Propagation of gravitational waves in a magnetized plasma

P. G. Macedo* and A. H. Nelson
Department of Applied Mathematics and Astronomy, University College, Cardiff, United Kingdom
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The propagation of gravitational waves parallel and perpendicular to a magnetic field in a collisionless plasma is considered. In the parallel case weak cyclotron damping of the gravitational waves exists, while in the perpendicular case there is coupling between gravitational and electromagnetic waves due to the generation of currents by the gravitational wave.

I. INTRODUCTION

In this paper we study the propagation of gravitational waves in a tenuous plasma containing a constant uniform magnetic field. We use a kinetic description for the plasma, and point out the existence of small cyclotron damping of the gravitational waves, as well as coupling with electromagnetic waves.

Two cases are considered: the case where the wave perturbation propagates in the direction parallel to the magnetic field and the case of the perpendicular propagation. There have been several studies using a kinetic theory formulation of the propagation of gravitational waves in material media,[1-3] as well as studies of the coupling between gravitational and electromagnetic waves in a magnetized vacuum[4,5] and near charged black holes,[6] but to our knowledge there has been no analysis of the propagation of these waves in a magnetized plasma.

As well as the coupling found in a magnetized vacuum we find another type of coupling, which involves the generation of electric currents in the plasma by the perturbation in the charged-particle trajectories due to the gravitational wave.

II. THE FORMALISM

We consider a tenuous collisionless plasma containing a constant uniform magnetic field $\vec{B}_0$.

We describe the plasma by a particle distribution function from each species, labeled with index j ($j=i,e$ for the ions and electrons, respectively), viz., $f_j(x^\alpha,p^\alpha)$, where x^α and p^α are the four-position and four-momentum coordinates in phase space. The particles move in the fields described by the metric $g_{\mu\nu}(x^\alpha)$ and the electromagnetic field tensor $F_{\mu\nu}(x^\alpha)$, which together with the distribution functions obey the set of coupled Einstein, Vlasov, and Maxwell equations.

In the unperturbed state, we assume the characteristic length of the background curvature (due to the energy-momentum of the plasma plus the unperturbed electromagnetic field) to be much larger than the wavelength (λ) of the waves considered, which means that for the purposes of this study we can consider the unperturbed space to be flat, i.e.,

$$g^{(0)}_{\mu\nu}=\eta_{\mu\nu}\ , \tag{1}$$

where $n_{\mu\nu}$ is the Minkowski metric.

Using a reference frame where the unperturbed magnetic field $\vec{B}_0$ is in the xz plane, and makes an angle θ with the z direction, the unperturbed Maxwell tensor will be

$$F_{(0)\mu\nu}=c\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & -B_0\cos\theta & 0\\ 0 & B_0\cos\theta & 0 & -B_0\sin\theta\\ 0 & 0 & B_0\sin\theta & 0\end{bmatrix}. \tag{2}$$

We also assume the unperturbed distribution functions are Maxwellian, i.e.,

$$f_0=Ae^{-\beta_\alpha p^\alpha}\ , \tag{3}$$

where

$$\beta_\alpha=\frac{U_\alpha}{kT}\ .$$

U_α is the four-velocity of the rest frame of the plasma, k is Boltzmann's constant, and T is the temperature of the plasma.

If we consider the plasma to be perturbed slightly, we can write

$$f=f_0+\bar{f},\ \ g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\ ,$$

$$F_{\mu\nu}=F_{(0)\mu\nu}+\bar{F}_{\mu\nu}\ ,$$

where the overbar denotes a perturbation, and we use the usual symbol $h_{\mu\nu}$ for the perturbation of the metric.

