Radiation in the Einstein universe and the cosmic background

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It is shown that the cosmic background radiation is not at all uniquely or scientifically relatively economically indicative of a "big bang." Specifically, essentially any temporally homogeneous theory in the Einstein universe is consistent with the existence of a cosmic background radiation (CBR) conforming to the Planck law; in particular, the chronometric cosmology is such. It is noted that the Einstein universe appears particularly natural as a habitat for photons by virtue of the absence of infrared divergences and of the absolute convergence of the trace for associated Gibbs-state density matrices. These features are connected with the closed character of space in the Einstein universe, and facilitate the use of the latter in modeling local phenomena, in place of Minkowski space with periodic boundary conditions or the like, with minimal loss of covariance or effect on the wave functions. In particular, the Einstein universe may be used in the analysis of the perturbation of a Planck-law spectrum due to a local nonvanishing isotropic angular momentum of the CBR, of whatever origin. The estimated distortion of the spectrum due to such a kinematically admissible effect is in very good agreement with that observed by Woody and Richards, which is opposite in direction to those earlier predicted by big-bang theories. The theoretical analysis involves a preliminary treatment of equilibria of linear quantum fields with supplementary quasilinear constraints.

I. INTRODUCTION

The cosmic background radiation (CBR) has often been construed as a very strong indication for, if not actual proof of, the existence of a "big bang."[1] Actual scientific arguments have, however, been detailed only within the framework of expanding-universe models. With increasing question as to the validity of the Hubble law,[2] the basis for this postulate has eroded, and the rapidity and unexpected character of the evolution that Friedmann models required for reconciliation with quasar observations[3] also suggest caution in the acceptance of the expanding-universe model, or may even dictate some urgency in the consideration of alternatives.

It should consequently be of interest to return to first principles and analyze the CBR without any special cosmological hypotheses, as much as possible on the basis of general kinematics and universally accepted verities such as Maxwell's equations.

The Einstein universe[4] is the maximal space-time within which Maxwell's equations remain intact, and to which solutions of these equations (or related equations such as those of Yang and Mills) in Minkowski space canonically extend.[5] Moreover, it contains Minkowski, de Sitter, and anti-de Sitter space-times as open submanifolds with their given causal structures.[6] It also appears to provide a satisfactory account of the overall gravitational structure of the universe.[7] The full causal and covariance features of the Yang-Mills etc. equations are expressible only in the Einstein universe, in the conformally invariant form known as the universal cosmos; in particular, global covariance under a 15-parameter group of causal transformations.[8] Finally the Einstein universe represents the simplest conceptual alteration of ordinary Minkowski space, since it may be realized by the replacement of Euclidean three-space E^3 by the spherical space S^3; and in its causally covariant form as the universal cosmos is the only space-time other than Minkowski space that embodies spatial and temporal isotropies, together with global causality and separability of time from space.

The possible form of background radiation in the Einstein universe is treated without any special assumption other than the temporal and spatial homogeneities of the underlying dynamics. Temporal homogeneity is meant here in the sense not only of the absence of any time origin, but also of the conventional identification of the energy with the infinitesimal generator of the time evolution group. In the latter respect it differs fundamentally from the "steady-state universe" conception of Hoyle; there is conservation of energy if temporal homogeneity holds. Assuming that the total energy of the background radiation in the universe is conserved, and that this radiation is in a maximally random state as measured by its entropy, an explicit and rigorous analysis confirms that the spectrum will be Planckian, with an assumption in the nature of ergodicity to the effect that its long-term interaction with the matter in the universe is not extremely special. The strength of the interaction is of no consequence (as long as it is not identically zero) by virtue of the unlimited time over which it may act in the Einstein universe.

The present treatment is in part an amplification of an earlier condensed one,[10] which was too brief to include proofs, respond to natural questions that might be raised from the big-bang point of view, or treat specifically the interpretation within the chronometric model. Here the problem is placed in a more general setting in order to clarify the fundamental issues involved. In particular it deals with equilibrium states of linear quantum fields with semiquadratic constraints; this is an abstraction from physical considerations involved in the analysis of a possible local conserved isotropic total angular momentum in the CBR. This kinematically admissible photon state is discussed in greater detail, and it is shown that the distortion in the CBR spectrum reported by Woody and Richards[11] is consistent with the perturbation of a pure Planck law by a local effect of this type. The efficiency from a purely information-theoretic standpoint of the analytic expression obtained[10] has in the meantime been treated.[12] Also published in the meantime have been models for the Woody-Richards effect within the framework of the big-bang model.[13] Purely information-theoretically, these models involve many more parameters than does that deriving from a locally nontrivial isotropic angular momentum in the CBR, in which only a single parameter is involved, for all possible values of which the predicted effect is in the direction observed. Typically the big-bang explanations involve the postulation of highly energetic processes at large red-shifts, and appear relatively remote from the possibility of empirical substantiation beyond that of mere consistency with the Woody-Richards effect.

