String cosmologies

Patricio S. Letelier
Departamento de Física, Universidade de Brasília, 70.910 Brasília, DF, Brazil
(Received 21 December 1982; revised manuscript received 7 April 1983)

A model of a cloud formed by massive strings is studied in the context of the usual general relativity. This model is used as a source of Bianchi type I and "Kantowski-Sachs" type of cosmological models. Some global properties of the above-mentioned cosmological models are studied for the equation of state (energy density) $\propto$ (tension density). Particular models that can be explicitly integrated are also studied. The possibility that during the evolution of the Universe the strings disappear leaving only particles is examined.

I. INTRODUCTION

Recently we studied a gauge-invariant model of a cloud formed by geometric strings and used this model as a source of the gravitational field. In particular, we solved the Einstein equations for a plane-symmetric, a spherically symmetric, and particular cases of cylindrically symmetric space-times.[1,2] We had two main reasons to study the above-mentioned model. First, as a test of consistency, for some particular field theories based on string models and other models that use strings as basic elements we must have a reasonable behavior of the gravitational field produced by these strings. Second, we point out that the universe can be represented by a collection of extended objects (galaxies). So a "string dust" cosmology gives us a model to investigate this fact. Also, the existence of strings in the early universe can be used to introduce density fluctuations that might shed some light on the problem of galaxy formation.[3]

The presence of the strings in the early universe can be explained using grand unified theories[4,5] (GUTs). In spontaneously broken gauge theories the spontaneously broken symmetry can be restored at a temperature T greater than some critical temperature T_c. In standard "big bang" cosmological models T_c will be exceeded in the very earliest stages of the evolution of the universe. A phase transition will occur, as the universe cools below T_c, in which a multiplet of scalar fields (Higgs fields) develops a vacuum expectation value $\langle \phi \rangle = \eta$. Such phase transition can result in the development of different kinds of vacuum structures depending on the structure and topology of the gauge group. One possibility is that these vacuum structures give origin to strings in spacetime; these strings sometimes are called vortices.[6]

In this paper we study a new model of a string cloud, in which the strings that form the cloud are massive strings instead of geometrical strings. Each massive string is formed by a geometric string with particles attached along its extension. Hence, the strings that form the cloud are the generalization of Takabayasi's realistic model of strings[7] that we call p strings.[8] This is the simplest model wherein we have particles and strings together. In principle we can eliminate the strings and end up with a cloud of particles. This is a desirable property of a model of a string cloud to be used in cosmology, since the strings are not observed at the present time of the evolution of the universe.

The cosmological models that we study are of the Bianchi type I and of the "Kantowski-Sachs" type. We have chosen these models because they are supposed to be a reasonable representation of the universe in early epochs and they are simple enough that many of their features can be studied exactly.

In Sec. II we present the model of a string cloud, in particular, we study the energy-momentum tensor for the cloud. In Sec. III we examine the integrability condition and the reality conditions (energy conditions) for a cloud of strings coupled to the Einstein equations. In Secs. IV and V we study some global properties as well as particular solutions to the Einstein equations coupled to a cloud of massive strings for Bianchi type I and Kantowski-Sachs type of cosmological models, respectively. Special attention is paid to the evolution of the density of particles and the density of tension for each particular case. Finally in Sec. VI we discuss some of the results obtained.

II. THE STRING-CLOUD MODEL

In this section we present a generalization of the model of string clouds studied in Ref. 1. The energy-momentum tensor for a cloud of geometric strings is[9]

$$T_s^{\mu\nu}=\bar{\lambda}\Sigma^{\mu\alpha}\Sigma_\alpha{}^\nu \,, \tag{2.1}$$

where $\bar{\lambda}$ is the geometric string's proper-energy density[10] and $\Sigma^{\mu\nu}$ is the simple-surface-forming bivector that spans the string's world sheet. The conditions for $\Sigma^{\mu\nu}$ to be simple and surface forming are,[11] respectively,

$$\Sigma^{\mu[\alpha}\Sigma^{\beta\gamma]}=0 \,, \tag{2.2}$$

$$\nabla_\mu\Sigma^{\mu[\alpha}\Sigma^{\beta\gamma]}=0 \,, \tag{2.3}$$

where the square brackets denote antisymmetrization in the enclosed indices. Also, we have