Einstein's weak-field equations, using a Lorentz gauge, are then

$$\Box^2 h_{\mu\nu}=-\frac{8\pi G}{c^4}T_{\mu\nu}\ , \tag{4}$$

where

$$T^{\mu\nu}=T^{\mu\nu}_{(\text{em})}+T^{\mu\nu}_{(\text{matter})}=g_{\alpha\beta}F^{\alpha\mu}F^{\beta\nu}-\tfrac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$$
$$+\sum_j\int_P\frac{f_j^*}{m_j}p^\mu p^\nu d^4p\ , \tag{5}$$

and where $\int_P$ denotes integration over all the four-momentum space and

$$f_j^*(x^\alpha,p^i)=f_j(x^\alpha,p^\alpha)\delta(p^\mu p_\mu-m_j{}^2c^2)\ . \tag{6}$$

Here m_j and e_j are, respectively, the rest mass and charge of the j-species particles. Maxwell's equations are

$$\frac{\partial}{\partial x^{\mu}}(\sqrt{-g}F^{\mu\nu})=-\frac{\sqrt{-g}}{c}J^{\nu}$$
$$=\frac{\sqrt{-g}}{c}\sum_j e_j \int_P \frac{p^{\nu}}{m_j} f_j^* d^4p \qquad (7)$$

and

$$\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}+\frac{\partial F_{\lambda\mu}}{\partial x^{\nu}}+\frac{\partial F_{\nu\lambda}}{\partial x^{\mu}}=0 , \qquad (8)$$

where $g=\det g_{\mu\nu}=-1+O(h^2)$ in the transverse traceless (TT) gauge which we will use.[7,8] Note that provided the phase velocity of the wave in the plasma is only slightly different from c, the use of the TT gauge is a good approximation.[7]

Finally Vlasov's equations are

$$p^{\alpha}\frac{\partial f_j}{\partial x^{\alpha}}+\left[\frac{e_j}{c}F^{\alpha}{}_{\beta}p^{\beta}-\Gamma^{\alpha}{}_{\beta\gamma}p^{\beta}p^{\gamma}\right]\frac{\partial f_j}{\partial p^{\alpha}}=0 , \qquad (9)$$

where $\Gamma^{\alpha}{}_{\beta\gamma}$ is the Christoffel symbol.

There have been many[9–13] studies of the propagation of electromagnetic waves in magnetized plasmas using a kinetic description, including the effects of both Landau and cyclotron dampings. They fall into two categories, those using a method due to Landau which involves a Fourier transform in space coordinates and a Laplace transform in time to consider the initial-value problem,[9,12] and those due to van Kampen which uses a "method of stationary solutions," which essentially involves a Fourier analysis in both space and time coordinates but which can solve the initial-value problem as well.[10,11,13]

In this paper we use this second method, which seems to us more appropriate in that it retains the symmetry between space and time.

If we Fourier analyze the solution of the linearized equations with respect to all four-coordinates x^{α}, then one Fourier component of the solution will be

$$\begin{aligned}\bar{f}_j&=g_j(p^{\alpha})e^{ik_{\alpha}x^{\alpha}} ,\\ \bar{F}_{\mu\nu}&=c_{\mu\nu}e^{ik_{\alpha}x^{\alpha}} ,\\ \bar{h}_{\mu\nu}&=e_{\mu\nu}e^{ik_{\alpha}x^{\alpha}} ,\end{aligned} \qquad (10)$$

$c_{\mu\nu}$ and $e_{\mu\nu}$ being constants, corresponding to a plane wave. We assume (without loss of generality) that the plane wave propagates in the $x^3\equiv z$ direction, i.e.,

$$k^{\alpha}=(k^0,0,0,k^3) . \qquad (11)$$

$c_{\mu\nu}$ has the components

$$c_{\mu\nu}=\begin{bmatrix}0 & \tilde{E}_x & \tilde{E}_y & \tilde{E}_z\\ -\tilde{E}_x & 0 & -c\tilde{B}_z & c\tilde{B}_y\\ -\tilde{E}_y & c\tilde{B}_z & 0 & -c\tilde{B}_x\\ -\tilde{E}_z & -c\tilde{B}_y & c\tilde{B}_x & 0\end{bmatrix} , \qquad (12)$$

where $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ are the amplitudes of the perturbation in the electric and magnetic fields.

