a_b\mathscr{R}^b_c\,|_{z=z_0}=0\ . \tag{A7}$$

The matrix of generators is nilpotent, which implies the nilpotency of the Hessian at the stationary point. The solution of the master equation is therefore an example of an infinite-stage gauge theory.

Let $r_\pm$ be the rank of the Hessian at the stationary point:

$$r_\pm=\text{rank}_\pm\left.\frac{\partial_l\partial_r S}{\partial z^a\partial z^b}\right|_{z=z_0}\ ,\quad r=r_+ + r_-\ . \tag{A8}$$

Then r_+ (r_-) bosonic (fermionic) equations of motion are solvable with respect to small disturbances of z. But r is also the number of linearly independent generators among the $\mathscr{R}$. More precisely, r_- (r_+) bosonic (fermionic) Noether identities among the (A6) are solvable with respect to $\partial S/\partial z$. It follows from the nilpotency relation (A7) that

$$r\le N$$

always.

The solution S of the master equation is called proper[1] if

$$r=N\ .$$

If the solution is proper, then all zero-eigenvalue eigenvectors of $\mathscr{R}^a_b(z_0)$ are linear combinations of $\mathscr{R}^b_c(z_0)$. In other words, the solution is proper only if its Hessian at the stationary point has no other zero-eigenvalue eigenvectors except those contained in itself.[20]

There is generally no correlation between rows and columns in which the invertible minor of a matrix lies. However, for the Hessian of the proper solution such a correlation exists. Define

$$z^*_a=\zeta_{ab}z^b\ ,\quad \zeta_{ab}\zeta^{bc}=\delta^c_a\ ,$$

and let the equations of motion be solvable with respect to N variables z^A_1; let z^A_2 be the remaining N variables. Then the invertible minor of the Hessian is

$$\frac{\partial_l\partial_r S}{\partial z^*_{2A}\partial z^B_1}\ . \tag{A9}$$

There can be a particular case in which the equations of motion are solvable with respect to the commuting (in the sense of antibrackets) set of variables,

$$(z^A_1,z^B_1)=0\ .$$

This is equivalent to

$$\{z^*_2\}=\{z_1\}\ .$$

It can be verified that under the boundary conditions for the master equation, which we impose, we are always dealing with this particular case.

Expression (3.9) is nothing other than the functional integral for the gauge action $S(z)$. In spite of the infinite-stage character of this theory, we were able in Ref. 1 to obtain the quantization rules for $S(z)$ in the special class of gauges (3.5) or, equivalently, (3.4). The most general expression for the functional integral in this class of gauges is

$$\mathscr{Z}=\int\exp\left[\frac{i}{\hbar}W(z)\right]\delta(\chi_A(z))J^{1/2}\prod_a dz^a\ , \tag{A10}$$

where $W(z)=S(z)+O(\hbar)$ is given by Eqs. (3.11)–(3.14),

$$(\chi_A,\chi_B)=0\ ,\quad A=1,\dots,N\ , \tag{A11}$$

and J is the Jacobian of a canonical transformation $\Phi,\Phi^*\to\Phi',\Phi'^*$ which converts the commuting χ's into antifields (or fields):

$$J=\text{Ber}\frac{\partial(\Phi',\Phi'^*)}{\partial(\Phi,\Phi^*)}\ ,\Phi'^*_A=\chi_A(z)\ . \tag{A12}$$

Here we have used some properties of canonical transformations in the space of fields and antifields, derived in Ref. 11.

If $S(z)$ is a *local* action of field theory, then the terms $O(\hbar)$ in W are proportional to $\delta(0)$ and can be omitted (see Sec. III). In this case expression (A10) gives the explicit Feynman rules for $S(z)$. The necessity of requiring $S(z)$ to be the proper solution is seen from (A10). Indeed, if the rank of the Hessian of S is smaller than N, then N δ functions in (A10) will be insufficient to remove the degeneracy of the functional integral.

In conclusion, we shall derive the consequences of the master equation for the coefficients of the expansion of S in terms of antifields:

$$S(\Phi,\Phi^*)=\mathscr{S}+\sum_{n=1}^{\infty}\Phi^*_{A_n}\cdots\Phi^*_{A_1}S^{A_1\cdots A_n}(\Phi)\ . \tag{A13}$$

The coefficients in (A13) possess the generalized symmetry

$$S^{A_1\cdots A_n}=(S_{\text{sym}})^{A_1\cdots A_n}\ ,$$

which for any quantity $X^{A_1\cdots A_n}$ is defined as

$$(X_{\text{sym}})^{A_1\cdots A_n}\equiv\epsilon^{A_1\cdots A_n}_{B_1\cdots B_n}X^{B_1\cdots B_n}\ ,$$

$$n!\epsilon^{A_1\cdots A_n}_{B_1\cdots B_n}\equiv\left[\frac{\partial_l}{\partial\Phi^*_{A_1}}\cdots\frac{\partial_l}{\partial\Phi^*_{A_n}}\Phi^*_{B_n}\cdots\Phi^*_{B_1}\right]\ .$$

The substitution of (A13) into the master equation yields the following relations for the coefficients:

$$\frac{\partial_r \mathscr{S}}{\partial \Phi^A} S^A = 0 \, ,$$

$$\frac{\partial_r S^{A_1 \cdots A_n}}{\partial \phi^A} S^A + (n+1)(-1)^{\epsilon_{n0}} \frac{\partial_r \mathscr{S}}{\partial \Phi^A} S^{AA_1 \cdots A_n} = (X_{\text{sym}})^{A_1 \cdots A_n} = 0 \, , \quad n \geq 1 \, ,$$

$$\epsilon_{no} \equiv \epsilon_{nk} \mid_{k=0} \, ,$$

where

$$X^{A_1} \equiv 0 \, ,$$

$$X^{A_1 \cdots A_n} \equiv \sum_{k=1}^{n-1} (n-k+1)(-1)^{\epsilon_{nk}} \frac{\partial_r S^{A_1 \cdots A_k}}{\partial \Phi^A} S^{AA_{k+1} \cdots A_m} \, , \quad n \geq 2 \, ,$$

$$\epsilon_{nk} \equiv n - k + \sum_{m=k+1}^{n} \epsilon_{A_m} \, ,$$

$$\epsilon_A \equiv \epsilon(\Phi^A) \, .$$

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$$\operatorname{rank} \mathscr{R} + \operatorname{rank} \lambda = K \, .$$

However, in the presence of Grassmann numbers this theorem is not valid. One can only say that

$$\operatorname{rank} \mathscr{R} + \operatorname{rank} \lambda \leq K \, .$$

Nevertheless under the boundary conditions on the master equation which we impose, the equality holds, as we verify in the text. The reason is that the above-mentioned theorem of classical linear algebra is fulfilled for all matrices

$$\frac{\partial_l \partial_r \mathscr{S}}{\partial \phi^i \partial \phi^j} \, , \; R^i_{\alpha_0} \, , \; Z^{\alpha_0}_{1\alpha_1} \, , \; Z^{\alpha_1}_{2\alpha_2} \, , \ldots \, ,$$

which enter the boundary conditions on the master equation. This is guaranteed by the postulates of Sec. II.