$$\Sigma^{\mu\nu}\Sigma_{\mu\nu}<0 \,. \tag{2.4}$$

The energy-momentum tensor for a cloud of particles is

$$T_p^{\mu\nu}=\rho_p u^\mu u^\nu \,, \tag{2.5}$$

where ρ_p and u^μ are the cloud rest energy density and the cloud of particle's four-velocity, respectively. Also, we have

$$u^\mu u_\mu = 1 \ . \tag{2.6}$$

From the fact that the strings' world sheets are two-dimensional timelike surfaces and (2.2) we get[12]

$$\Sigma^{\mu\nu} = v^\mu \bar{x}^\nu - v^\nu \bar{x}^\mu \ , \tag{2.7a}$$

$$v^\mu v_\mu > 0, \ \ \bar{x}^\mu \bar{x}_\mu < 0 \ . \tag{2.7b}$$

It is always possible to define a new vector $\bar{v}^\mu$ such that

$$\bar{v}^\mu \bar{v}_\mu > 0 \ , \tag{2.8a}$$

$$\bar{v}^\mu \bar{x}_\mu = 0 \ . \tag{2.8b}$$

The vector $\bar{v}^\mu$ is related to v^μ and $\bar{x}^\mu$ by

$$\bar{v}^\mu = v^\mu - (\bar{x}_\alpha v^\alpha / \bar{x}^\beta \bar{x}_\beta) \bar{x}^\mu \ . \tag{2.9}$$

Note that (2.8a) follows from (2.9), (2.4), and (2.7).

The energy-momentum tensor for a system formed by a cloud of particles and a cloud of strings is

$$T^{\mu\nu} = T_s^{\mu\nu} + T_p^{\mu\nu} \ . \tag{2.10}$$

From (2.1), (2.5), (2.7), (2.9), and (2.8) we get

$$T^{\mu\nu} = \rho_p u^\mu u^\nu + \bar{\lambda}(-\bar{x}_\alpha \bar{x}^\alpha \bar{v}^\mu \bar{v}^\nu - \bar{v}_\alpha \bar{v}^\alpha \bar{x}^\mu \bar{x}^\nu) \ . \tag{2.11}$$

Now we shall specialize (2.11) for the case that the particles are attached to the strings. In this case we have that the particles four-velocity u^μ is parallel to the four-vector that describes the motion of each element of strings, i.e., $\bar{v}^\mu$. In other words, we have particles placed at the same position of each string element moving with the same velocity. For a discussion of this point see Ref. 8. From the condition $u^\mu || \bar{v}^\mu$ and (2.6) we get

$$u^\mu = \bar{v}^\mu / (\bar{v}^\alpha \bar{v}_\alpha)^{1/2} \ . \tag{2.12}$$

Defining

$$x^\mu \equiv \bar{x}^\mu / (-\bar{x}^\alpha \bar{x}_\alpha)^{1/2} \ , \tag{2.13}$$

$$\lambda \equiv -\bar{\lambda} \bar{x}_\alpha \bar{x}^\alpha \bar{v}_\beta \bar{v}^\beta \ , \tag{2.14}$$

$$\rho \equiv \rho_p + \lambda \ , \tag{2.15}$$

we can cast the energy-momentum (2.11) as

$$T^{\mu\nu} = \rho u^\mu u^\nu - \lambda x^\mu x^\nu \ . \tag{2.16}$$

Note that $\lambda \geq 0$ and $\rho \geq 0$; we also have

$$u^\mu u_\mu = -x^\mu x_\mu = 1 \ , \tag{2.17}$$

$$u^\mu x_\mu = 0 \ . \tag{2.18}$$

The density ρ is the rest energy density of the cloud of strings with particles attached to them (p strings). The vector u^μ describes the cloud four-velocity and x^μ represents a direction of anisotropy, i.e., the strings' direction. λ is the cloud strings' tension density.