Also, using the Lorentz-gauge condition $h^{\mu\nu}{}_{,\nu}=0$ and since we can, to the approximation of this analysis, use the TT gauge conditions, $e_{\mu\nu}$ reduces to an approximate form

$$e_{\mu\nu}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & A_+ & A_\times & 0\\ 0 & A_\times & -A_+ & 0\\ 0 & 0 & 0 & 0\end{bmatrix} . \qquad (13)$$

Linearizing Eqs. (5), (7), (8), and (9) with respect to the perturbed quantities, the first-order equations in the perturbations are then

$$[(k_3)^2-(k_0)^2]h_{\mu\nu}=-\frac{8\pi G}{c^4}\left[\sum_j \frac{\eta_{\mu\alpha}\eta_{\nu\beta}}{m_j}\int_P f_j^* p^{\alpha}p^{\beta}d^4p+\bar{F}_{\mu\alpha}F_{(0)\nu}{}^{\alpha}+F_{(0)\mu}{}^{\alpha}\bar{F}_{\nu\alpha}\right.$$
$$\left.-\tfrac{1}{2}\eta_{\mu\nu}\bar{F}_{\alpha\beta}F_{(0)}^{\alpha\beta}+h^{\alpha\beta}F_{(0)\mu\alpha}F_{(0)\nu\beta}-\tfrac{1}{2}\eta_{\mu\nu}h^{\alpha\beta}F_{(0)\alpha\gamma}F_{(0)\beta}{}^{\gamma}-\tfrac{1}{4}h_{\mu\nu}F_{(0)\alpha\beta}F_{(0)}^{\alpha\beta}\right] , \qquad (14)$$

$$i(k_0\eta^{0\alpha}+k_3\eta^{3\alpha})(\eta^{\nu\beta}\bar{F}_{\alpha\beta}+h^{\nu\beta}F_{(0)\alpha\beta})=-\sum_j \frac{e_j}{cm_j}\int_P f_j^* p^{\nu}d^4p , \qquad (15)$$

$$k^{\lambda}\bar{F}_{\mu\nu}+k^{\nu}\bar{F}_{\lambda\mu}+k^{\mu}\bar{F}_{\nu\lambda}=0 , \qquad (16)$$

and

$$i(k_0p^0+k_3p^3)\bar{f}_j+\frac{e_j}{c}F_{(0)}{}^{\alpha}{}_{\beta}p^{\beta}\frac{\partial\bar{f}_j}{\partial p^{\alpha}}=\left[\Gamma_{\beta}{}^{\alpha}{}_{\gamma}p^{\beta}p^{\gamma}-\frac{e_j}{c}(h^{\alpha\sigma}F_{(0)\sigma\beta}+\eta^{\alpha\sigma}\bar{F}_{\sigma\beta})\right]\frac{\partial f_{oj}}{\partial p^{\alpha}} . \qquad (17)$$

Owing to the axial symmetry of the physics around the direction of the unperturbed magnetic field, we make a coordinate transformation in p space to cylindrical coordinates $p_0, p_{\|}, p_{\perp}$, and ϕ, i.e.,

$$\begin{aligned}p_1&=p_{\|}\sin\theta+p_{\perp}\cos\theta\cos\phi ,\\ p_2&=p_{\perp}\sin\phi ,\\ p_3&=p_{\|}\cos\theta-p_{\perp}\sin\theta\cos\phi .\end{aligned} \qquad (18)$$

Using (18), (12), and (13) in (14), (15), (16), and (17), we obtain the set of equations

$$A_+=\frac{\chi}{[(k_0)^2-(k_3)^2]+\frac{3}{2}\chi c^2B_0{}^2\cos^2\theta}\left[\sum_j\int_P\frac{g_j^*}{m_j}(p_{\|}\sin\theta+p_\perp\cos\theta\cos\phi)^2d^4p-c^2B_0\tilde{B}_x\sin\theta\right], \tag{19}$$

$$A_\times=\frac{\chi}{[(k_0)^2-(k_3)^2]+\chi c^2B_0{}^2(\frac{3}{2}\cos^2\theta+\frac{1}{2}\sin^2\theta)}\left[\sum_j\int_P\frac{g_j^*}{m_j}(p_{\|}\sin\theta+p_\perp\cos\theta\cos\phi)p_\perp\sin\phi\,d^4p-c^2B_0\tilde{B}_y\sin\theta\right], \tag{20}$$

