PHYSICAL REVIEW D VOLUME 28, NUMBER 10 15 NOVEMBER 1983

Quantum cosmological singularities

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(Received 21 March 1983)

The problem of gravitationally induced spacetime collapse is studied in the framework of quantum cosmology. We find that whether quantum collapse occurs is effectively predetermined, on the *classical* level, by the choice of time. The crucial distinction is between "fast" and "slow" times, that is, between times which give rise to complete or incomplete classical evolution, respectively. We conjecture that *unitary slow-time quantum dynamics is always nonsingular,* while *unitary fast-time quantum dynamics inevitably leads to collapse.* These contentions are supported by an analysis of the dust-filled Friedmann-Robertson-Walker universes in two choices of time: a cosmic time defined by the velocity potential for the dust and an intrinsic time linked to the expansion. Indeed, these quantum models avoid the classical singularity in the slow matter-time gauge but collapse in the fast geometric-time gauge. We also investigate the qualitatively different forms—unitary and contractive—that the slow-time quantum evolution may take and explore their implications regarding quantum singularity avoidance. One surprising result is that, contrary to widespread belief, this phenomenon does *not* depend upon the choice of boundary conditions.

I. INTRODUCTION

Cosmological singularities are among the most exotic and puzzling phenomena of contemporary physics, since they constitute a stage in the evolution of the Universe in which conditions are so extreme that the presently known laws of physics must break down. Moreover, according to the famous theorems of Hawking and Penrose, such singularities are bound to occur in all spacetimes which satisfy physically plausible restrictions on their causal structure and matter content.

The inevitability of spacetime collapse poses a fundamental dilemma which has yet to be resolved. One view, that a loss of predictive power is unacceptable, has prompted an extensive search for physical mechanisms capable of eliminating the offending singularities.[1] Quantum effects in particular have repeatedly been conjectured to provide an escape from the classical collapse predicament. On the semiclassical level, for example, attention has centered on phenomena such as negative vacuum stresses, particle production, and the presence of massive scalar fields, all of which induce violations of the various positive-energy conditions which appear in the singularity theorems. While initially promising, such attempts at preventing the formation of singularities have ultimately proven ineffectual.

The alternate view, advocated by Misner,[2] is that singularities form an essential element of classical cosmology, a necessary "absolute zero of time" for the Universe. Wheeler[3] has extended this to the quantal domain with his "rule of unanimity": "Given that all solutions of the equations of motion run into a singularity (or are free of singularity) except a set of measure zero. Then all solutions of the corresponding quantum-mechanical problem are singular (or free of singularity)."

A satisfactory solution to the collapse problem must likely await the development of a complete quantum theory of gravity. Meanwhile, research has focused upon the homogeneous cosmologies, since one is then able to include the quantum effects of both matter *and* gravity in the analysis. Although this is achieved at the expense of "freezing out" all but a finite number of degrees of freedom, quantum cosmology has enjoyed considerable success (for reviews, see Refs. 4 and 5).

Even within the framework of these simplified models, the issue of quantum gravitational collapse remains unsettled. There are three reasons for this impasse: (1) ambiguities inherent in the canonical quantization procedure; (2) the lack of precise general criteria for determining whether the quantized models actually collapse; and (3) different choices of time lead to inequivalent results.

The first problem is particularly vexing, since usually the minisuperspaces for the homogeneous cosmologies are curved and the corresponding Hamiltonians are quite complicated.[4–6] These features give rise to ambiguities in defining the quantum evolution as well as the quantum state space and its inner product. Devastating factor-ordering problems also appear. Fortunately, many of these difficulties can be circumvented, or at least controlled, by means of the geometric Kostant-Souriau quantization procedure.[7] This technique—essentially a rigorous global generalization of the canonical quantization method—has already proven useful in quantum cosmology.[8]

Difficulty (1) is essentially a technical matter whereas (2) is more serious conceptually. The usual criterion for the nonsingularity of a quantum model is that the wave function vanishes at the classical singularity.[5,9–11] While intuitively it would seem that this boundary condition ". . . makes the probability amplitude for catastrophic three-geometries vanish, and hence gets the physicist out of his classical collapse predicament,"[9] its fundamental significance has always been clouded in controversy.[5,12] As it turns out (cf. Sec. III C), this boundary condition has little, if anything, to do with quantum singularity avoidance. Instead, Blyth and Isham[12] suggest studying the transition rate from an initial state to a singular configuration. In practice, however, this does not enable one to clearly differentiate between a collapsed and a nonsingular situation.

It seems better yet to examine the expectation values of observables which classically vanish at the singularity, as proposed by Lund[13] and Gotay and Isenberg.[8] We therefore adopt the definition that a quantum state ψ is singular if and only if $\langle \psi, [Qf]\psi \rangle = 0$ for any quantum observable Qf whose classical counterpart f vanishes at the singularity. This test for quantum collapse is at least as convincing as any other and has the advantage that it is entirely straightforward to check.

The last obstacle (3) is the most fundamental—and formidable—of all. Not only do different choices of time typically result in unitarily inequivalent quantizations, they also yield divergent answers to the quantum collapse question. For example, Misner,[14] Ryan,[4] and Lapchinskii and Rubakov,[15] using time variables corresponding to the isotropic part of the expansion of their models, find no avoidance of the classical singularity whatsoever. The same is true for Gotay and Isenberg[8] who employ various clocks tied to the extrinsic geometry and the matter content of their models. On the other hand, DeWitt,[9] Liang,[10] Lund,[13] Lapchinskii and Rubakov,[15] and Demaret[16] conclude that quantum fluctuations prevent the occurrence of a singularity in systems quantized in certain other geometric- and matter-time gauges.

In view of these difficulties, it is hardly surprising that no consensus regarding quantum singularities has emerged. In this paper we carefully analyze the quantum collapse problem in some of the very simplest cosmological models: dust-filled Friedmann-Robertson-Walker universes. These models are sufficiently straightforward that (1) above is not a serious problem, enabling us to concentrate upon the more interesting issues (2) and (3). We quantize these models in two choices of time: a cosmic time defined by the velocity potential for the dust (Sec. III) and an intrinsic time related to the expansion of these universes (Sec. V). We find that in the matter-time gauge the quantum models unquestionably avoid the singularity, while the models definitely collapse when quantized in the geometric-time gauge.

The controlling factor here is the choice of time—this very choice effectively determines, on the *classical* level, the qualitative behavior of the quantum models. The essential difference between these two time gauges which leads to such incompatible quantum behaviors is that the matter clock is "slow" in the sense that the corresponding classical dynamics is incomplete, whereas the intrinsic clock is "fast," that is, it gives rise to complete classical dynamics.[17] More generally, we conjecture that quantum dynamics in a slow-time gauge is always nonsingular, while that in a fast-time gauge inevitably leads to quantum collapse.

We also briefly investigate the qualitatively different forms that the quantum dynamics may take (Sec. III D). Certain possibilities along these lines, e.g., Klein-Gordon versus Schrödinger evolution, are well known and have been extensively discussed in the literature.[4,6] Here, we concentrate on a somewhat unfamiliar alternative: that of "unitary" versus "contractive" quantum dynamics. This choice also has a significant influence upon quantum gravitational collapse which, however, is secondary to that due to the choice of time.

II. THE CLASSICAL DUST-FILLED FRIEDMANN-ROBERTSON-WALKER MODELS

The cosmological models we will analyze are the dust-filled Friedmann-Robertson-Walker (FRW) universes. These classically collapsing spacetimes are ideal for our purposes, since they are well understood classically and have been extensively studied quantum mechanically.[4,5,9,13,15,16,18] Moreover, these models are roughly compatible with the current observations and symmetry assumptions of cosmology.

In this section we set up the classical Hamiltonian apparatus for these models and perform several Arnowitt-Deser-Misner (ADM) reductions. We base our treatment on Schutz's theory of a relativistic perfect fluid[19,20] which, in turn, generalizes Seliger and Whitham's velocity-potential version of perfect-fluid hydrodynamics. Schutz's formulation has the attractive feature that it imparts dynamic degrees of freedom to the fluid. This is particularly important in the case of the FRW models, since otherwise there is no dynamics to discuss.[21]

A. ADM Hamiltonian formalism (Ref. 22)

The homogeneous and isotropic FRW universes are described by the metrics

$$g = -N^2(t)dt^2 + e^{2\mu(t)}d\Sigma^2 ,$$

where $d\Sigma^2$ is the line element for a three-manifold of constant curvature $k = +1$, 0, or -1. These cases correspond to spherical, flat, and hyperbolic spacelike sections, respectively.