To end this section we want to point out that the condition (2.3) is equivalent to any of the following relations:

$$\epsilon_{\delta\alpha\beta\gamma} u^\beta x^\gamma (u'^\alpha - \dot{x}^\alpha) = 0 \ , \tag{2.19a}$$

$$\dot{x}^\mu - u'^\mu = u_\alpha \dot{x}^\alpha u^\mu + x_\alpha u'^\alpha x^\mu \ , \tag{2.19b}$$

$$H^\mu_\alpha (u^\beta \partial_\beta x^\alpha - x^\beta \partial_\beta u^\alpha) = 0 \ , \tag{2.19c}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol. The overdot and prime denote directional derivatives in the directions of u^μ and x^μ, respectively, i.e., $(\dot{\ }) = u^\mu \nabla_\mu (\ )$ and $(\ )' = x^\mu \nabla_\mu (\ )$. The tensor H^μ_α is the projection "operator" that projects in the directions that are perpendicular to both x^μ and u^μ, i.e., in the directions that are perpendicular to the string's world sheets,

$$H^\mu_\alpha = \delta^\mu_\alpha - u^\mu u_\alpha + x^\mu x_\alpha \ . \tag{2.20}$$

Some properties of this operator are

$$H^\mu_\nu u^\nu = H^\mu_\nu v^\nu = 0 \ , \tag{2.21a}$$

$$H^\mu_\alpha H^\alpha_\nu = H^\mu_\nu, \ \ H_{\mu\nu} = H_{\nu\mu} \ , \tag{2.21b}$$

$$H^\alpha_\alpha = 2, \ \ \det(H^\mu_\nu) = 0 \ . \tag{2.21c}$$

III. EINSTEIN EQUATIONS COUPLED TO A CLOUD OF STRINGS

The Einstein equations for a cloud of strings are

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -(\rho u_\mu u_\nu - \lambda x_\mu x_\nu) \ . \tag{3.1}$$

The contracted Bianchi identity for (3.1) is equivalent to

$$\nabla_\mu (\rho u^\mu) - \lambda x'^\nu u_\nu = 0 \ , \tag{3.2a}$$

$$\nabla_\mu (\lambda x^\mu) - \rho \dot{u}^\nu x_\nu = 0 \ , \tag{3.2b}$$

$$H^\mu_\nu (\rho \dot{u}^\nu - \lambda x'^\nu) = 0 \ . \tag{3.2c}$$

Equations (3.2) are the evolution equations for the cloud of strings; these equations are also the integrability conditions for Eq. (3.1).

Equation (3.1) and its integrability condition (3.2) are a system of 14 equations for the unknowns $g_{\mu\nu}$, ρ, λ, u^μ, and x^μ, i.e., 17 unknowns [u^μ and x^μ have only five independent components due to Eqs. (2.17) and (2.18)]. Two additional equations are obtained from the fact that u^μ and x^μ, at each point, fit together to form a world sheet, this fact is described by any of Eqs. (2.19). So, we still need one more equation to have a well-defined mathematical problem. An example of an equation that we can choose to close the system is a state equation for the string cloud. Examples of state equations for strings are the following: (a) the state equation for a cloud of geometric (Nambu) strings[1] is

$$\rho = \lambda \ \ (\rho_p = 0) \ . \tag{3.3}$$

(b) The equation of state for a cloud of Takabayasi strings is

$$\rho = (1+\omega)\lambda \ \ (\rho_p = \omega\lambda) \ , \tag{3.4}$$

where ω is a constant, such that $\omega > 0$. This equation is a direct consequence of Takabayasi's model of "realistic" strings.[7]

(c) A more general "barotropic" equation is

$$\rho = \rho(\lambda) \ \ (\rho_p = \rho - \lambda) \ . \tag{3.5}$$

The equations of state are restricted by the energy conditions.[13] We find that the weak energy condition as well as the strong energy condition give us $\rho \geq \lambda$ with $\lambda \geq 0$ or $\rho \geq 0$ with $\lambda < 0$. The dominant energy condition implies $\rho \geq 0$ and $\rho^2 \geq \lambda^2$. Thus, these energy conditions do not restrict the sign of λ. For $\lambda<0$, Eq. (3.1) is the Einstein equation for an anisotropic fluid with pressure different from zero only along the direction x^{μ}.