$$\tilde{B}_z=0, \tag{21}$$

$$k_3\tilde{E}_x=k_0c\tilde{B}_y, \tag{22}$$

$$k_3\tilde{E}_y=-k_0c\tilde{B}_x, \tag{23}$$

$$\tilde{E}_x=-\frac{k_0}{(k_0)^2-(k_3)^2}\left[i\sum_j\frac{e_j}{cm_j}\int_P g_j^*(p_{\|}\sin\theta+p_\perp\cos\theta\cos\phi)d^4p-ck_3B_0A_\times\sin\theta\right], \tag{24}$$

$$\tilde{E}_y=-\frac{k_0}{(k_0)^2-(k_3)^2}\left[i\sum_j\frac{e_j}{cm_j}\int_P g_j^*p_\perp\sin\phi\,d^4p+ck_3B_0A_+\sin\theta\right], \tag{25}$$

$$\tilde{E}_z=\frac{i}{ck_0}\sum_j\frac{e_j}{m_j}\int_P g_j^*(p_{\|}\cos\theta-p_\perp\sin\theta\cos\phi)d^4p, \tag{26}$$

$$i[k_0p^0+k_3(p_{\|}\cos\theta-p_\perp\sin\theta\cos\phi)]g_j+e_jB_0\frac{\partial g_j}{\partial\phi}=-\psi_j(p^\alpha), \tag{27}$$

where we have used (21) in (19) and

$$\chi=\frac{8\pi G}{c^4},\quad g_j^*=g_j\delta(p^\alpha p_\alpha-m_j{}^2c^2),\quad d^4p=p_\perp dp^0d\phi dp_{\|}dp_\perp$$

since the unperturbed distribution functions f_{0j} are assumed to be isotropic and we take $U^\alpha=(-c,0,0,0)$, it follows that

$$\frac{\partial f_{0j}}{\partial p^1}=\frac{\partial f_{0j}}{\partial p^2}=\frac{\partial f_{0j}}{\partial p^3}=0. \tag{28}$$

Therefore the only nonvanishing component of $\partial f_{0j}/\partial p^\alpha$ is $\partial f_{0j}/\partial p^0$, so that the function $\psi_j(p^\alpha)$ appearing on the right-hand side of (27), which is the third term of the left-hand side of (17), becomes

$$\psi_j(p^\alpha)=\Bigg\{\frac{e_j}{c}[\tilde{E}_x(p_{\|}\sin\theta+p_\perp\cos\theta\cos\phi)+\tilde{E}_yp_\perp\sin\phi+\tilde{E}_z(p_{\|}\cos\theta-p_\perp\sin\theta\cos\phi)]$$
$$+\frac{ik_0}{2}\{A_+[(p_{\|}\sin\theta+p_\perp\cos\theta\cos\phi)^2-p_\perp{}^2\sin^2\phi]+2A_xp_\perp\sin\phi(p_{\|}\sin\theta+p_\perp\cos\theta\cos\phi)\}\Bigg\}\frac{\partial f_{0j}}{\partial p^0}. \tag{29}$$

Now, following Bernstein,[9] Vlasov's equation (27) can be integrated by dividing it by eB_0 and multiplying by the integrating factor

$$G_j(p^\alpha)=\exp\left[i\left[\frac{k_0p^0+k_3p_{\|}\cos\theta}{e_jB_0}\phi-\frac{k_3p_\perp\sin\theta}{e_jB_0}\sin\phi\right]\right] \tag{30}$$

so that it becomes

$$\frac{\partial}{\partial\phi}(G_jg_j)=-\frac{G_j\psi_j}{e_jB_0} \tag{31}$$

and therefore the solution of Vlasov's equation is

$$g_j=-\frac{1}{e_jB_0G_j}\int G_j\psi_jd\phi. \tag{32}$$

Now, substituting (32) into (19), (20), and (24)–(26), we obtain a set of coupled equations in A_+, $A_\times$, $\tilde{E}_x$, $\tilde{E}_y$, and $\tilde{E}_z$ which can be written as

$$D\begin{bmatrix}A_+\\A_\times\\\tilde{E}_x\\\tilde{E}_y\\\tilde{E}_z\end{bmatrix}=0, \tag{33}$$

where D is a 5×5 matrix. From the determinant of D we obtain the dispersion relation of the waves, and hence any possible damping, while Eq. (33) yields the coupling between the gravitational and electromagnetic waves for each mode.