Ideally of course the state of the CBR would be derived from that of the total system consisting of the CBR together with all matter in the universe, which on the usual statistical basis may be postulated to have a density matrix of the form $D=e^{-bH}/\mathrm{tr}e^{-bH}$, where H is the total energy operator: $H=H_{\mathrm{CBR}}+H_{\mathrm{matter}}+H_{\mathrm{interaction}}+H_{\mathrm{other}}$; H_{other} includes the energies of neutrinos, the x-ray background, and their interactions, as well as possible unknown components. Unfortunately the specification of all the components of H other than H_{CBR} would at best be highly speculative, in addition to which ultraviolet divergences in the treatment of interactions that are currently unresolved would foreclose the possibility of reliable calculations, even if analytic expressions for the components of H were given. Fortunately, special features of photons and of the CBR suggest the feasibility of a separate treatment of the CBR. Photons appear to lack direct interactions; and their indirect interaction via pair production appears unlikely to exert a major effect on the CBR. Another distinction between the CBR and matter constituents in the universe that is natural in the Einstein universe, and especially so in its chronometric interpretation, is that in their dynamical time scales, which may be very much greater for the CBR. The natural units in the Einstein universe are those in which $\hbar=c=R=1$, where R is the radius of the universe; these are conformally invariant.[14] The corresponding natural unit of time, which may be called an "eon," is that in which light traverses a distance of 1 radius. In Minkowski space 2π eons appears as the time interval from $-\infty$ to $+\infty$,[9] and appears comparable to the time scale on which stars and perhaps galaxies may evolve effectively of an order between 10^9 and 10^{11} years according either to the chronometric or Friedmann cosmologies. Light however, and especially the components of the CBR, may circulate around the universe for thousands of eons before interacting with the diffuse matter therein, as estimated for one specific possible mechanism.[15]

Thus the present treatment in which the CBR is treated as a separate entity that is approximately invariant under the isometry group of the Einstein universe, with matter serving only as a source and sink and diffusing mechanism, may be regarded as a quasiphenomenological one that appears justified by the indicated special features of photons and the CBR. The CBR is assumed to be itself in a maximally random state, with density matrix of the same form as D except for the replacement of H by H_{CBR}, which is reflective of its interaction with matter over many eons of time, and not directly reflective of observable matter during the past eon. Its temperature thus corresponds to the temperature of matter averaged over many eons, rather than over the time scale normally regarded as appropriate to stars or galaxies, and in addition must be presumed to involve in an essential way the dark matter in the universe, which is now fairly thought to be possibly of much greater mass than the luminous matter. This average temperature of matter would be expected to be far less than that of observable, luminous, matter, but appears inherently incapable of direct observation.

Finally it is interesting to note that the problems of the big-bang theory with isotropy and causality, which have led to the development of the inflationary-universe model, do not arise in general class of models based on the Einstein universe, including the chronometric theory.

II. MAXIMAL-ENTROPY EQUILIBRIUM STATES OF QUANTUM FIELDS WITH QUASILINEAR CONSTRAINTS

The case of a free field illustrates the essential ideas, and is the only one susceptible to a full and general mathematical treatment. For succinctness and logical clarity, a compact formulation[16] of basic principles is used here and familiarity with these principles will be assumed. The treatment is insensitive to the statistics, and only the case of bosons is considered here.

Let H denote the single-particle complex Hilbert space. Let G denote the fundamental symmetry group (whether simply temporal evolution, the Lorentz group, still larger group, etc.). For present purposes, "particle" is definable as a given unitary representation, say U, of G on H. Stability requires that U be a positive-energy representation, in the sense that if $g(t)$ is an arbitrary temporal one-parameter subgroup, then the one-parameter unitary group $U(g(t))$ has a non-negative generator. It will be assumed that this is the case, the temporal evolution one-parameter subgroup of G being assumed specified. There is then a unique free positive-energy quantization,[17] whose Hilbert space will be denoted as K. For any unitary operator V on H, there is a corresponding unitary operator on K, which will be denoted as $\Gamma(V)$. The mapping $V\to\Gamma(V)$ is a representation of the group $U(\mathsf{H})$ of all unitary operators on H, into the group $U(\mathsf{K})$ of all unitary operators on K.

Associated with the representations U and Γ of G and $U(\mathsf{H})$ are corresponding infinitesimal representations u and γ of the Lie algebras (or infinitesimal groups) $\mathscr{G}$ and $\mathscr{U}(\mathsf{H})$ of these groups, defined in the usual way: If X is any infinitesimal self-adjoint generator of G, then $u(X)=-i(d/dt)U(e^{itX})\,|_{t=0}$, and similarly for γ. In addition, as always,[18] these infinitesimal representations may be uniquely extended by associativity and linearity to the "enveloping algebra" $\mathscr{A}$, i.e., the algebra of all polynomials in the generators of G, unconstrained except by their given commutation relations. This extension will be defined so as to carry Hermitian elements of $\mathscr{A}$ into Hermitian operators; by definition it preserves the usual associative algebraic operations, and hence commutators, within the requisite factors of i.

Now some confusion may arise from the circumstance that the composition of the representations U and Γ, i.e., the mapping $g \rightarrow \Gamma(U(g))$ from G to operators on K has the canonical extension to $\mathscr{A}$ of the corresponding infinitesimal representation of G distinct from the application of γ to the extension of u. In other terms, denoting extensions to $\mathscr{A}$ by the superscript e, if Q is a Hermitian element of $\mathscr{A}$ (to be physically interpreted as a quantum number label), so that $u^e(Q)$ is a Hermitian operator in H and $\gamma(u^e(Q))$ a Hermitian operator in K, this operator is distinct from $(\gamma \circ u)^e(Q)$. The latter operator will be called the *field Q*, and the form the *particle Q;* they are the same when Q is simply a generator of G, and their restrictions to the single-particle subspace are the same in any event. Both have canonical features, commute with all symmetries that commute with Q, carry positive operators into positive operators, etc. Which if either is appropriate to represent Q in a given physical context depends on the specifics of theory and/or observation.