IV. BIANCHI TYPE-I COSMOLOGICAL MODELS

The metric for Bianchi type-I cosmological models is[14]

$$ds^2=dt^2-\alpha^2 d\xi^2-\beta^2 d\eta^2-\gamma^2 d\zeta^2 , \quad (4.1)$$

where α, β, and γ are functions of t only. For the metric (4.1) we have

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=0, \quad \mu\neq\nu . \quad (4.2)$$

From Eqs. (4.2), (3.1), (2.17), (2.18), and (4.1) we conclude that

$$u^{\mu}=u_{\mu}=(1,0,0,0) \quad (4.3)$$

and that x^{μ} must be taken along either of the directions $\partial/\partial\xi$, $\partial/\partial\eta$, or $\partial/\partial\zeta$. Thus, without losing generality we choose $x^{\mu}||\partial/\partial\xi$, i.e.,

$$x^{\mu}=(0,\alpha^{-1},0,0) . \quad (4.4)$$

Equations (4.1), (4.3), and (4.4) tell us that the Bianchi identities (3.2) reduce to the single equation

$$\dot{\rho}+\frac{\dot{\alpha}}{\alpha}(\rho-\lambda)+\left[\frac{\dot{\beta}}{\beta}+\frac{\dot{\gamma}}{\gamma}\right]\rho=0 , \quad (4.5)$$

where the overdots indicate differentiation with respect to t. Note that as a consequence of the Einstein equation ρ and λ are functions of t only. The condition (2.19) is satisfied identically.

The Einstein equations (3.1) with $\mu=\nu$ are equivalent to

$$\frac{\dot{\alpha}\dot{\beta}}{\alpha\beta}+\frac{\dot{\beta}\dot{\gamma}}{\beta\gamma}+\frac{\dot{\gamma}\dot{\alpha}}{\gamma\alpha}=\rho , \quad (4.6)$$

$$\frac{\ddot{\beta}}{\beta}+\frac{\ddot{\gamma}}{\gamma}+\frac{\dot{\beta}}{\beta}\frac{\dot{\gamma}}{\gamma}=\lambda , \quad (4.7)$$

$$\alpha\ddot{\gamma}+\dot{\alpha}\dot{\gamma}+\ddot{\alpha}\gamma=0 , \quad (4.8)$$

$$\alpha\ddot{\beta}+\dot{\alpha}\dot{\beta}+\ddot{\alpha}\beta=0 . \quad (4.9)$$

Equation (4.5) is a consequence of (4.6)—(4.9). Thus, we have four equations for the five unknowns α, β, γ, λ, and ρ. We have two simple ways to close the system of equations (4.6)—(4.9): first, to assume an equation of state that relates ρ with λ in any of the forms discussed in the preceding section; and second, to give an explicit functional form one of the functions, ρ, λ, α, β, or γ and then to solve the system (4.6)—(4.9) for the rest of the unknowns. We shall follow both approaches in this section.

Let us assume that ρ and λ are related by Takabayasi's equation of state, $\rho=(1+\omega)\lambda$. The system of Eqs. (4.6)—(4.9) can be better studied by making the change of functions

$$x=\frac{\dot{\alpha}}{\alpha}, \quad y=\frac{\dot{\beta}}{\beta}, \quad z=\frac{\dot{\gamma}}{\gamma} . \quad (4.10)$$

After some simple algebra one finds that the system of equations (4.6)—(4.9) is equivalent to

$$\rho=xy+yz+zx\geq 0 , \quad (4.11)$$

$$\dot{x}=-x^2+\frac{\omega}{2(1+\omega)}yz-\frac{2+\omega}{2(1+\omega)}x(y+z) , \quad (4.12)$$

$$\dot{y}=-y^2+\frac{2+\omega}{2(1+\omega)}zx-\frac{\omega}{2(1+\omega)}y(z+x) , \quad (4.13)$$

$$\dot{z}=-z^2+\frac{2+\omega}{2(1+\omega)}xy-\frac{\omega}{2(1+\omega)}z(x+y) . \quad (4.14)$$

Thus, we have that (4.6)—(4.9) for Takabayasi's equation of state reduces to an autonomous system of differential equations of the quadratic type.[15] Note that (4.12)—(4.14) has only the critical point $(x,y,z)=(0,0,0)$. This system of equations can be easily studied numerically. The limit cases $\omega\to 0$ and $\omega\to\infty$ are interesting. In the first case we have geometric strings and in the second [(4.12)—(4.14)] reduces to the case of a cloud of pure particles that can be explicitly integrated.[16] It is also interesting to write Eq. (4.5) using the new functions (4.10), we find

$$\frac{\dot{\rho}}{\rho}=-\left[\frac{\omega}{1+\omega}x+y+z\right] . \quad (4.15)$$

From equations (4.13), (4.14), and (4.10) we get

$$(y-z)R^3=C_0 , \quad (4.16)$$

where C_0 is an arbitrary constant and

$$R^3\equiv\alpha\beta\gamma . \quad (4.17)$$

R can be interpreted as the universe mean radius.[16] Thus (4.16) tells us that when the universe expands the anisotropy decreases in the directions that are perpendicular to the strings.