III. CASE OF PARALLEL PROPAGATION

When the wave travels parallel to the unperturbed magnetic field $\vec{B}_0$ (i.e., $\theta=0$), Eqs. (19), (20), and (24)–(26) become

$$A_+=\alpha\sum_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}\frac{g_j^*}{m_j}p_\perp{}^3\cos^2\phi\, d\phi\, dp_\perp dp_{\|}dp^0 , \tag{34}$$

$$A_\times=\alpha\sum_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}\frac{g_j^*}{m_j}p_\perp{}^3\frac{\sin2\phi}{2}d\phi\, dp_\perp dp_{\|}dp^0 , \tag{35}$$

$$\widetilde{E}_x=-i\beta\sum_j e_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}\frac{g_j^*}{cm_j}p_\perp{}^2\cos\phi\, d\phi\, dp_\perp dp_{\|}dp^0 , \tag{36}$$

$$\widetilde{E}_y=-i\beta\sum_j e_j \int_{\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}\frac{g_j^*}{cm_j}p_\perp{}^2\sin\phi\, d\phi\, dp_\perp dp_{\|}dp^0 , \tag{37}$$

$$\widetilde{E}_z=-i\gamma\sum e_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}\frac{g_j^*}{cm_j}p_\perp p_{\|}d\phi\, dp_\perp dp_{\|}dp^0 , \tag{38}$$

where

$$\alpha=\frac{\chi}{(k_0)^2-(k_3)^2+\frac{3}{2}\chi c^2B_0{}^2} ,\qquad \beta=\frac{k_0}{(k_0)^2-(k_3)^2},\ \ \gamma=-\frac{1}{k_0} , \tag{39}$$

and from (29), (30), and (32) we have

$$g_j=i\left\{\frac{e_j}{c}\left[\frac{\widetilde{E}_z}{\xi_j}p_{\|}+\frac{p_\perp}{2}\left[\frac{\widetilde{E}_l}{\zeta_j}e^{i\phi}+\frac{\widetilde{E}_r}{\eta_j}e^{-i\phi}\right]\right] + i\frac{k_0}{4}p_\perp{}^2\left[\frac{A_l}{\mu_j}e^{2i\phi}+\frac{A_r}{\nu_j}e^{-2i\phi}\right]\right\}\frac{\partial f_{0j}}{\partial p^0} . \tag{40}$$

In Eq. (40) we have used

$$\widetilde{E}_l=\widetilde{E}_x-i\widetilde{E}_y,\ \ \widetilde{E}_r=\widetilde{E}_x+i\widetilde{E}_y , \tag{41}$$

$$A_l=A_+-iA_\times,\ \ A_r=A_++iA_\times , \tag{42}$$

with the subscripts r and l denoting right and left circularly polarized modes, respectively, and

$$\begin{aligned}&\xi_j=k_0p^0+k_3p^3 ,\ \ \zeta_j=k_0p^0+k_3p^3+e_jB_0 ,\\ &\eta_j=k_0p^0+k_3p^3-e_jB_0 ,\\ &\mu_j=k_0p^0+k_3p^3+2e_jB_0,\ \ \nu_j=k_0p^0+k_3p^3-2e_jB_0 .\end{aligned} \tag{43}$$

If now we assume the plasma to be nonrelativistic (i.e., the random velocity of a particle $v \ll c$), we can write

$$p_\perp=m_jv_\perp,\ \ p_{\|}=m_jv_{\|},\ \ p^0=cm_j .$$

Using k for k_3, and defining the wave frequency $\omega=-k_0c$, we can define the new parameters, $u=\omega/k$ (phase velocity of the waves) and $u_{Lj}=\omega_{Lj}/k$, where ω_{Lj} is the Larmor frequency of species j, i.e.,

$$\omega_{Lj}=\frac{e_jB_0}{m_j} .$$

Equations (43) then become

$$\begin{aligned}&\xi_j=km_j(v_{\|}-u),\ \ \zeta_j=km_j(v_{\|}-u+u_{Lj}) ,\\ &\eta_j=km_j(v_{\|}-u-u_{Lj}) ,\\ &\mu_j=km_j(v_{\|}-u+2u_{Lj}) ,\\ &\nu_j=km_j(v_{\|}-u-2u_{Lj}) .\end{aligned} \tag{44}$$