The matter content will (initially) consist of a perfect fluid with pressure p, density of total mass-energy ρ, and specific enthalpy η. Schutz takes as his basic variables three scalar fields, φ, θ, and s, where only the specific entropy s has any direct physical significance; in terms of these potentials, the fluid's four-velocity u is written

$$u = -\frac{1}{\eta}(d\varphi + \theta\, ds) . \tag{2.1}$$

All thermodynamic quantities may be related to η and s through the equation of state $p = p(\eta, s)$, while η itself is given as a function of the three velocity potentials via the normalization condition $g(u,u) = -1$.

The Lagrangian for the gravitational field and the perfect fluid is

$$\mathscr{L} = (R + p)(-g)^{1/2} ,$$

where R is the scalar curvature of the spacetime metric g. A $(3+1)$ split followed by the Legendre transform yields the canonical momenta

$$p_\mu = -\frac{12}{N} e^{3\mu}\dot{\mu}$$

and

$$p_\varphi = \frac{1}{\eta}(\rho+p)u^0 N e^{3\mu} , \tag{2.2}$$

along with two initial-value constraints

$$p_\theta = 0 , \quad p_s = \theta p_\varphi$$

on the fluid variables. These constraints may be used to eliminate the canonical pair (θ, p_θ) from the formalism; once this is accomplished, the Lagrangian may be written in ADM form:

$$\mathscr{L} = p_\mu \dot{\mu} + p_\varphi \dot{\varphi} + p_s \dot{s} - N(\mathscr{H} + \mathscr{E}) , \tag{2.3}$$

where

$$\mathscr{H} = -(\tfrac{1}{24} e^{-3\mu} p_\mu{}^2 + 6k e^{\mu}) \tag{2.4}$$

is the vacuum gravity super-Hamiltonian, and

$$\mathscr{E} = N^2 e^{3\mu}[(\rho+p)(u^0)^2 + p g^{00}] \tag{2.5}$$

is the coordinate energy density measured by a comoving observer. Finally, varying the lapse N in (2.3) gives $\mathscr{H} + \mathscr{E} = 0$. Taking this "super-Hamiltonian constraint" into account, these systems have (in general) two dynamic degrees of freedom.

We now specialize to the case of dust.[23] Then $p = 0$, $\eta = 1$, and φ is the only velocity potential different from zero. Substituting p_φ given by (2.2) into (2.5), employing the normalization condition on the four-velocity (2.1) and combining with (2.4), the super-Hamiltonian constraint reduces to

$$p_\varphi - \tfrac{1}{24} e^{-3\mu} p_\mu{}^2 - 6k e^{\mu} = 0 . \tag{2.6}$$

The dust-filled FRW models are left with a single degree of freedom.

B. Choices of time, reductions, and classical dynamics

The vanishing of the Hamiltonian $N(\mathscr{H} + \mathscr{E})$ indicates that our models are in parametrized form and consequently admit a reduction via a "choice of time" followed by an elimination of the super-Hamiltonian constraint (2.6).[4–6] Usually many such reductions are possible, depending upon the specific model under consideration and one's objectives; Blyth and Isham[12] provide a nice illustration of this.

For classically collapsing universes it is essential to differentiate between various choices of time. We define a time variable t to be *dynamically admissible* if it is *a priori* bounded neither above nor below. The motive for this distinction is quantum mechanical: Only reductions corresponding to dynamically admissible time gauges are compatible with the requirement that the quantum dynamics be generated by a self-adjoint Hamiltonian. Indeed, suppose that upon quantization the reduced Hamiltonian determined by a choice of time t is represented by a self-adjoint operator. This operator generates the quantum evolution, which is then parametrized by the classical time t. By self-adjointness, however, the quantum evolution is defined for *all* times t, $-\infty < t < \infty$, and this is inconsistent with the very definition of t unless t is dynamically admissible. In summary, certain choices of time—although perfectly acceptable classically—may not be suitable quantum mechanically, and for this reason we restrict consideration to dynamically admissible times henceforth.

It is also convenient to group all dynamically admissible choices of time into two categories. We call a time variable t a *fast time* if the singularities always occur at either $t = -\infty$ or $t = \infty$. If this is not the case, then t is said to be a *slow time.* Note that fast-time dynamics is necessarily *complete,* whereas slow-time dynamics is always *incomplete.* The dynamics of a system in a fast-time gauge may thus be viewed as a "regularization" of the dynamics in a slow-time gauge. The distinction between fast and slow times will be particularly important on the quantum-mechanical level.

Returning to the FRW models, one useful clock is provided by the dust itself: we may choose $t = -\varphi$. By (2.3) and (2.6), this fixes $N = -1$ so that this time is effectively cosmic time for these universes. The reduced phase space is then R^2 with coordinates (μ, p_μ); solving the constraint (2.6) for the effective Hamiltonian $H = p_\varphi$ yields

$$H(\mu, p_\mu) = \tfrac{1}{24} e^{-3\mu} p_\mu{}^2 + 6k e^{\mu} . \tag{2.7}$$

Because of (2.2) and (2.1), H must always be non-negative.

This is greatly simplified if we make the canonical tranformation $\mathsf{R}^2 \to (0,\infty) \times \mathsf{R}$ given by

$$x = \tfrac{4}{3}\sqrt{6} e^{3\mu/2} , \quad p_x = \frac{\sqrt{6}}{12} e^{-3\mu/2} p_\mu .$$

The dynamics of the system then becomes that of a particle on the half-line $(0,\infty)$ with Hamiltonian

$$H(x, p_x) = p_x{}^2 + K x^{2/3} , \tag{2.8}$$

where $K = \frac{3}{2}\sqrt[3]{6}k$. From (2.8) it is easy to ascertain the qualitative features of the dynamics, observing that the line $x = 0$ represents classically singular states. Depending upon initial conditions, the $k = -1, 0$ models have either an initial singularity followed by continual expansion or a prolonged contractive phase culminating in collapse. The $k = +1$ model begins with a big bang, expands to some maximum radius, and then collapses.

The time $t = -\varphi$ is slow. An example of a fast time is $t = \mu$; for this choice reduction yields the phase space $\mathsf{R} \times (0,\infty)$ with coordinates (φ, p_φ). Upon taking the negative square root, the effective Hamiltonian $H = -p_\mu$ becomes

$$H(\varphi, p_\varphi) = 2\sqrt{6} e^{3t/2}(p_\varphi - 6k e^{t})^{1/2} \tag{2.9}$$

and is unavoidably time dependent. The $k = -1, 0$ models have initial singularities at $t = -\infty$ and expand thereafter. The $k = +1$ model also has a big bang at $t = -\infty$, but one must redefine t at the point of maximum expansion to keep the time running monotonically.

These two reductions—one corresponding to the matter-time gauge $t = -\varphi$ and the other to the intrinsic-time gauge $t = \mu$—have the same physical content, at least classically. In the following sections we will quantize the FRW models in both choices of time; surprisingly, the results will differ drastically. In particular, we will see that the quantization in the $t = -\varphi$ gauge *cannot* be reconciled with the unanimity principle, while that in the $t = \mu$ gauge *can.* Ultimately, this anomaly will be traced to the fact that $t=-\varphi$ is a slow time whereas $t=\mu$ is fast. This classically innocuous distinction will actually turn out to be the single most important factor governing the occurrence of quantum gravitational collapse.

III. QUANTIZATION IN THE MATTER-TIME GAUGE

We begin our analysis by quantizing the FRW models in the matter-time gauge $t=-\varphi$. We work on the classical phase space $(0,\infty)\times\mathrm{R}$ with coordinates (x,p_x).[24] Since the configuration space is the half-line $(0,\infty)$, the quantum Hilbert space is $L^2(0,\infty)$ with the inner product

$$\langle\phi,\psi\rangle=\int_0^\infty \bar{\phi}(x)\psi(x)dx \ .$$

The quantum "position" and "momentum" operators are

$$Qx=x \ , \quad Qp_x=-i\hbar\frac{d}{dx} \ .$$

Note that Qx is a positive self-adjoint operator, in contrast to Qp_x which has no self-adjoint extensions on $L^2(0,\infty)$.[25] Finally, the quantized Hamiltonian (2.8) is

$$QH=-\hbar^2\Delta+Kx^{2/3} \ , \tag{3.1}$$

where $\Delta=d^2/dx^2$.