It may be helpful to recall the basic properties of u and especially its statistical interpretation, as originally developed in an invariant Hilbert-space context by Cook.[19] If B is any given self-adjoint operator in H, representing a single-particle observable, $\gamma(B)$ is simply the infinitesimal self-adjoint generator of the one-parameter unitary group on K, $\Gamma(e^{itB})$. It follows that $\gamma(B)$ is positive when B is, and that γ preserves commutators. If L is any closed linear subspace of H, the "number of particles with wave function in L" is the operator $\gamma(P)$, where P is the projection of H onto L. This representation of the particle number combines with the spectral resolution of B to exhibit directly the statistical interpretation of $\gamma(B)$. Taking for simplicity the case when B has a discrete eigenbasis $e_1, e_2, \ldots,$ with eigenvalues $\lambda_1, \lambda_2, \ldots,$ then the occupation number n_j of the state e_j is $\gamma(P_j)$, where P_j is the operator in H projection onto the one-dimensional subspace spanned by e_j. The corresponding quantum field observable is then expressible in the familiar form $\gamma(B) = \lambda_1 n_1 + \lambda_2 n_2 + \cdots$.

It is relevant also that the additivity properties of γ extend to the formation of composite systems. Thus if H' is another single-particle space, and Γ denotes the corresponding representation of $U(\mathsf{H}')$ into $U(\mathsf{K}')$, K' being the state vector space for the quantum field of H'-particles, then Γ satisfies an exponential law connected with the formation of composite systems: $\Gamma(U \oplus U') = \Gamma(U) \otimes \Gamma(U')$, U and U' being arbitrary unitary operators on H and H'. This exponential law is equivalent to the *additive* property relative to *direct* sums of self-adjoint operators for γ: $\gamma(B \oplus B') = \gamma(B) \otimes I + I \otimes \gamma(B')$, where the I's denote the identity operators on K and K', respectively.

On the other hand, despite a fundamental role played by γ as a mapping transferring a given single-particle observable to a corresponding one on the quantized field of such particles, its application to the single-particle total squared angular momentum does not yield the total squared angular momentum of the quantum field. The result may rather be described as the total squared angular momentum of all the quanta of the field. Due to possible forms of interference between these quanta, this is not the same as the square of the total angular momentum of the field as a whole. In states in which such interference is absent, the expectation values of these distinct operators will correspondingly coincide. In particular, it is reasonable to expect that in maximally chaotic states, the quanta will be stochastically independent and the interference will cancel out, resulting in the identity of the expectation values of the operators.

To clarify the relation between the two versions of the quantum field total squared angular momentum, consider these operators in the particle representation for K. In this K appears as the direct sum of the symmetrized tensor powers of H:

$$\mathsf{K} = \mathsf{H}^0 \oplus \mathsf{H} \oplus \mathsf{H} \otimes' \mathsf{H} \oplus \mathsf{H} \otimes' \mathsf{H} \otimes' \mathsf{H} \oplus \cdots ,$$

where $\otimes'$ denotes the symmetrized tensor (or direct) product, and H^0 is a one-dimensional space spanned by the vacuum state vector. The two operators $\gamma(u^e(Q))$ and $\beta^e(Q)$, where $Q = -R_1{}^2 - R_2{}^2 - R_3{}^2$ is the element in $\mathscr{A}$ that represents the total squared angular momentum, agree in the 0- and 1-particle subspaces H^0 and H, but not in the 2- and higher-particle subspaces. To determine the difference explicitly, note that for any X in $\mathscr{G}$, $\beta(X)$ takes the form, relative to the decomposition of K into n-particle subspaces,

$$\beta(X) = \gamma(u(X)) = 0 \oplus u(X) \oplus (u(X) \otimes I' + I \otimes' u(X)) \oplus \cdots ,$$

whence noting that $u^e(X^2) = u(X)^2$,

$$\gamma(u^e(X^2)) = 0 \oplus u(X^2) \oplus (u(X)^2 \otimes' I + I \otimes' u(X)^2) \oplus \cdots .$$

On the other hand,

$$\beta(X)^2 = 0 \oplus u(X)^2 \oplus ((u(X)^2 \otimes' I + 2u(X) \otimes u(X) + I \otimes u(X)^2) \oplus \cdots .$$

Thus

$$\beta^e(X^2) - \gamma(u^e(X^2)) = 0 \oplus 0 \oplus 2u(X) \otimes' u(X) \oplus \cdots .$$

The higher-particle subspaces involve similar terms consisting of symmetrized tensor products of two of the $u(X)$ with factors consisting of the identity I. Substituting $X = R_j$, $j = 1,2,3$; and summing gives the difference between the field and particle total squared angular momentum.

In an isotropic (rotationally invariant) state, the $u(R_j)$ will have spectra that are symmetric around 0, and the same is true of the $u(R_j) \otimes' u(R_j)$; thus the expectation value of the difference appears *a priori* to be as likely to be positive as to be negative. More importantly, if in a given state E of the quantum field (whether pure or mixed) the basis vectors $e_1, e_2, \ldots,$ are stochastically independent in an appropriate sense, then the two operators do in fact have identical expectation value. We say the basis vectors are stochastically independent in case the occupation numbers $n_1, n_2, \ldots,$ are stochastically independent as random variables. They are representable as such, relative to the given state E, by taking, e.g., the expectation value of $\exp[i(n_1t_1+n_2t_2+\cdots)]$ where the $t_1,t_2,\ldots,$ are arbitrary variables, to be the characteristic function of the distribution of $n_1,n_2\ldots$ (for any finite set of n_j). Stochastic independence is equivalent to the factorizability of this characteristic function into the product of the expectation values of the $\exp[in_jt_j]$. In an isotropic state the expectation values of the $u(M_j)$ vanish, and so if E has this additional property, the two distinct versions of the total squared angular momentum of the field have the same expectation values.