Equation (4.15) can be integrated and we get

$$\rho\alpha^{-\lambda/\rho}R^3=C_1 , \quad (4.18)$$

where C_1 is another arbitrary constant. Thus, when the radius of the universe increases $\rho\alpha^{-\lambda/\rho}$ decreases.

From Eqs. (4.12) and (4.13), and Eqs. (4.12) and (4.14) we obtain, respectively,

$$[(y-x)R^3]^{\cdot}=\lambda R^3 , \quad (4.19a)$$

$$[(z-x)R^3]^{\cdot}=\lambda R^3 . \quad (4.19b)$$

Equations (4.19) tell us how the anisotropy on the planes (ξ,η) and (ξ,ζ) evolves.

Particular exact solutions to the Einstein equations (4.6)—(4.9) can be found using the second method described above, i.e., first α is given as an explicit function of t, then β and γ are computed from (4.8) and (4.9). The densities ρ and λ are determined by (4.6) and (4.7). Note that in this case the integration of the Einstein equations reduces to the integration of a single second-order linear equation, Eq. (4.8) or Eq. (4.9). The fact that ρ and λ are computed from (4.6) and (4.7) does not guarantee that the energy conditions studied in Sec. III will be satisfied, they must be checked case by case.

Let us assume

$$\alpha=at+b, \quad a\neq 0 , \quad (4.20)$$

where a and b are arbitrary constants. The general solutions to (4.8) and (4.9) with an α given by (4.20) are

$$\beta=\frac{c_1}{a}\ln(at+b)+d_1\ , \quad (4.21a)$$

$$\gamma=\frac{c_2}{a}\ln(at+b)+d_2\ , \quad (4.21b)$$

where c_1, c_2, d_1, and d_2 are arbitrary constants. The metric (4.1) as well as Eqs. (4.6)—(4.9) are invariant under a change of the origin of time. Equations (4.6)—(4.9) are also invariant under the multiplication of the metric coefficients α, β, and γ by constants. Note that, in the metric (4.1), these constants can be absorbed by redefining the variables ξ, η, and ζ. Thus without losing generality we can redefine the constants a, b, c_1, c_2, d_1, and d_2 to end up with

$$\alpha=t\ , \quad (4.22)$$

$$\beta=\ln(t/t_1)\ , \quad (4.23a)$$

$$\gamma=\ln(t/t_2)\ , \quad (4.23b)$$

where t_1 and t_2 are new arbitrary constants. From (4.6) and (4.7) we get

$$\rho=\frac{1+\ln(t^2/t_1t_2)}{t^2\ln(t/t_1)\ln(t/t_2)}\ , \quad (4.24)$$

$$\lambda=\frac{1-\ln(t^2/t_1t_2)}{t^2\ln(t/t_1)\ln(t/t_2)}\ . \quad (4.25)$$

For this model the expansion coefficient, the scalar shear, and the scalar vorticity reduce to[17]

$$\theta=\frac{1}{t}\left[1+\frac{\ln(t^2/t_1t_2)}{\ln(t/t_1)\ln(t/t_2)}\right]\ , \quad (4.26)$$

$$\sigma^2=\frac{1}{t^2}\{1+[\ln(t/t_1)]^{-2}+[\ln(t/t_2)]^{-2}-\tfrac{1}{3}(t\theta)^2\}\ , \quad (4.27)$$

$$\omega=0\ . \quad (4.28)$$

The determinant associated with the metric (4.1), i.e., $-(\alpha\beta\gamma)^2$, is null for $t=0$, t_1, and t_2. These three instants represent not only metric singularities, but real singularities as the behavior of ρ, λ, θ, and σ indicates. Alas, this solution does not satisfy the energy conditions for all $t\in]0,\infty[$. When $t_0\equiv t_1=t_2$ we have $\beta=\gamma$. In this case the spacetime has plane symmetry and Eqs. (4.24) and (4.25) reduce to