Our task thus amounts to studying the quantum mechanics of a particle on a half-line moving in a potential $V(x)=Kx^{2/3}$. This is not as straightforward as it may appear, however. One problem is due to the form of the quantum Hamiltonian: Because of the exponent $\frac{2}{3}$ in the potential, little can be said about the properties of QH in general. In particular, the Schrödinger equation cannot be analytically solved. Consequently, when explicit illustrations and calculations are required we will set $k=0$. This is not that severe a restriction, since we are only interested in the qualitative behavior of these models near the classical singularity and in this respect they are roughly similar. Regardless, our conclusions will ultimately be independent of the choice of the parameter k.

The second problem is somewhat more subtle, and concerns the fact that the configurations of our universe lie on the half-line. One might offhandedly expect that the quantum mechanics of such a "particle" would be similar to that of a particle on the entire line $(-\infty,\infty)$. This is actually *not* the case; the two are substantially different. A number of the most surprising differences are discussed as they appear in the text; the point is that one must pay careful attention to the functional analysis.[26]

A. Unitary quantum dynamics

Although the quantum Hamiltonian (3.1) is not essentially self-adjoint on $C_0^\infty(0,\infty)$, it does have an infinite number of self-adjoint extensions $QH_\alpha=-\hbar^2\Delta+Kx^{2/3}$ on the domains

$$D_\alpha=\{\psi\in H^2(0,\infty)\,|\,\psi'(0)=\alpha\psi(0)\} \ , \tag{3.2}$$

where $\alpha\in(-\infty,\infty]$. Here, $H^2(0,\infty)$ denotes the Sobolev space consisting of those $\psi\in L^2(0,\infty)$ with $\psi\in C^1(0,\infty)$, ψ' absolutely continuous, $\psi''\in L^2(0,\infty)$ locally, and $QH_\alpha[\psi]\in L^2(0,\infty)$. Since each QH_α is self-adjoint, it generates a one-parameter unitary group $\{T_\alpha(t)\,|\,-\infty<t<\infty\}$ on $L^2(0,\infty)$, where

$$T_\alpha(t)=\exp\left[-\frac{i}{\hbar}tQH_\alpha\right] . \tag{3.3}$$

This defines the quantum evolution

$$\psi(t_0+t)=T_\alpha(t)\psi(t_0) \tag{3.4}$$

for all $\psi\in L^2(0,\infty)$.

At this point we make some observations.

(i) Each self-adjoint extension QH_α generates genuinely different dynamics. But which value of the parameter $\alpha\in(-\infty,\infty]$ should one choose? In other words, which of the boundary conditions (3.2) gives the correct quantum dynamics? DeWitt,[9] in an effort to avoid the singularity at $x=0$, set the boundary condition $\psi(0)=0$ (i.e., $\alpha=\infty$). But, as we shall see shortly, such considerations are largely irrelevant. In any case, there appears to be no *a priori* justification for choosing any one value of α to the exclusion of the others and so we shall treat them democratically. We will return to this point in Sec. III C.

(ii) The time t parametrizing the quantum evolution (3.4) is none other than the *slow* classical time $-\varphi$. But, by self-adjointness, the quantum evolution persists *eternally.* A fundamental "paradox" thus begins to emerge: How does one reconcile *incomplete* classical dynamics with *complete* quantum dynamics? The fact that the quantum evolution is defined for all t, $-\infty<t<\infty$, is a strong indication that the Universe cannot collapse quantum mechanically. We will soon find that this is actually true.

We now turn to the $k=0$ model. In this case, our problem becomes that of determining the quantum dynamics of a *free* particle on the half-line and consequently we are able to say more about the properties of the Hamiltonians $QH_\alpha=-\hbar^2\Delta$.

Each operator QH_α has a continuous spectrum of $[0,\infty)$ corresponding to the eigendistributions

$$\frac{\alpha}{(\alpha^2+\lambda)^{1/2}}\left[\sin\sqrt{\lambda}x+\frac{\sqrt{\lambda}}{\alpha}\cos\sqrt{\lambda}x\right] . \tag{3.5}$$

For $\alpha\in(-\infty,0)$, there is also a point spectrum consisting of the single eigenvalue $\{-\hbar^2\alpha^2\}$ corresponding to the bound state $e^{\alpha x}$. These states, which have no classical counterparts, are evidently the $k=0$ analogs of Misner's closed FRW "quantum puff" universe.[6,12]

Reed and Simon (Sec. X.1 of Ref. 26) have given a nice interpretation of the dynamics generated by each QH_α. Following them, consider a plane wave $e^{-i\kappa x}$ with momentum $\hbar\kappa$ moving towards the left. Ignoring the behavior at infinity, note that this state is not in D_α for any α, since it does not satisfy the boundary condition (3.2). With an eye on (3.5), however, observe that the wave

$$e^{-i\kappa x}+\delta_\alpha(\kappa)e^{i\kappa x} \tag{3.6}$$

is in D_α near zero, where

$$\delta_\alpha(\kappa)=\frac{i\kappa+\alpha}{i\kappa-\alpha} \ . \tag{3.7}$$

Since $|\delta_\alpha(\kappa)|=1$, it follows that QH_α *generates the dynamics in which an incoming plane wave is reflected at the origin with a phase shift determined by* α. In particular, the boundary condition $\alpha=\infty$ corresponds to an infinite potential wall at $x=0$, with phase shift π, while $\alpha=0$ corresponds to a symmetric extension of the half-line $(0,\infty)$ to the entire line $(-\infty,\infty)$.[15]

It is of course difficult to be more explicit about the quantitative features of the dynamics in such a general setting. To get a better feeling for our quantum universes, we examine the motion of a wave packet.

B. Motion of a wave packet

Fix $\beta=b+iB$, with $b>0$, and consider the normalized initial state

$$\psi(x,0)=\left[\frac{8b}{\pi}\right]^{1/4} e^{-\beta x^2}. \tag{3.8}$$

To evolve this wave packet we must decide upon an appropriate boundary condition. Since $\psi'(0)=0$, $\psi\in D_0$ and it is natural to choose $\alpha=0$. Using the spectral theorem in the guise of the Fourier cosine transform [cf. (3.5) with $\alpha=0$], $\psi(x,0)$ may be expanded:

$$\psi(x,0)=\left[\frac{8b}{\pi}\right]^{1/4}(4\pi\beta)^{-1/2}\int_0^\infty e^{-\lambda/4\beta}\lambda^{-1/2}\cos\sqrt{\lambda}x\,d\lambda\,.$$

Then (3.3) and (3.4) give

$$\psi(x,t)=\left[\frac{8b}{\pi}\right]^{1/4}(4\pi\beta)^{-1/2}\times\int_0^\infty \exp[-(1+4i\hbar\beta t)\lambda/4\beta]\lambda^{-1/2}\cos\sqrt{\lambda}x\,d\lambda\,,$$

which integrates to

$$\psi(x,t)=\left[\frac{8b}{\pi}\right]^{1/4}(1+4i\hbar\beta t)^{-1/2}\times\exp[-\beta x^2/(1+4i\hbar\beta t)]\,. \tag{3.9}$$

Note that $\psi(x,t)\neq0$ for all x and t.

To extract the physics from this we compute some expectation values, recalling that the observable $x\propto e^{3\mu/2}$ measures the expansion of our model universe. From (3.9) the expectation value $\langle Qx(t)\rangle=\langle\psi(t),[Qx]\psi(t)\rangle$ of the position is

$$\langle Qx(t)\rangle=(2\pi b)^{-1/2}[1-8B\hbar t+16(b^2+B^2)\hbar^2t^2]^{1/2}\,. \tag{3.10}$$

Similarly, one has

$$\langle Qp_x(t)\rangle=2\hbar(2\pi b)^{-1/2}\times\left[\frac{ib+4(b^2+B^2)\hbar t-B}{[1-8B\hbar t+16(b^2+B^2)\hbar^2t^2]^{1/2}}\right]. \tag{3.11}$$

Since $\langle Qp_x(t)\rangle$ is complex, the mean value of the momentum is $\mathrm{Re}\langle Qp_x(t)\rangle$; this is in accord with the observation that $(d/dt)\langle Qx(t)\rangle=2\,\mathrm{Re}\langle Qp_x(t)\rangle$.