This result is basically a general one, and may be formulated as such in the following manner. Let Q be a given element of the enveloping algebra of G, assumed at most quadratic in the generators of G. Then $\gamma(u(Q))$ may be termed the *field-Q* and $\beta(Q)$ the (total) *particle-Q*. A state E of the quantum field may be termed *factorizable* in case there exists a basis $e_1,e_2,\ldots,$ for the single-particle space that are stochastically independent relative to E. Then

Theorem. If E is a factorizable state of the quantum field in which every generator of G has vanishing expectation value, and if Q is any quadratic element of the enveloping algebra of G, then the field-Q and the total-particle-Q have the same expectation values in the state E.

We now turn to the question of the form of an equilibrium state of the quantum field, subject to constraints in addition to temporal invariance, and determined by maximal randomness. Under rather general conditions, given a Hilbert space L, and a non-negative self-adjoint operator A (physically interpretable as the energy) such that $e^{-\beta A}$ has absolutely convergent trace for some nonvanishing value of β, there exists a density matrix D of absolutely convergent trace that maximizes $-\mathrm{tr}(D\log D)$ subject to the constraint $\mathrm{tr}DA=E_0$, where E_0 is given and sufficiently large. D may in fact be taken to have the Gibbs form $D=e^{-\beta A}/\mathrm{tr}(e^{-\beta H})$, where β now depends on E_0, and is the unique density matrix with the indicated properties under suitable further general conditions. We emphasize the generality of this result; it applies irrespective of any postulated forms of interaction, assuming only that they are of such a character as to effectively maximize randomness. In practice, as long as there is a randomizing influence that is not too special, these special forms constituting a set of probability zero typically in problems of this type, the density matrix takes the indicated form. The Hilbert space L does not need to be that associated with a quantum field; there need be no connection with photons or any other specific field or particle.[20]

This is stressed because in the literature the Planck law is commonly derived with the use of assumptions regarding mechanisms of thermalization, boundary conditions, special properties of light, etc. It is important here that no specific property of light is required other than its representation by Maxwell's equations, its Bose-Einstein character, and the fact that it is stochastically emitted and absorbed by matter via a temporally invariant interaction.

While it might seem supererogatory to discuss the close relation of the present manifestly invariant [basis-independent, transforming under the full unitary group $\mathsf{U}(\mathsf{H})$] approach to quantum statistics to the textbook one, confusion about this relation appears in some recent literature, and the treatment will serve also to provide notation used later. This will be done in the context of an arbitrary self-adjoint operator A in the single-particle Hilbert space H. Physically A is to be interpreted as the energy; it will be assumed to be positive, and that is has the eigenvalues $a_1,a_2,\ldots,$ arranged in nondecreasing order, with corresponding eigenvectors $e_1,e_2,\ldots$; the eigenvalues may have arbitrary finite multiplicities. The corresponding Bose-Einstein quantum field has its Hilbert space K representable as the infinite tensor product of the quantizations K for each of the one-dimensional Hilbert spaces H spanned by e_j, in such a way that the total field energy $H=\gamma(A)$ in K is the direct sum of the energies H_j (tensor produced with the obvious identity operators on the other K_j's). This energy H_j in K_j is the quantization (via γ) of the simple operator of multiplication by a_j in H_j.

By the "harmonic oscillator," or in rigorous invariant form, "real wave representation"[21] of boson fields, H_j is unitarily equivalent to $L_2(R^1)$ in such a way that H_j is unitarily equivalent to a_jN, where N is the one-dimensional harmonic-oscillator Hamiltonian, having the eigenvalues $0,1,2,\ldots,$ each with multiplicity 1. In this unitary equivalence of K with the infinite tensor product of the K_j, $e^{-\beta H}$ is correspondingly the infinite tensor product of the $e^{-\beta H_j}$. It follows that $\mathrm{tr}e^{-\beta H}=\prod_j \mathrm{tr}e^{-\beta H_j}$. From the spectrum of the harmonic oscillator,

$$\mathrm{tr}e^{-cN}=1+e^{-c}+e^{-2c}+\cdots=(1-e^{-c})^{-1}\ .$$

Defining $\phi(\beta)=\mathrm{tr}e^{-\beta H}$, it follows that $\psi(\beta)=\prod_j\psi(a_j\beta)$ where $\psi(c)=(1-e^{-c})^{-1}$. From this the distribution of the number n_j of particles in the state e_j is readily determined: its characteristic function takes the form

$$f(t)=\mathrm{tr}(e^{itN}e^{-\beta a_jN})/\mathrm{tr}(e^{-\beta a_jN})\ .$$

This implies that the expected value of

$$n_j \text{ is } -a_j{}^{-1}\psi'(a_j\beta)/\psi(a_j\beta)=(e^{\beta a_j}-1)^{-1}\ ,$$

which in turn implies that the expected energy spectrum takes the form $\lambda N_\lambda(e^{\beta\lambda}-1)^{-1}$, where N_λ is the multiplity for the eigenvalue λ of A. Note that no special assumptions regarding wave equations or the spatio-temporal labeling of the vectors in H are required here; a still more general analysis is included in Kon.[22]

Now consider how the foregoing result is modified in the presence of additional constraints. Using Lagrange multipliers, the maximal entropy state subject to given expectation values on the self-adjoint operators $S_1,S_2,\ldots,$ has a density matrix of the form $Ce^{-\beta H-c_1S_1-c_2S_2-\cdots}$, where C is a normalizing constant, and β and the c_j are to be determined from the given expectation values. If in particular it is proposed to constrain the total angular momentum (isotropically), the question arises of whether the field or particle quantities should be substituted for the operator S_1, or whether neither is appropriate. This question can be given a definite answer only when the physical mechanism by which the equilibrium is attained is determined, at least to the extent of the knowledge of the correlations of the angular momenta of the quanta of the field with those of the system with which interaction takes place, or with each other. However, if there are *a priori* physical indications, whether empirical or otherwise, that the equilibrium state is factorizable, then by the Theorem, the field and particle constraints are equivalent.