$$\rho=\frac{1+2\ln(t/t_0)}{t^2[\ln(t/t_0)]^2}\ , \quad (4.29)$$

$$\lambda=\frac{1-2\ln(t/t_0)}{t^2[\ln(t/t_0)]^2}\ . \quad (4.30)$$

For a value of the constant t_0 such that $0<t_0<e$, the evolution of the matter during $\sqrt{et_0}>t>t_0$ presents the following properties: (a) All the energy conditions are satisfied during the considered time interval. (b) For $t\simeq t_0$ we have $\rho\simeq\lambda$, i.e., the matter behaves as a cloud of geometric strings. (c) When $t\to\sqrt{et_0}$ the cloud of strings tension density λ goes to zero. Thus we remain only with a cloud of particles, and $\rho\to\rho_p$. When $t>\sqrt{et_0}$, the string phase of the universe disappears because λ becomes negative, i.e., we have an anisotropic fluid of particles. For $t>\sqrt{et_0}$ we also have that all the energy conditions are satisfied, since $\rho>0$ and $\rho>|\lambda|$. The critical instant of time $t_c=\sqrt{et_0}$ may be calculated by knowing the critical temperature T_c given by GUTs.

Let us assume $a=0$ and $b\neq 0$ in (4.20) or better let us take $b=1$ since we do not lose generality in doing so. Solutions to (4.8) and (4.9) are

$$\beta=t;\ \ \gamma=t-t_0\ . \quad (4.31)$$

The dynamical quantities ρ and λ for this case reduce to

$$\rho=\lambda=\frac{1}{t(t-t_0)}\ . \quad (4.32)$$

This particular model presents two essential singularities at $t=0$ and $t=t_0$. Equation (4.32) tells us that the cloud is formed by geometric strings only.

V. KANTOWSKI-SACHS-TYPE COSMOLOGICAL MODELS

The metric for Kantowski-Sachs-type cosmological models is[18]

$$ds^2=dt^2-\Lambda^2dr^2-K^2(d\theta^2+\sin^2\theta\, d\phi^2)\ , \quad (5.1)$$

where Λ and K are functions of t only. As in the preceding case we find that the Einstein equations (3.1) for the metric (5.1) impose on u^μ and x^μ the restrictions

$$u^\mu=u_\mu=(1,0,0,0)\ , \quad (5.2)$$

$$x^\mu=(0,\Lambda^{-1},0,0)\ . \quad (5.3)$$

Equations (5.1)—(5.3) tell us that the Bianchi identities (3.2) reduce to the single equation

$$\dot\rho+\frac{\dot\Lambda}{\Lambda}(\rho-\lambda)+2\frac{\dot K}{K}\rho=0\ . \quad (5.4)$$

The condition (2.19) is satisfied identically and Eq. (3.1) can be written as

$$2\frac{\dot\Lambda}{\Lambda}\frac{\dot K}{K}+\frac{1+\dot K^2}{K^2}=\rho\ , \quad (5.5)$$

$$2\frac{\ddot K}{K}+\frac{1+\dot K^2}{K^2}=\lambda\ , \quad (5.6)$$

$$K\ddot\Lambda+\dot K\dot\Lambda+\ddot K\Lambda=0\ . \quad (5.7)$$

Equation (5.4) is a consequence of Eqs. (5.5) and (5.6). Thus, we have three equations for the four unknowns K, Λ, ρ, and λ. Hence, as in the preceding case, to close the system of differential equations (5.5)—(5.7) we need either a new equation, e.g., an equation of state, or to know a function, e.g., K. In this section we shall follow both approaches to solve the above-mentioned system of equations.

Defining new variables, the system of equations (5.5)—(5.7) together with Takabayasi's equation of state can be reduced to a plane autonomous system of differential equations. These new variables are defined by[19,20]

$$-\Omega+\beta=\ln\Lambda\ , \quad (5.8)$$

$$\Omega+\beta/2=-\ln K\ . \quad (5.9)$$

The variable Ω is related to the universe mean radius $R^3=K^2\Lambda$, by

$$R^3=e^{-3\Omega} . \tag{5.10}$$

And the variable β is related to the scalar shear σ, by

$$\sigma=(\tfrac{2}{3})^{1/2}\left[\frac{\dot K}{K}-\frac{\dot\Lambda}{\Lambda}\right] \tag{5.11}$$

$$=-(\tfrac{3}{2})^{1/2}\dot\beta . \tag{5.12}$$

The variable x is defined by

$$x\equiv 3\rho/\theta^2=\rho/3\dot\Omega . \tag{5.13}$$

This variable is the dynamical importance of the string cloud.[19] The dynamical importance of the shear is[19]

$$\frac{\sigma}{\theta}=\frac{1}{\sqrt{6}}\beta' , \tag{5.14}$$

where the prime denotes differentiation with respect to the variable Ω that will be used as a new time variable.