The initial state $\psi(x,0)$ represents a wave packet centered on $\langle Qx(0)\rangle=(2\pi b)^{-1/2}$ with momentum $\mathrm{Re}\langle Qp_x(0)\rangle=-2\hbar(2\pi b)^{-1/2}B$. If $B<0$ the wave packet is moving towards the right —that is, the universe is expanding. Equations (3.10) and (3.11) imply that this model will continue to expand with ever increasing momentum.

For $B>0$ the universe is initially contracting. But $\langle Qx(t)\rangle$ is strictly positive always; in fact, $\langle Qx(t)\rangle$ falls to a minimum value

$$(2\pi b)^{-1/2}[1-B^2/(b^2+B^2)]$$

with t equals the *turn-around time*

$$T=B/4(b^2+B^2)\hbar\,,$$

and *thereafter increases without bound.* This behavior is reflected by (3.11), since $\mathrm{Re}\langle Qp_x(t)\rangle<0$ for $0<t<T$, $\mathrm{Re}\langle Qp_x(T)\rangle=0$, and $\mathrm{Re}\langle Qp_x(t)\rangle>0$ for $t>T$.

Without any doubt this wave packet avoids the singularity. Although surprising, perhaps, this behavior is entirely consistent with the physical interpretation of the dynamics generated by QH_0 given in subsection A above. In these terms, the wave packet with $B>0$ starts off moving towards the singularity at $x=0$. But all the while the leading edge of the packet is being reflected at the origin, and these reflected waves contribute positively to the momentum which therefore increases from its initial negative value. For small times, very little of the wave packet has been reflected, so $\langle Qx(t)\rangle$ decreases. For longer times $t\simeq T$, approximately "half" of the wave packet has been reflected, $\langle Qx(t)\rangle$ attains its minimum and $\mathrm{Re}\langle Qp_x(t)\rangle$ momentarily vanishes. Afterwards, the wave packet has been almost totally reflected and is now traveling towards the right; $\langle Qx(t)\rangle$ and $\mathrm{Re}\langle Qp_x(t)\rangle>0$ are increasing.

This aside, we turn to the classical limit. Corresponding to the wave packet (3.8) with $B>0$ is the classical initial state $x(0)=(2\pi b)^{-1/2}$ and $p_x(0)=-2\hbar(2\pi b)^{-1/2}B$. From (2.8), this contracting universe evolves according to

$$x(t)=(2\pi b)^{-1/2}(1-4B\hbar t)\,, \tag{3.12}$$

$$p_x(t)=-2\hbar(2\pi b)^{-1/2}B \tag{3.13}$$

and thus collapses after a time $T_c=1/4B\hbar$.

Comparing (3.12) with (3.10) for times $t\simeq0$ one finds that $\langle Qx(t)\rangle\gtrsim x(t)$, so that in the "classical epoch" $t\ll T$ the quantum dynamics mirrors the classical dynamics, just as one expects. For later times $t\simeq T<T_c$, however, quantum effects begin to manifest themselves rather forcefully: the quantum model decelerates, "bounces," and then expands while the classical model uniformly contracts to the singularity. This behavior is displayed in Fig. 1. Note that for a very classical initial state (i.e., $b\to0$), the classical/quantum correspondence becomes quite exact for $t<T$. In this case the quantum turn-around time T is very nearly equal to the classical collapse time T_c; the quantum universe bounces at the "last moment."

The disparity between the classical and quantum predictions becomes even more vivid when one compares (3.11) and (3.13). Classically, of course, momentum is conserved. But this is clearly impossible on the quantum level because of the reflection mechanism at the origin which is built into the dynamics. Nonetheless, in the classical regime the correspondence is not all that bad: for $t\simeq0$, (3.11) yields

$$\mathrm{Re}\langle Qp_x(t)\rangle\simeq-2\hbar(2\pi b)^{-1/2}B(1-4b^2\hbar t/B)\,,$$

which is reasonably consistent with (3.13). The correspon-

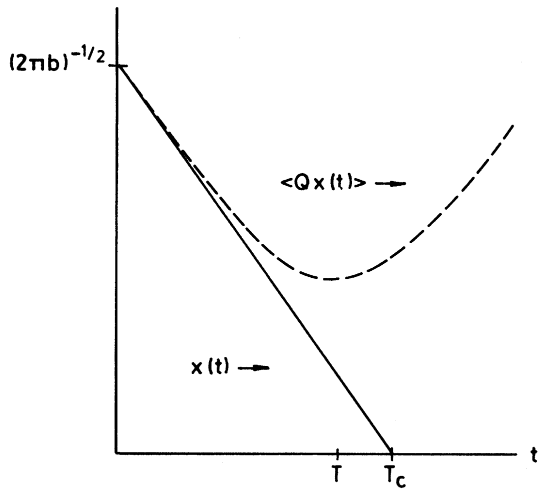

FIG. 1. Classical/quantum correspondence for the wave packet (3.9).

dence is even better if $\psi(x,0)$ is very classical, so that reflection at the origin will not be significant until times $t \simeq T$. Then $\mathrm{Re}\langle Qp_x(t)\rangle \simeq -2\hbar(2\pi b)^{-1/2}B$ is essentially constant for $t < T$.

Some corroboration for this phenomenon of quantum singularity avoidance was provided by DeWitt[9] when, in his WKB analysis of the $k=+1$ dust-filled model, he found that ". . . the packet rebounds repeatedly from the collapsed state . . . ." Our findings are further substantiated by Lapchinskii and Rubakov's analysis[15] of the $k=+1$ radiation-dominated model in a different, but still slow, matter-time gauge. One is thus led to wonder whether this behavior is in any way typical, or rather just an "accident"? It turns out, as will be shown in the next section, that this behavior *is* typical: *Every* evolving state must either "bounce" or at least avoid the singularity in the sense that $\langle Qx(t)\rangle \neq 0$ for all finite t.

C. Quantum singularity avoidance

We now turn to a general discussion of the quantum dynamics of the FRW models, dropping the restriction $k=0$.

Let $\psi(x,t_0)$ be any initial state in $L^2(0,\infty)$ and fix a self-adjoint Hamiltonian QH_α. We presume that the corresponding evolving state $\psi(x,t)$ is such that the expectation value

$$\langle Qx(t)\rangle = \int_0^\infty x\,|\psi(x,t)|^2 dx \tag{3.14}$$

is defined for all $t \in \mathbb{R}$. Since Qx is a *positive* operator and since classically $x=0$ represents the singularity, the expectation value (3.14) is a good indicator of quantum gravitational collapse. That is, we regard the state $\psi(x,t)$, for some finite time t, as being quantum mechanically singular if and only if $\langle Qx(t)\rangle = 0$.

Our first rather unexpected conclusion is that *no nontrivial initial state $\psi(x,t_0)$ can quantum mechanically evolve into a singularity.* Indeed, suppose that $\langle Qx(t)\rangle = 0$ for some $|t| < \infty$. Then (3.14) yields $\psi(x,t)=0$ which, by (3.4) and unitarity, implies that $\psi(x,t_0)=0$. This seemingly trivial result, which was first intimated by Lund,[13] has far-reaching implications. It means that within this dynamical framework *quantum gravitational collapse is strictly forbidden.* We emphasize that the sole underlying dynamical cause of this circumstance is the unitarity of the $T_\alpha(t)$ or, equivalently, the self-adjointness of the quantum Hamiltonian.

Another surprising fact is that *this phenomenon of quantum singularity avoidance is independent of the particular choice of boundary conditions:* the above conclusions hold for any $\alpha \in (-\infty,\infty]$. This observation enables us to clear up a certain amount of confusion in the literature regarding the role of boundary conditions in quantum cosmology.