Since the particle total square angular momentum is a linear form in the occupation numbers, its analytical consequences are derivable by the method just indicated. If this results in a state that is factorizable, then *a fortiori* the resulting state is maximal entropy within the class of all *factorizable* states satisfying the given constraints, as well as within the larger class not constrained to be factorizable. The question of whether the field or particle angular momenta should be substituted for S then becomes moot, and analytically the particle quantity is more convenient. A concrete exemplification of the results is given for the case of the quantized photon field on the Einstein universe in the next section.

III. PHOTONS IN THE EINSTEIN UNIVERSE

The previous considerations are here applied to the case in which the single-particle space consists of all normalizable free photon wave functions. This Hilbert space H is essentially the same whether one treats photons in Minkowski space M_0 or the Einstein universe M.[23,24] However here we take the Einstein universe as fundamental, and the quantum numbers that are appropriate in this connection will be used. Angular momentum quantum numbers are essentially the same as those in Minkowski space-time, but otherwise there are differences of order R^{-1}, where R denotes the radius of the universe in laboratory units.[14,23]

To avoid possible confusion, it should be noted that the conformal group of present relevance, which is the universal cover of the matrix group SU(2,2) is that of the four-dimensional Einstein universe; as an infinitesimal group, this is the same as that of Minkowski space-time. The full conformal groups of the spheres S^3 or of R^3 play a quite limited role because they do not extend to M or M_0, respectively, except for isometries and other special transformations. This may be especially confusing because of the fortuitous circumstances that the conformal group of M is locally O(2,4), while that of S^3 is O(1,4), which is obviously a subgroup of O(2,4); however, the isomorphism of O(1,4) with this subgroup of O(2,4) does not necessarily carry a given element of O(1,4) into an element of O(2,4) that acts in the same way on the spacelike section S^3 of M at Einstein time $t=0$, or even leaves any such spacelike section at a fixed Einstein time invariant, although it does so in certain cases (see below).

It should be recalled that the time $\times$ space separation in the Einstein universe, and that in the Minkowski space-time that osculates at a given point, agree only at the fixed time in question, say $t=x_0=0$. As time evolves, the respective time $\times$ space decompositions begin to differ, and at very large times differ quite considerably.[5]

To summarize the presently essential points regarding photons in the Einstein universe, we note that this universe has an Einstein metric of the form dt^2-ds^2 where t ranges over the reals, representing the time, and ds is the element of distance on the space S^3, both t and s being in radians; this entails choosing units so that $c=1$, and also the radius R of S^3 is 1; in addition, the remaining freedom in units will be fixed by choosing $\hbar=1$. These units are invariant under the group G of all causality-preserving transformations on M, a 15-parameter group locally isomorphic to SU(2,2). The Einstein energy is defined as the self-adjoint generator of time evolution in M, for any species of field on which time displacement acts unitarily. This applies in particular to the case of photons. The normalizable solutions of the Maxwell equations in M transform unitarily under arbitrary transformations in G, and the Einstein energy A is discrete with lowest eigenvalue 2.

A complete set of quantum numbers is given by the Einstein energy and helicity, and the total angular momentum and one component thereof.[23,24] Each energy value n occurs with a multiplicity $2(n^2-1)$, so that the total expected energy at frequency n, in the equilibrium state of the preceding section, is $2n(n^2-1)(e^{\beta n}-1)^{-1}$. The energy in the frequency range from n to n' is given by the corresponding sum, which is extremely fine grained in the observable frequency range and consequently differs by an unobservably small amount from the corresponding integral, $\int_n^{n'} 2\nu^3(e^{\beta\nu}-1)^{-1}d\nu$, i.e., the usual Planck law is obtained.

It is interesting that in the Einstein universe not only is the quantization of the frequencies enforced by the discrete spectrum of the Einstein energy, but the photon has nonvanishing Einstein mass, i.e., infimum of the Einstein energy, of wave functions subjected to arbitrary causal (conformal) transformations. This may appear paradoxical until it is realized that under scale transformations the Einstein energy does not itself scale. With $R=c=1$ this mass is $2\hbar$, making possible a conceptual definition of $\hbar$ in terms of the Einstein photon rest mass. This nonvanishing rest mass also accounts for the absence of infrared divergences in the Einstein universe.[25] The present analysis shows that in addition the density matrix under consideration is rigorously of absolute convergent trace, without limitation of space to a subregion with reflecting walls or periodic boundary conditions, such as is often done in Minkowski space in order to achieve a discrete spectrum and an absolutely convergent trace. Elsewhere[26] it is shown that certain interacting quantum field divergences are absent in the Einstein universe. A similar analysis applies in the case of massless fermions.