From Eqs. (3.4) and (5.4)–(5.14), after some algebra, we find[21]

$$\beta''=\tfrac{1}{2}\beta'\left[4+\beta'-\frac{\omega}{1+\omega}x\right]+\frac{2x}{1+\omega}-2 , \tag{5.15}$$

$$x'=x\left[2+\beta'-(\beta')^2+\frac{\omega}{1+\omega}(1-\beta'-x)\right] . \tag{5.16}$$

Thus, the system of Eqs. (3.4) and (5.4)–(5.7) is equivalent to the plane autonomous system of differential equations[22] (5.15) and (5.16) in the variables (x,β'). The system (5.15) and (5.16), in the general case $\omega\neq 0$, has three nodal critical points: $P_1(0,-2)$, $P_2(0,1)$, and $P_3(0,2)$. But, for $\omega=0$ the last point is a degenerate critical point. The existence of these critical points can be used to visualize the solutions to (5.15) and (5.16) as integral curves in the Poincaré phase plane[20,22] (x,β'). Numerical integration of (5.15) and (5.16) can be easily performed.

For the particular case of a cloud of geometric strings, i.e., $\omega=0$, the integral curves of (5.15) and (5.16) can be explicitly found, they are

$$x=x_0(1-\beta')^{-4/3}(2+\beta')^{-2/3} . \tag{5.17}$$

Exact particular solutions to the Einstein equations (5.5)–(5.7) can be found by using a method similar to the one described in Sec. IV, i.e., first one gives a function K in an explicit way, this function must be such that Eq. (5.7) allows an explicit integration for Λ. Then, from (5.5) and (5.6) one can compute λ and ρ. Equation (5.7) is equivalent to the corresponding equation for Bianchi type-I cosmological models, but the densities ρ and λ are given by different expressions, so the behavior of these quantities will be different.

Let us assume

$$K=t/a , \tag{5.18}$$

where a denotes an arbitrary constant. Also, the constants b, c, and t_0 will be used in this section. From (5.18) and (3.7) we get

$$\Lambda=b+\ln t , \tag{5.19}$$

and from (5.5) and (5.6) we obtain

$$\rho=t^{-2}\left[1+a^2+\frac{2}{b+\ln t}\right] , \tag{5.20}$$

$$\lambda=(1+a^2)t^{-2} . \tag{5.21}$$

The scalar kinematical quantities for this model reduce to

$$\theta=\frac{1+2b+2\ln t}{t(b+\ln t)} , \tag{5.22}$$

$$\sigma=\left[\frac{2}{3}\right]^{1/2}\frac{b-1+\ln t}{t(b+\ln t)} , \tag{5.23}$$

$$\omega=0 . \tag{5.24}$$

Now we have a model with two essential singularities at $t=0$ and $t=e^{-b}$. The density of particles for this model is

$$\rho_p=2t^{-2}(b+\ln t)^{-1} . \tag{5.25}$$

Near the singularity $t=e^{-b}$ the particles will "dominate" the strings $(\rho_p\gg\lambda)$ and near $t=0$ we have $\rho\simeq\lambda$, i.e., geometric strings. Hence, we have a model of geometric strings that evolves to a particle-dominated era.

Let us assume

$$K=a . \tag{5.26}$$

From (5.7) we find

$$\Lambda=t . \tag{5.27}$$

Note that the Eqs. (5.5) and (5.6) are invariant under the transformations $t\to t-t_0$ and $\Lambda\to c\Lambda$. Thus, there is no loss of generality in choosing the particular solution (5.27). From (5.5) and (5.6) we get

$$\rho=\lambda=\frac{1}{a^2} . \tag{5.28}$$

And, from (5.26) and (5.27) we find

$$\theta=(\tfrac{3}{2})^{1/2}\sigma=\frac{1}{t} , \tag{5.29}$$

$$\omega=0 . \tag{5.30}$$

This particular solution represents a cloud of geometric strings with constant density and a singularity at $t=0$.