Traditionally, boundary conditions of the form $\psi(0,t)=0$ for all times t have been interpreted as a criterion for the nonsingularity of a model[9–11] while, conversely, a state with $\psi(0,t)\neq 0$ for some t was taken to represent a collapsed state.[11] These are attractive ideas but their physical meaning has remained obscure, since $|\psi(0,t)|^2$ is merely a probability density—a point emphasized on numerous occasions.[5,8,12,13,15] In view of the above results, these interpretations of the boundary conditions are now seen to be, if not entirely wrong, then at least quite misleading. For, consider once again the evolving state (3.9); certainly this wave packet is nonsingular, even though $\psi(0,t)\neq 0$ for all t.

The upshot is that in this context *the boundary conditions are just not relevant to the issue of quantum gravitational collapse.* Perhaps the role of the boundary conditions themselves should be put in a somewhat different perspective. It seems inappropriate to use them as a means of selecting "physically acceptable" states. Rather, all states in $L^2(0,\infty)$ should be thought of as being physically admissible, not just those belonging to some D_α [which is, after all, only dense in $L^2(0,\infty)$]. From this standpoint, the boundary conditions properly serve *only* to determine the quantum dynamics by fixing a quantum Hamiltonian.

Our next task is to confront the most important consequence of the above analysis, namely, the breakdown of the unanimity principle. The most dramatic manifestation of this failure is that quantum gravitation collapse cannot occur although, ultimately, the failure originates in the completeness of the quantum dynamics.

How is one to cope with this state of affairs? If one views Wheeler's related "rule of unanimity" as being fundamental, then the inescapable conclusion is that either the classical or the quantum models must be "wrong."

If the quantum models accurately describe nature then, presumably, Einstein's equations can no longer be valid near the classical singularity. In this case, one must search for reasonable alternatives and/or modifications to Einstein's theory; some possibilities along these lines are listed in note (1). Even without an entirely new classical theory, one can still search for a sort of regularization of the given classical dynamics which will yield results more in agreement with the unanimity principle. The goal of such a regularization in this instance would be to produce complete classical dynamics.

One such possibility was suggested by DeWitt[27] and applied to the FRW models by Lund,[13] Brill,[18] and Demaret.[16] The idea is very simple: To complete the classical dynamics, it suffices to let the configuration variable x range over all of R rather than just the half-line $(0,\infty)$. One then interprets the classical state $(-|x|,p_x)$ as being physically identical to the state $(|x|,-p_x)$. As Lund explains, "Classically, a particle moving from positive to negative x represents a collapsing (dust) shell reexploding back into the same universe from which it came, and not appearing suddenly into another universe otherwise disconnected from the first." This completion has the attractive feature that it effectively incorporates the quantum reflection behavior at the origin—corresponding to $\alpha=0$—into the classical dynamics.

Unfortunately, this regularization is fatally flawed. The problem is still that the classical singularity is not avoided in any sense: the classical model continuously transits through $x=0$, i.e., it actually collapses and then reexpands. But $x=0$ is a real physical infinite density singularity and here one necessarily loses all predictive power. In effect, this completion gives mathematical life to a model which has physically perished, and so is unacceptable on physical grounds. This type of regularization also suffers from two other defects: it is coordinate-dependent and it has a limited range of applicability.

So this straightforward sort of completion will not work. Nonetheless, as we shall discover in Sec. V, there exists another type of classical regularization which does, although the physical implications of this completion will be much different than one might expect.

Let us consider the other alternative, namely, that the quantum models are somehow incorrect. To understand what this might mean, recall that the impossibility of quantum gravitational collapse is due entirely to the unitarity of the quantum dynamics. If we are to regain the unanimity principle, it is therefore necessary to drop the requirement that the quantum Hamiltonian be self-adjoint.

Renouncing this postulate opens up a whole Pandora's box of possibilities. Even so, non-self-adjoint Hamiltonians are familiar in other contexts, e.g., decaying systems and absorption processes. In particular, if one thinks of the classical dynamics as describing a kind of "absorption (of the Universe by the singularity) process," then a non-self-adjoint quantum Hamiltonian is very natural physically—in fact, more so than a self-adjoint one.

Quantum evolution with a non-self-adjoint Hamiltonian becomes even more attractive in light of the following observation. Consider once again the sample calculation of subsection B above. The wave packet (3.9) actually "bounces" in the sense that $\langle Qx(t)\rangle\to\infty$ as $t\to\infty$. As a *first step* towards obtaining a more reasonable classical limit, we ask whether an initially contracting state can *asymptotically collapse* in the sense that $\langle Qx(t)\rangle\to 0$ as $t\to\infty$. Such an evolving state is still always nonsingular, strictly speaking, but its behavior is not nearly so outrageous as that of a state which bounces. Unfortunately, conservation of probability may rule out this possibility in general. For, suppose that the evolving normalized state $\psi(x,t)\to\Psi(x)$ pointwise as $t\to\infty$ and that both $|\psi(x,t)|^2$ and $x|\psi(x,t)|^2$ are dominated in $L^1(0,\infty)$ for all t. Then by the dominated convergence theorem one has

$$0=\lim_{t\to\infty}\int_0^\infty x\,|\psi(x,t)|^2dx$$
$$=\int_0^\infty x\times\lim_{t\to\infty}|\psi(x,t)|^2dx=\int_0^\infty x\,|\Psi(x)|^2dx\ ,$$

which implies that $\Psi(x)=0$. But then

$$0=||\Psi||^2=\int_0^\infty\lim_{t\to\infty}|\psi(x,t)|^2dx$$
$$=\lim_{t\to\infty}||\psi(t)||^2=1\ ,$$

so that subject to these restrictions an evolving state cannot asymptotically collapse. However, asymptotic collapse *is* possible if we do not require that probability be conserved. For a better classical limit, then, the quantum Hamiltonian should not generate a unitary one-parameter group, but rather a *contraction semigroup.* In the next subsection we study the "decaying" quantum dynamics so obtained.

D. Contractive quantum dynamics

We begin by reviewing the properties of contraction semigroups and then specialize to the FRW models; our analysis here is intended to be illustrative rather than exhaustive. The basic references are Reed and Simon (Sec. X.8) and Dunford and Schwartz (Sec. VII.1).[26]

Let $\mathscr{H}$ be a Hilbert space. A *strongly continuous semigroup* is a family of bounded operators $\{T(t)\,|\,0\le t<\infty\}$ on $\mathscr{H}$ satisfying

(i) $T(0)=I$,

(ii) $T(s)T(t)=T(s+t)$, $s,t\in[0,\infty)$,

(iii) For each $\psi\in\mathscr{H}$, the map $t\to T(t)\psi$ is continuous on $[0,\infty)$.

If $\{T(t)\,|\,0\le t<\infty\}$ is a strongly continuous semigroup then there exist constants $M>0$ and ω such that

$$||T(t)||\le Me^{\omega t} \tag{3.15}$$

for all $t\ge 0$. If $\omega\le 0$ and $M=1$ the semigroup is said to be *contractive.*

Strongly continuous semigroups are thus the natural generalizations of unitary one-parameter groups. If $\psi(t_0)\in\mathscr{H}$ represents a quantum state at time t_0, then it will still evolve according to $\psi(t_0+t)=T(t)\psi(t_0)$ for $t\ge 0$ but now, in view of (3.15), probability is not necessarily conserved. In particular, if the semigroup $\{T(t)\}$ is *strictly* contractive, i.e., $\omega<0$, then the system must "decay" in the sense that

$$\lim_{t\to\infty}||T(t)\psi||=0$$

for all $\psi\in\mathscr{H}$. Finally, note that the dynamics is + complete, that is, defined for all $t\ge 0$.

As with unitary one-parameter groups, all strongly continuous semigroups arise as the "exponentials" $T(t)=e^{-tA}$ of a certain class of operators A on $\mathscr{H}$. Conversely, there is a general theorem—due to Hille, Yosida, and Phillips—which delineates the properties which an operator A must possess if it is to generate a strongly continuous semigroup e^{-tA}. For our purposes, we need only the following weaker result from Reed and Simon (Sec. X.8): A closed operator A generates a contraction semigroup if both A and its adjoint A^* are accretive, in which case one has

$$e^{-tA}\psi = \lim_{n\to\infty}\left[\left[I+\frac{t}{n}A\right]^{-n}\psi\right]. \tag{3.16}$$

Here, A is *accretive* provided $\mathrm{Re}\langle A\psi,\psi\rangle \geq 0$ for all ψ in the domain of A.