Now consider how the spectrum is affected by the assumption of the existence of a substantial isotropic conserved angular momentum, above and beyond that which follows from the foregoing density matrix. If the south pole of the Einstein universe is taken as the point of observation, then the analog of space translations in Minkowski space are, in the Einstein universe, rotations around the north pole. Indeed in the vanishing curvature limit in which the Einstein universe becomes flat and essentially the Minkowski space-time, these rotations become exactly space translations. Thus there is no invariant distinction between rotations and translations in the Einstein universe; indeed, the antipodal mapping on S^3, which is in the connected group of isometries, exchanges these transformations, together with the north and south poles, and the locally observable angular and linear momenta. It follows that even the pure Planck law in the Einstein universe involves a nonvanishing expected squared angular momentum; its contribution to the energy is precisely equal to that of the expected squared "linear" momentum, defined here by the sum of the squares of the generators of rotations around the antipode of the point of observation; and the individual components of angular momentum have vanishing expectation value. These results follow immediately from the invariance of the Planck law density matrix under arbitrary rotations on S^3, or more exactly, the unique conformal transformations on the Einstein universe that extend space rotations.

However, our concern here is with a total angular momentum in excess of the level automatically implied by the Gibbs state of the photon field on M. Such angular momentum will be considered relative to the point of observation, taken as the south pole. Isotropy, assumed here, implies that each component of angular momentum will have vanishing expectation value. We ask for an equilibrium state in which the constituent photons are independent, as they are in the case just considered, and as might be expected in a maximally chaotic state constrained only in energy and angular momentum. This expectation is however no guarantee of independence; each concrete physical application must be considered on its own merits. But such an assumption of independence is often made, appears natural, and most important, is kinematically entirely admissible. Once it is made the expectation values of the field-total angular momentum and the particle total angular momentum are the same, and analysis[10] may be based on a constraint for the latter leading to the modified spectrum

$$F(\nu)=2\nu k^{-1}\ln\frac{1-e^{-\beta\nu-k\nu^2}}{1-e^{-\beta\nu}} .$$

Here k is a positive constant related to the expected total angular momentum.

In the present theoretical generality, there is no obstacle to this law being applicable either globally or locally, local regions being modeled by an Einstein universe in the same fashion as a local region in Minkowski space may be modeled by a box with periodic boundary conditions. The features of a discrete positive-energy spectrum in the Einstein universe together with an absolutely convergent trace for the putative Gibbs state density matrix make it unnecessary to enforce any boxlike limitation. The Einstein universe has besides these advantages one that is especially relevant here of being rotationally invariant in local usage. The connection with a local flat observing space may be made through the stereographic projection, which is conformal and does not affect the angular momenta, and carries the origin in Minkowski space into the south pole in the version applicable here.[24] There is a question here of scale, of the region considered in relation to the universe as a whole, of the frequency ranges in which the spectrum is treated, and of the assumed level of the conserved total angular momentum in relation to the kinematically admissible maximum. If the region is small, the frequencies high in natural units, and the level of total angular momentum is small compared to the maximum, the approximation of the local space-time region by an Einstein universe appears physically reasonable. These conditions will hold in the application made in the following section.

IV. THE WOODY-RICHARDS ANOMALY

The measurements of Woody and Richards[11] indicate a perturbation of the Planck law spectrum opposite in direction to that predicted within the expanding universe framework.[12,27] The report of Woody and Richards was concurrent with research[22] directed in part toward the determination of an upper limit of possible isotropic angular momentum of the locally observable CBR. This limit was to be deduced from the apparent agreement within observable limits of the empirical CBR spectrum with that for a pure blackbody law, which represented the observational situation prior to the report of Woody and Richards. The completion of this research showed that the predicted effect was a distortion of the spectrum toward an earlier and higher rise followed by a subsequent earlier decline, in very good agreement with the reported measurements. The overall level of isotropic angular momentum is here a free parameter, up to an energy-dependent limit, but any positive level displaces the spectrum in the observed direction; and a level of the order of 10% the kinematically admissible maximum level is quite sufficient to produce a distortion of the order of that observed.

Many conceivable mechanisms could be involved in the association of an excess of isotropic angular momentum with local conditions near the point of observation of the CBR, on a variety of scales, and it would be dubiously scientific to speculate on them. It should however be noted that if galaxies are typical sites for such an excess, then observers (presumably located in galaxies) would tend to observe the effects throughout the universe, although in intergalactic regions none would be present. With present uncertainties regarding galactic evolution, the dominant mechanism and basic time scale of production of the CBR, etc., a phenomenological and statistical standpoint appears preferable to a speculative one. As association of the Woody-Richards effect with vicinities of galaxies cannot reliably be either established or disproved at this time, but it would appear theoretically quite tenable on the bases of kinematical possibility, maximal entropy statistics, and the reported observations.

The possible association of the Woody-Richards effect with localized regions of high isotropic angular momentum relative to their centers is obviously physically totally distinct from a universe wide effect permeating intergalactic space. The mathematical treatment of each of these hypotheses is however similar. The discussion by Wright[28] and in part by Goebel[29] of the latter hypothesis elaborate the evident mathematical feature of association of angular momentum with a specific origin, and in no way impugn the former physical hypothesis or the mathematical treatment of either.