VI. DISCUSSION

We have studied cosmological models generated by a cloud of strings with particles attached to them. For some particular models, we have that a certain "critical" instant of time, the density of tension that characterizes the strings is zero or it is completely dominated by the density of particles. Thus, we have models of a universe that evolve from a pure geometric string-dominated era or a massive string-dominated era to a particle-dominated era, with or without a remnant of strings.

In the early universe (string-dominated era) the strings might produce fluctuations in the density of particles. One may speculate that as the strings vanish and the particles become important, the fluctuations will grow in such a way that finally we shall end up with galaxies. As the strings disappear the spacetime anisotropy introduced by them will also disappear.

[1]P. S. Letelier, Phys. Rev. D 20, 1294 (1979).

[2]See also, P. S. Letelier, Nuovo Cimento 63B, 519 (1981); J. Stachel, Phys. Rev. D 21, 2171 (1980), this paper contains earlier references in this area.

[3]Ya. B. Zeldovich, Mon. Not. R. Astron. Soc. 192, 663 (1980); A. Vilenkin, Phys. Rev. Lett. 46, 1169 (1981); 46, 1496(E) (1981) and references therein.

[4]T. W. B. Kibble, J. Phys. A 9, 1387 (1976); A. E. Everett, Phys. Rev. D 24, 858 (1981); A. Vilenkin, *ibid.* 24, 2082 (1982).

[5]See also, Ya. B. Zeldovich *et al.*, Zh. Eksp. Teor. Fiz. 67, 3 (1974) [Sov. Phys. JETP 40, 1 (1975)]; J. Ellis, in *Gauge Theories and Experiments at High Energies,* proceedings of the 21st Scottish Universities Summer School, 1980, edited by K. C. Bowler and D. G. Sutherland (SUSSP Publications, Edinburgh, 1981), p. 201.

[6]H. B. Nielson and P. Olesen, Nucl. Phys. B61, 45 (1973).

[7]T. Takabayasi, in *Quantum Mechanics, Determinism, Causality, and Particles,* edited by M. Flato *et al.* (Reidel, Dordrecht-Holland, 1976), p. 179.

[8]P. S. Letelier, J. Math. Phys. 19, 1908 (1978).

[9]The notation and conventions used in this paper are essentially the same of Ref. 1. The use of a different notation will be explicitly indicated.

[10]Note that the λ used in this paper is the $\rho/\sqrt{-\gamma}$ of Ref. 1.

[11]See, for instance, J. A. Schouten, *Ricci-Calculus* (Springer, Berlin, 1954), pp. 23 and 81.

[12]We shall not consider the strings' boundaries. At the geometric strings' boundaries we can have $\bar{v}^{\mu}\bar{v}_{\mu}=0$.

[13]See, for instance, S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge, England, 1974), p. 88 and references therein.

[14]See, for instance, M. P. Ryan Jr., and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton, New Jersey, 1975), p. 201.

[15]See, for instance, R. Reissig *et al.*, *Non-linear Differential Equations of Higher Order* (Nordhoff, Leyden, 1974), p. 236, and references therein.

[16]K. C. Jacobs, Astrophys. J. 155, 379 (1969).

[17]The definitions of the kinematical quantities used in this paper are the commonly used ones in cosmology, see, for instance, Ref. 14, p. 51. For a different approach, see J. Stachel, in *Relativity and Gravitation,* proceedings of the Third Latin-American Symposium, edited by S. Hojman *et al.* (Universidad Nacional Autónoma de México, 1982), p. 241.

[18]R. Kantowski and R. K. Sachs, J. Math Phys. 7, 443 (1976); see also Ref. 14, pp. 168–171.

[19]C. B. Collins, J. Math. Phys. 18, 2116 (1977).

[20]See also, C. B. Collins and G. F. R. Ellis, Phys. Rep. 56, 67 (1979).

[21]The method used to derive Eqs. (5.15) and (5.16) is presented in Ref. 19.

[22]A simple introduction to plane autonomous systems is presented in G. Birkhoff and G. C. Rota, *Ordinary Differential Equations* (Ginn, Boston, 1962), Chap. VI.