For the FRW universes the Hilbert space is $L^2(0,\infty)$ and the relevant operator is iQH defined on $C_0^\infty(0,\infty)$. Its closure, also denoted by iQH, has domain consisting of those $\psi \in H^2(0,\infty)$ with $\psi(0)=0=\psi'(0)$. This operator is accretive, but its adjoint $(iQH)^* = -iQH$ on $H^2(0,\infty)$ is not, so we must impose boundary conditions in order to obtain accretive restrictions of $(iQH)^*$. Since iQH has no boundary values at infinity these boundary conditions must take the form

$$\psi'(0) = \alpha\psi(0), \tag{3.17}$$

where $\alpha \in \mathbb{C}$ or $\alpha = \infty$. We denote by D_α those $\psi \in H^2(0,\infty)$ satisfying (3.17) for a fixed α, and by QH_α the operator QH with domain D_α.

Since

$$\mathrm{Re}\langle [iQH_\alpha]\psi,\psi\rangle = -(\mathrm{Im}\alpha)\,|\psi(0)|^2 \quad \text{for } \psi\in D_\alpha,$$

iQH_α will be accretive provided $\mathrm{Im}\alpha \leq 0$. Furthermore, since

$$\langle[-iQH]\phi,\psi\rangle = i[\bar\phi'(0)-\alpha\bar\phi(0)]\psi(0) + \langle\phi,[iQH_\alpha]\psi\rangle$$

for $\psi \in D_\alpha$, one has that $(iQH_\alpha)^* = -iQH_{\bar\alpha}$ on $D_{\bar\alpha}$ which is therefore also accretive if $\mathrm{Im}\alpha \leq 0$. Reintroducing Planck's constant, it follows that each $(i/\hbar)QH_\alpha$ with $\mathrm{Im}\alpha\leq 0$ generates a contraction semigroup $\exp[-(i/\hbar)tQH_\alpha]$.

These results are very similar to those given in subsection A above, except now α may be *complex*. Note that $D_\alpha \neq D_{\bar\alpha}$ unless $\mathrm{Im}\alpha = 0$, so that the quantum Hamiltonians QH_α are not symmetric in general. If $\mathrm{Im}\alpha=0$ we are back in the self-adjoint case.

We now revert to the $k=0$ model. To elucidate the physics of these contraction semigroups, consider once again a plane wave $e^{-i\kappa x}$ incident from the right. Mimicking the analysis in subsection A, we find that the wave (3.6) is in D_α near zero, with $\delta_\alpha(\kappa)$ given by (3.7). Now, however, the reflection coefficient is

$$|\delta_\alpha(\kappa)|^2 = \frac{\kappa^2+|\alpha|^2+2\kappa(\mathrm{Im}\alpha)}{\kappa^2+|\alpha|^2-2\kappa(\mathrm{Im}\alpha)} \leq 1$$

so that the reflection is no longer total when $\mathrm{Im}\alpha<0$. Consequently, QH_α *generates the dynamics in which a plane wave is reflected at the origin with attenuation determined by* $\mathrm{Im}\alpha$. This is exactly the behavior we require in view of the comments at the end of the previous subsection; it is the partial absorption of the incoming wave at $x=0$ that makes asymptotic collapse possible. Once we have explicitly described the quantum dynamics we will work out the details for a wave packet.

Fixing α with $\mathrm{Im}\alpha<0$ and denoting the Laplacian on the domain D_α by Δ_α, (3.16) gives the evolution

$$\psi(x,t) = \lim_{n\to\infty}\left[\left[I-\frac{i\hbar t}{n}\Delta_\alpha\right]^{-n}\psi(x,0)\right]. \tag{3.18}$$

To determine $\psi(x,t)$ we must therefore compute the operator

$$\left[I-\frac{i\hbar t}{n}\Delta_\alpha\right]^{-1} = \lambda R(\lambda;\Delta_\alpha), \tag{3.19}$$

where $\lambda = -(n/\hbar t)i$ and $R(\lambda;\Delta_\alpha)$ is the resolvent of Δ_α. From Dunford and Schwartz (Sec. XIII.3), we calculate

$$R(\lambda,\Delta_\alpha)\psi(x) = \frac{1}{2\sqrt{\lambda}}[e^{\sqrt{\lambda}x}+\Gamma_\alpha(\lambda)e^{-\sqrt{\lambda}x}] \times \int_x^\infty \psi(s)e^{-\sqrt{\lambda}s}ds + \frac{e^{-\sqrt{\lambda}x}}{2\sqrt{\lambda}} \times \int_0^x \psi(s)[e^{\sqrt{\lambda}s}+\Gamma_\alpha(\lambda)e^{-\sqrt{\lambda}s}]ds \tag{3.20}$$

for $\lambda\in(-\infty,0)i$, where $\Gamma_\alpha(\lambda) = (\sqrt{\lambda}-\alpha)/(\sqrt{\lambda}+\alpha)$.

Now consider the normalized state

$$\psi(x,0) = \sqrt{2b}\,e^{-\beta x},$$

where both $b=\mathrm{Re}\beta$ and $B=\mathrm{Im}\beta$ are positive, representing an initially contracting universe. Since $\psi(x,0)\in D_\alpha$ if $\alpha=-\beta$, it is natural to fix α in this manner (note then that $\mathrm{Im}\alpha = -B<0$). This choice also greatly simplifies our calculations, since this wave function is then an eigenstate of QH_α. Substituting $\psi(x,0)$ into (3.20) and (3.19) gives

$$\left[I-\frac{i\hbar t}{n}\Delta_\alpha\right]^{-1}\psi(x,0) = \sqrt{2b}\left[1-\frac{i\hbar t}{n}\beta^2\right]^{-1}e^{-\beta x},$$

so that (3.18) becomes, not surprisingly,

$$\psi(x,t) = \sqrt{2b}\;e^{i\hbar t\beta^2}e^{-\beta x}. \tag{3.21}$$

Then $||\psi(x,t)||^2 = e^{-4bB\hbar t}$ which vanishes as $t\to\infty$, just as we had hoped.

The expectation values of the position and the momentum are

$$\langle Qx(t)\rangle = e^{-4bB\hbar t}/2b \tag{3.22}$$

and

$$\mathrm{Re}\langle Qp_x(t)\rangle = -e^{-4bB\hbar t}B\hbar. \tag{3.23}$$

Clearly this wave packet asymptotically collapses, in accord with the contractive nature of the dynamics. It is worthwhile comparing the behavior of this wave packet with that of the evolving state (3.9) from the standpoint of the reflection/attenuation mechanism at the origin. As with (3.9), the leading edge of (3.21) is reflected at $x=0$, but now only *partially*. Because of this absorption, the positive momentum carried by the reflected waves is no longer sufficient to offset the contraction and turn the wave packet around. The net result is that the model continually contracts, albeit at an exponentially decreasing rate.

For the classical evolution with the initial conditions

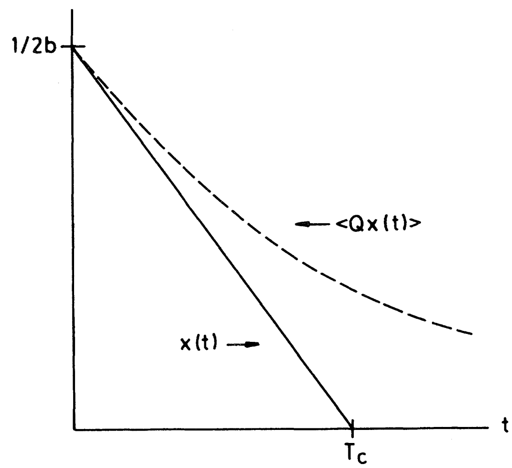

FIG. 2. Classical/quantum correspondence for the wave packet (3.21).