The effect of observation at a point other than the center of the postulated isotropic angular momentum is readily computed and seen to be beyond observational limits of detection within a region of the order of magnitude in size of a galaxy. More specifically, if O is the origin and if P is another point, then P is obtainable from O by a rotation of S^3 that is infinitesimally a linear combination of the L_{j4} in the notation of Ref. 9; more exactly, there is a unique conformal transformation on the Einstein universe that effects this rotation and does not affect the time. It should be realized that in the flat limit in which R becomes ∞, the L_{j4} deform into the Minkowskian infinitesimal space displacements $\partial/\partial x_j$; by choosing an appropriate system of coordinates it will be no essential loss of generality to take $j=1$. Suppose then that the one-parameter group $O(s)=e^{sL_{14}}$ in the universal cover of SU(2,2) carries O into P. The effect on the observed density matrix D is to transform it by the quantum field action $\Gamma(O(s))$ corresponding to $O(s)$. Thus the density matrix as observed at P takes the form $D(s)=\Gamma(O(s))^{-1}D\Gamma(O(s))$.

The effect of this on the observed spectrum is to replace K in the density matrix $D=ce^{-\beta H-\gamma K}$, where c is a constant, by its transform under $O(s)$. Now $e^{sL_{14}}$ transforms L_{12} into $\cos s L_{12}+\sin s L_{24}$, and has comparable actions on the other L_{ij}. It follows that within terms of order s^3, K is to be replaced by $K+sA+s^2B$, where A is similar to its negative and will have vanishing expectation value in a maximally isotropic state, and B is the square of the Einstein linear momentum, or effectively the square of the frequency. Here natural units are used as earlier indicated. For small s the linear effect on the spectrum thus should vanish, and at most a second-order effect in the distance s be present. But if s is the radius of a galaxy in units of the radius of the universe, the term proportional to s^2 would appear far below the limit of measurability.

It may finally be noted that Kapusta[30] has based a criticism of the treatment[10] of the Woody-Richards anomaly on the claim that the constraint imposed was nonadditive. However, as noted in this treatment the angular momentum constraint is indeed additive; see also Sec. II. It appears quite possible that, mathematically, a constraint on the expected square of the angular momentum of the quantum field, and a constraint on the total angular momentum of the particle constituting the field, yield similar maximal entropy density matrices, in the presence of a maximal isotropy assumption, since any difference would represent physically a correlation between the free photons into which the overall maximal entropy state (representing the CBR) may be resolved. Since the context of the universe is unprecedented, and the gestation times and/or periods of the CBR and matter in the universe are not observed and can only be inferred from theoretical models, an extrapolation from various forms of chemical thermodynamics or particle theory is relatively uncertain. The argument of Sec. II together with a quasiphenomenological and statistical standpoint that recognizes the apparent randomness, isotropy, and lack of material self-interaction of the CBR, as presently developed, seems reasonable and more conservative than the elaboration of scenarios that may be highly model dependent and of relatively limited empirical falsifiability.

V. CAUSALITY AND ISOTROPY IN THE CHRONOMETRIC THEORY

The chronometric theory can be considered to be based on the Einstein universe, but at a fundamental level is invariant under the full essential group $\widetilde{SU}(2,2)$ of causal (or conformal) transformations of the Einstein universe, and not merely its isometry group, and has other special features. However, the theoretical invariance under $\widetilde{SU}(2,2)$ of the evolutionary equations is not carried over to the state of the universe, which determines a particular inertial frame in which the kinetic energy is minimal, and thereby a particular decomposition of the space-time into time $\times$ space. This represents the chronometric cosmos as an Einstein universe, which is static on the time scale of eons, and thereby makes possible a well-defined redshift-distance law, which has been systematically confronted with galaxy and quasar observations, as noted earlier. In principle the inertial frame of the chronometric cosmos evolves in time, but there is no evidence that its time scale is sufficiently rapid to be detectable in the extragalactic observations thus far made.[31] The dynamics of the inertial frame, and the relation to mass and gravitation have been considered elsewhere, and it will suffice here to consider aspects particularly relevant to the CBR.

The CBR is supposed to be the equilibrium state of free photons that have been scattered, reemitted, absorbed, etc., many times in the course of possible thousands of eons. Because of its relation to the matter in the universe, the inertial frame of the CBR would be expected to agree fairly closely with that of the matter, and indeed, while there may be some difference, it appears slight.[32] As an equilibrium state of a system that is invariant under the isometry group of the Einstein universe, it would be expected to be rotationally invariant, i.e., isotropic, at every point of the universe, to the extent that the inertial frame (of the CBR, or only slightly differently, of matter in the universe) is constant. This is of course as observed. A failure of isotropy at a significant level would be a considerable embarassment to the theory, as its existence has been to the big bang theory.

In the course of many eons all parts of the chronometric cosmos have had the time to affect one another causally, and there is every reason to expect the temperature of the CBR to be the same throughout the universe. Statistical fluctuations from place to place as well as from time to time should be predictable in accordance with the density matrix given earlier, again disregarding the apparently unmeasurably slow evolution of the inertial frame of the universe. Alternatively, the constancy of the temperature is implied by the spatial homogeneity of the density matrix, which is virtually *a priori* indicated by the invariance of the theory under the Einstein universe isometry group.

The CBR may be contrasted with the x-ray background, which may well come from discrete sources such as active galactic nuclei,[33] and develop on a wholly different time scale, similar in fact to that of the matter from which it may originate, quite possibly. Instead of a gestation period of thousands of eons, the x-ray background may develop over a period of one eon, or perhaps a few eons, the observed kink in the spectrum conceivably reflecting a loss of energy by the x-ray photons on their second and later circuits of the universe.