$x(0)=1/2b$ and $p_x(0)=-B\hbar$, (2.8) yields

$$x(t)=(1-4bB\hbar t)/2b$$

and

$$p_x(t)=-B\hbar\ ,$$

which agree with (3.22) and (3.23) for sufficiently small t. At the classical collapse time $T_c=1/4bB\hbar$, $\langle Qx(t)\rangle$ has fallen to $1/e$ times its original value. This classical-quantum correspondence is summarized in Fig. 2.

This analysis demonstrates that it is indeed possible to obtain a "better" correspondence limit by relaxing the requirement that the quantum dynamics be unitary. Still, this state of affairs is unsatisfactory, since it is not possible to guarantee that an initially contracting state will collapse in *finite* time as demanded by the unanimity principle.

IV. "PARADOXES" AND "RESOLUTIONS"

We have seen that whether the quantized FRW models collapse depends significantly upon the *qualitative* features of the quantum dynamics. Thus, collapse in finite time is prohibited as long as one insists upon unitarity, while asymptotic collapse can occur in the contractive case. Perhaps there exists some other dynamical framework—based on a notion more general yet than a contraction semigroup—which would yield an even better classical correspondence? Although this is a possibility, it seems unlikely that our problems can really be solved in this fashion.

A way out of this quandary is provided by the observation that our results on quantum gravitational collapse are also influenced by the classical choice of time. In fact, the underlying cause of the disparity between the behaviors of the classical and quantum models is that *the quantum dynamics persists eternally whereas the classical dynamics does not.* The classical evolution is incomplete simply because $t=-\varphi$ measures proper time. But quantum mechanically, the very choice of either unitary or contractive dynamics *guarantees* that the evolution is defined for at least all future times. Since it is therefore difficult to avoid complete (or at least + complete) quantum dynamics, it is apparent that this paradox of incomplete classical versus complete quantum evolution will arise whenever one makes a *slow* choice of time.

Thus, in the final analysis our problem really originates at the classical level. To circumvent it, one must choose a *fast* time. The corresponding classical dynamics is then complete and, upon quantization, the quantum dynamics will be complete as well. One might therefore expect results more in agreement with the unanimity principle if one quantizes in a fast-time gauge. This is in keeping with the suggestion of Lund[13] that one should always quantize on a geodesically complete minisuperspace. Here, however, the regularization necessary to produce complete classical evolution consists of modifying the choice of time rather than the minisuperspace itself.

This sort of regularization has profound implications regarding quantum gravitational collapse. Classically, of course, such a regularization is more cosmetic than substantive. The system is still singular, since asymptotic collapse in fast time is physically the same as collapse in finite slow time. Quantum mechanically, however, completeness in fast time has a quite different physical meaning than it does in slow time. As we have discovered, eternal quantum evolution in slow time is a strong indication that quantum collapse is impossible. But quantum completeness in fast time, being physically equivalent to quantum incompleteness in slow time, can only signal the presence of a singularity. Put somewhat differently, it is plausible that fast-time quantum dynamics incorporates quantum collapse in much the same way that self-adjoint slow-time dynamics is always nonsingular.

For the FRW models $t=\mu$ is a fast clock; we now quantize these models in this intrinsic-time gauge and check for quantum collapse.

V. QUANTIZATION IN THE INTRINSIC-TIME GAUGE

Since the choice $t=\mu$ is really appropriate only for expanding universes, we restrict consideration to the $k=-1,0$ models in this section. From Sec. II B we have the classical phase space $\mathrm{R}\times(0,\infty)$ with coordinates (φ,p_φ). Quantizing in the momentum representation we obtain the quantum Hilbert space $L^2(0,\infty)$ with inner product

$$\langle\phi,\psi\rangle=\int_0^\infty\bar{\phi}(p_\varphi)\psi(p_\varphi)dp_\varphi\ .$$

This representation diagonalizes the Hamiltonian (2.9) so that QH is just multiplication by H:

$$QH[\psi]=2\sqrt{6}e^{3t/2}(p_\varphi-6ke^t)^{1/2}\psi\ . \tag{5.1}$$

Note that QH is not a single operator, but rather a time-dependent family $\{QH(t)\}$ of operators, one for each $t\in\mathrm{R}$. Each $QH(t)$ is a positive self-adjoint operator on $L^2(0,\infty)$ with a purely continuous spectrum $\sigma(t)=[12\,|\,k\,|\,e^{2t},\infty)$.

We pattern our discussion of the quantum dynamics after those in Blyth and Isham[13] and Gotay and Isenberg.[8]

Since the $\{QH(t)\}$ commute, we can solve the Schrödinger equation by expanding in an evolving complete set of states $\{\psi_E(t)\}$ which are simultaneous eigendistributions of $QH(t)$ at all times. Thus, if

$$QH(t_0)[\psi_E]=E\psi_E \tag{5.2}$$

at some reference time t_0, then there will exist numbers $E(t)$, with $E(t_0)=E$, such that

$$QH(t)[\psi_E]=E(t)\psi_E \ . \tag{5.3}$$

Consequently, the states ψ_E evolve according to

$$\psi_E(t)=\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}E(s)ds\right]\psi_E(t_0) \ . \tag{5.4}$$

To evolve an arbitrary state $\psi\in L^2(0,\infty)$, one first performs a spectral decomposition

$$\psi(p_\varphi,t_0)=\int_{\sigma(t_0)}\phi(E)\psi_E(p_\varphi,t_0)dE$$

with respect to $QH(t_0)$ at time t_0; then

$$\psi(p_\varphi,t)=\int_{\sigma(t_0)}\phi(E)\psi_E(p_\varphi,t)dE \ , \tag{5.5}$$

where $\psi_E(p_\varphi,t)$ is given by (5.4).

From (5.1), we find that (5.2) has the distributional solutions

$$\psi_E(p_\varphi,t_0)=C_E e^{-3t_0}\delta\left[p_\varphi-\left[6ke^{t_0}+\frac{E^2}{24}e^{-3t_0}\right]\right] \ . \tag{5.6}$$

Substituting (5.6) into (5.3) then yields

$$E(t)=2\sqrt{6}e^{3t/2}\left[\frac{E^2}{24}e^{-3t_0}+6k(e^{t_0}-e^{t})\right]^{1/2} \ .$$

Because $E(t)$ is continuous the integral in (5.4) exists for all t. Equations (5.4) and (5.5) then imply that the quantum dynamics is unitary. In particular, when $k=0$ (5.4) becomes

$$\psi_E(t)=\exp\left[\frac{2i}{3\hbar}E(1-e^{3(t-t_0)/2})\right]\psi_E(t_0) \ .$$

By unitarity the quantum systems must evolve to the $t=-\infty$ limit and, since classically $t=-\infty$ represents the initial singularity, it follows that the quantum models *must* asymptotically collapse. But asymptotic collapse in this fast intrinsic time means that the quantum systems become singular in *finite* proper time.

This can be verified directly. Since the classical Hamiltonian $H(t)\to 0$ as $t\to-\infty$ and as $QH(t)$ is a positive operator, the expectation value $\langle\psi(t),[QH(t)]\psi(t)\rangle$ is a good indicator of quantum gravitational collapse. More precisely, the quantum models asymptotically collapse if

$$\lim_{t\to-\infty}\langle\psi(t),[QH(t)]\psi(t)\rangle=0 \tag{5.7}$$

for all evolving states $\psi(t)\in L^2(0,\infty)$ for which the limit in (5.7) is defined. We show that this is always the case.

Applying (5.5), (5.4), and (5.1), we compute

$$\langle\psi(t),[QH(t)]\psi(t)\rangle=\langle\psi(t_0),[QH(t)]\psi(t_0)\rangle=2\sqrt{6}\int_0^\infty e^{3t/2}(p_\varphi-6ke^{t})^{1/2}|\psi(p_\varphi,t_0)|^2dp_\varphi \ .$$

If $k=0$ then (5.7) is immediate. For $k=-1$, consider the family of functions $\{I_t(p_\varphi)\}$, where $I_t(p_\varphi)$ is the integrand in the above expression at time t. This family converges pointwise to zero as $t\to-\infty$ and $I_t(p_\varphi)<I_{t_0}(p_\varphi)$ for each p_φ if $t<t_0$. Since in addition $I_{t_0}\in L^1(0,\infty)$ by assumption, the dominated convergence theorem implies that

$$\lim_{t\to-\infty}\int_0^\infty I_t(p_\varphi)dp_\varphi=\int_0^\infty\lim_{t\to-\infty}[I_t(p_\varphi)]dp_\varphi$$

vanishes and (5.7) follows. Thus the FRW models, when quantized in a fast-time gauge, *unquestionably collapse*. In fact, the behavior of these systems is essentially the same as that of their classical counterparts.

VI. ON QUANTUM GRAVITATIONAL COLLAPSE

Our analysis of the dust-filled Friedmann-Robertson-Walker models demonstrates that the problem of quantum gravitational collapse is really rather subtle. The resolution of this problem and, more generally, the validity of Wheeler's rule of unanimity, depend critically upon two factors: the classical choice of time and the qualitative choice of quantum dynamics.

Although we have concentrated here upon a certain class of models, the substance of our conclusions holds for any homogeneous cosmology. For suppose that one quantizes a classically collapsing universe in a slow-time gauge, and that the resulting dynamics is generated by a self-adjoint Hamiltonian. Then quantum gravitational collapse is impossible, at least in the sense that the quantum formalism is well-behaved at the classical singularity and observables that classically vanished at the singularity have nonvanishing quantum expectation values there.[13] Indeed, if the quantum operators which serve as indicators of collapse are positive (as any such "good" indicator must be), then the same arguments as in Sec. III C show that the expectation values of these operators can never vanish in finite time. The results of Secs. IV and V, on the other hand, indicate that unitary fast-time quantum evolution inexorably leads to collapse. In fact, this version of quantum cosmology does not alter the classical behavior near the singularity in any significant way.

In light of these observations, we put forward the following conjecture regarding quantum gravitational collapse.

Conjecture: *(i) Self-adjoint quantum dynamics in a slow-time gauge is always nonsingular; within this dynamical framework, quantum gravitational collapse is strictly forbidden. (ii) Self-adjoint quantum dynamics in a fast-time gauge is always singular; within this dynamical framework, quantum gravitational collapse is inevitable.*

Much of the literature on quantum cosmology to date supports this conjecture. This is especially true of the work of Misner, Nutku, Ryan, *et al.*, on various Bianchi models which is summarized by Ryan[4] and MacCallum.[5] These investigations utilize Misner's fast intrinsic-time formalism and without exception conclude that there is, as Misner[14] put it, ". . . no suggestion of anything which would allow a contracting closed universe to pass through a quantum phase and emerge as an expanding universe." Gotay and Isenberg's analyses of quantized Robertson-Walker universes filled with a scalar field in fast extrinsic- and matter-time gauges[8] lead to the same conclusion, as does Brill's study of a certain fast-time FRW model.[18] Thus, (ii) above is confirmed for a wide range of both cosmological models and fast-time gauges.

Within the slow-time dynamical framework, we find support for (i) above in the work of DeWitt[9] and Lund[13] on the FRW universes and Demaret[16] on several Bianchi models. In particular, the results of Lapchinskii and Rubakov's matter-time quantization of a FRW model[15] are remarkably consistent with those obtained here. Finally, nice heuristic comparisons of the behavior of the quantized FRW models in both fast and slow choices of time are given by Ryan[4] and Brill.[18]

We therefore believe that our conjecture is quite plausible. Ultimately, it means that one's determination of whether quantum gravitational collapse occurs depends crucially upon the manner in which one makes this determination, that is, how one sets up the classical model and to what extent one regularizes it prior to quantization, as well as how one fixes the precise form of the quantum dynamics.

The decisive factor here is the choice of time; to a large degree, this very choice *classically predetermines* whether quantum gravitational collapse can occur. However, the picture is not entirely as black and white as this may suggest; the qualitative choice of quantum dynamics introduces some gray. Thus, as illustrated in Secs. III C and III D, it is possible to have asymptotic collapse—a phenomenon intermediate between the extremes of outright quantum bounce and quantum collapse—if one takes the slow-time quantum evolution to be contractive rather than unitary. Yet finer tuning can be achieved by setting various boundary conditions. This has quantitative consequences but does not seem to have a significant bearing on the qualitative behavior of the models. These observations help to reconcile the heretofore bewildering array of different answers to the quantum collapse question.

This brings us to the ultimate issue of which classical/quantum formalism is actually correct. Without conclusive physical evidence and beyond very general and somewhat philosophic considerations,[2] such as requiring a reasonable classical limit,[3] there are no precise criteria to help us decide. This circumstance is due in large part to the very nature of quantum cosmological models—because of their simplicity, they just do not furnish enough insight into the physical mechanisms which underlie quantum gravitational phenomena. Such insight can only be provided by a much more complete consistent quantum theory of the gravitational field and its interactions than is currently available. In the meantime, one must proceed by "hints, guesses, and model calculations"; quantum cosmology must remain speculative rather than definitive. Only (the choice of) time will tell.

ACKNOWLEDGMENTS

One of us (M.J.G.) would like to thank P. Binding, P. J. Browne, E. Couch, and J. Śniatycki for helpful conversations. This work was partially supported by a Natural Sciences and Engineering Research Council of Canada operating grant to The University of Calgary.

[1]Lists of these efforts, including references and some discussion, may be found in L. Parker, in *Asymptotic Structure of Space-Time,* edited by F. P. Esposito and L. Witten (Plenum, New York, 1977); J. D. Barrow and R. A. Matzner, Phys. Rev. D 21, 336 (1980); N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982), Sec. 7.4; Ref. 11.

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[17]We emphasize that the crucial distinction here is between fast and slow times and *not,* as Lapchinskii and Rubakov[15] intimate, between geometric and matter-linked ones.

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[21]Schutz's method should be compared with that of Ryan,[4] which introduces no degrees of freedom for the fluid, and DeWitt,[9] which endows the fluid with "internal" degrees of freedom.

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[23]This restriction simplifies the ADM reductions in the next subsection. It is also possible to carry out such reductions for more general perfect fluids, in particular those obeying the equation of state $p=(\gamma-1)\rho$ with $1\leq\gamma\leq 2$, but these are complicated. Details may be found in Ref. 20. The radiation-dominated case $\gamma=\frac{4}{3}$ is also discussed in Ref. 15.

[24]One could equally well quantize the phase space R^2 with coordinates (μ,p_μ). In this case (cf. Ref. 7), the quantum Hilbert space is $L^2(\mathrm{R},e^{3\mu/2})$ with the inner product

$$\langle\phi,\psi\rangle=\int_{-\infty}^{\infty}\bar{\phi}(\mu)\psi(\mu)e^{3\mu/2}d\mu ,$$

and the operator corresponding to the Hamiltonian (2.7) is

$$\tilde{Q}H=-\frac{\hbar^2}{24}e^{-3\mu}\left[\frac{d^2}{d\mu^2}-\frac{3}{2}\frac{d}{d\mu}\right]+6ke^{\mu} .$$

This quantization is unitarily equivalent to the one given in the text. Indeed, the linear transformation U given by

$$(U\phi)(\mu)=(24)^{1/4}\phi(\tfrac{4}{3}\sqrt{6}e^{3\mu/2})$$

is a unitary isomorphism of $L^2(0,\infty)$ onto $L^2(\mathrm{R},e^{3\mu/2})$ which intertwines the quantum Hamiltonians: $QH=U^{-1}(\tilde{Q}H)U$.

[25]Although it is a basic tenet of quantum mechanics that a physical observable should be represented by a self-adjoint operator, Qp_x is not and, indeed, *should not* be self-adjoint. For if it were, then it would generate a one-parameter group of translations $\psi(x)\rightarrow\psi(x+t)$ which is impossible, since translation to the right $(t\leq 0)$ is not defined on $L^2(0,\infty)$. On the other hand, translation to the left $(t\geq 0)$ is defined, so that one really has a (contraction) semigroup. Furthermore, the generator of this semigroup (in the sense of Sec. III D) is just $-d/dx$. Thus, insofar as translations are defined, they are generated by Qp_x and this is entirely consistent with the non-self-adjointness of Qp_x.

[26]The relevant functional analysis may be found in Chapters VIII and X of M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, New York, 1972 and 1975), Vols. I and II, and Chapters VII and XIII of N. Dunford and J. T. Schwartz, *Linear Operators* (Interscience, New York, 1957 and 1963), Vols. I and II.

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