In keeping with the quasiphenomenological and statistical directions of this work, no hypothesis regarding the, no doubt, significantly interacting evolution of matter (apparently largely in the form of galaxies of one type or another) and of radiation (the CBR and the x-ray background) will be proposed, other than its invariance and causality implicit in the action of the essential group $\widetilde{SU}(2,2)$ of all causal transformations in the Einstein universe. Suffice it to write here, as relevant to an overall evaluation of the chronometric theory in relation to the CBR, that this theory, although nonparametric and nonevolutionary, has made numerous predictions regarding observable quantities, and these appear consistent with all clearly objective and statistically unbiased extragalactic observations.

VI. DISCUSSION

The CBR is consistent with the big bang hypothesis, but this hypothesis, or some variant thereof such as the inflationary universe, is not at all uniquely indicated by it. A temporally homogeneous universe with conservation of energy and consequent attainment of a large-scale equilibrium state of the universe would appear to be scientifically a relatively economical explanation of an observed approximately Planck law CBR. The nonvanishing coupling between matter and radiation, irrespective of the precise details of its mechanisms and strength, would be expected to thermalize the radiation in the course of the unlimited time that would be available for this process.

The chronometric cosmological theory is temporally homogeneous (nonevolutionary) and is quite consistent with systematic red-shift and other extragalactic observations; and it is *a priori* quite conceivable that there may be other such theories based on the Einstein universe. Unless it can be shown that a temporally homogeneous universe is not physically sustainable, and this has not been possible even in the specific, nonparametric case of the chronometric cosmology, a claim for the big bang theory that it is *the* natural or logical explanation for the CBR and its apparently Planck law spectrum would appear untenable.

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[1]See, e.g., D. N. Schramm and R. V. Wagoner in Annu. Rev. Nucl. Sci. 27, 37 (1977), who write (p. 41) "... the primary reason for believing that our universe did emerge from [a big bang] remains the 3-K background radiation," etc.; R. Weiss, Annu. Rev. Astron. Astrophys. 18, 489 (1980), who writes (p. 489) "... the [cosmic background] radiation satisfies almost beyond reasonable expectations the simple hypothesis that is a remnant of a ... primeval explosion," etc.

[2]See J. F. Nicoll *et al.*, Proc. Natl. Acad. Sci. USA 77, 6275 (1980); J. F. Nicoll and I. E. Segal, Astron. Astrophys. 115, 398 (1982); 118, 180 (1983); I. E. Segal, Astron. Astrophys. 123, 151 (1983) and literature cited therein.

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[4]A. Einstein, Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 1, 142 (1917).

[5]Y. Choquet-Bruhat *et al.*, J. Funct. Anal. (to be published).

[6]J. L. Tits, *Colloque sur la Théorie de la Relativité* (Centre Belge Recherches Mathematiques, C.N.R.S., Brussels, 1960), p. 107.

[7]See C. Møller, *The Theory of Relativity,* 2nd ed. (Clarendon, Oxford, 1972).

[8]For Maxwell's equations, infinitesimal conformal invariance in Minkowski space was first shown by E. Cunningham, Proc. Math. Soc. London 8, 77 (1910); H. Bateman, *ibid.* 8, 223 (1910). Such conformal aspects were reinterpreted and extended by H. Weyl in *Space, Time, and Matter* (Blackie and Son, Glasgow, 1922). Global invariance of wave equations on conformal space-time, i.e., the conformal compactification of Minkowski space, was treated by O. Veblen, Proc. Natl. Acad. Sci. USA 19, 462 (1933); 19, 503 (1933). Global causality of the universal cover of conformal space, which is conformal to the Einstein universe, was shown by H. Rudberg, Thesis, University of Uppsala, Uppsala, Sweden, 1958 (unpublished). Infinitesimal unitarity and in part global unitarity for massless wave equations on Minkowski space was shown by L. Gross, J. Math. Phys. 5, 687 (1964). A comprehensive treatment of unitarity and related aspects of conformally invariant wave equations on Minkowski space was given by H. P. Jakobsen and M. Vergne, J. Funct. Anal. 24, 52 (1977). General fields on the universal cosmos, to which the Einstein universe is conformal were treated by S. M. Paneitz and I. E. Segal, J. Funct. Anal. 47, 78 (1982); 49, 335 (1982), showing in particular that all solutions of the Maxwell equations on the Einstein universe are invariant under the center of the conformal group of this space-time, and hence in particular periodic in time with period 2π. These authors together with Y. Choquet-Bruhat showed further that every finite-energy solution of the Yang-Mills equations on Minkowski space extends uniquely to a global solution on the Einstein universe, provided Cauchy data are slightly regular and small at spatial infinity; similarly, every normalizable solution of Maxwell's equations in vacuum on Minkowski space extends uniquely to a global solution on the Einstein universe. A conformal imbedding of Minkowski space into the Einstein universe was given by R. Penrose in *Relativity, Groups, and Topology,* edited by C. DeWitt and B. DeWitt (Blackie and Son, Glasgow, 1954); this represents an explicit parametrization of the canonical imbedding of Minkowski space into conformal space, given in Weyl (*loc. cit.*) in a variant of the form used by Liouville in a similar connection, and in manifestly invariant form by Veblen.

[9]See I. E. Segal, *Mathematical Cosmology and Extragalactic Astronomy* (Academic, New York, 1976), and literature cited here.

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[13]See, e.g., R. J. Bontz, R. H. Price, and M. P. Haugan, Astrophys. J. 246, 592 (1981); J. Negroponte, M. Rowan-Robinson,

and J. Silk, *ibid.* 248, 38 (1981). See also M. Alexanian and B. Grinstein, Phys. Rev. Lett. 44, 359 (1980).

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[33]See, e.g., T. Maccacaro *et al.*, Astrophys. J. 266, L73 (1983), and literature cited there.