In the nonrelativistic case, the Maxwellian distribution functions f_{0j} take the usual form

$$f_{0j}=n_j\left[\frac{m_j}{2\pi\tau}\right]^{3/2}\exp\left[-m_j\frac{(v_{\|}{}^2+v_\perp{}^2)}{2\tau}\right] \tag{45}$$

with

$$\tau=kT ,$$

where n_j is the number density of particles of species j. Hence the derivative of f_{0j} with respect to p^0 is

$$F_{0j}=\frac{\partial f_{0j}}{\partial p^0} =-\frac{n_jc}{\tau}\left[\frac{m_j}{2\pi\tau}\right]^{3/2}\exp\left[-m_j\frac{(v_{\|}{}^2+v_\perp{}^2)}{2\tau}\right] . \tag{46}$$

Substituting (46) into (40) and taking account of the function $f_{0j}(v_{||},v_\perp)$ being even both in $v_\perp$ and $v_{||}$, the integrals over some of the terms in Eqs. (34)—(38) vanish and we are left with a system of equations whose matrix D is quasidiagonal, i.e.,

$$D=\begin{bmatrix} D_{11} & D_{12} & 0 & 0 & 0 \\ D_{21} & D_{22} & 0 & 0 & 0 \\ 0 & 0 & D_{33} & D_{43} & 0 \\ 0 & 0 & D_{34} & D_{44} & 0 \\ 0 & 0 & 0 & 0 & D_{55} \end{bmatrix} . \qquad (47)$$

Its form shows that there is no coupling between the electromagnetic and gravitational modes. This is due to the fact that a gravitational wave propagating along the direction of the unperturbed magnetic field does not generate electric currents in the plasma.

If we use the circularly polarized modes $\widetilde{E}_r$, $\widetilde{E}_l$, A_r, and A_l, then we have a diagonal matrix D', i.e., the system of equations becomes

$$\begin{bmatrix} D'_1 & 0 & 0 & 0 & 0 \\ 0 & D'_2 & 0 & 0 & 0 \\ 0 & 0 & D'_3 & 0 & 0 \\ 0 & 0 & 0 & D'_4 & 0 \\ 0 & 0 & 0 & 0 & D'_5 \end{bmatrix} \begin{bmatrix} A_r \\ A_l \\ \widetilde{E}_r \\ \widetilde{E}_l \\ \widetilde{E}_z \end{bmatrix} = 0 . \qquad (48)$$

The components of the matrix D'_i contain singular integrals which give rise to Landau damping (D'_5) and cyclotron damping [D'_i $(i=1,\ldots,4)$]. The electromagnetic modes have been considered by Van Kampen and Felderhoff, among others,[9–13] and we consider only the gravitational modes.

Owing to the similarity of the two circularly polarized modes, we calculate explicitly the frequency and damping for A_r only, and simply quote the result for A_l.

Substituting (40) into (34) and (35) and then using (42), we obtain

$$A_l=\alpha \sum_j \frac{\omega}{8cm_j} \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{2\pi} \frac{A_l}{\mu_j} F_{0j} p_\perp{}^5 d\phi \, dp_\perp dp_{||} , \qquad (49)$$

where only nonzero terms are retained.

If we define $H_j(v_{||})$ by

$$H_j(v_{||}) = \int_0^{\infty} F'_{0j}(v_{||},v_\perp) v_\perp{}^5 dv_\perp , \qquad (50)$$

where

$$F'_{0j}(v_{||},v_\perp) = F_{0j}(p_{||},p_\perp) ,$$

using this, together with α defined in (39) in (49), we obtain the dispersion relation from this mode by canceling A_l,

$$c^2k^2-\omega^2=\chi\left[\tfrac{3}{2}c^4B_0{}^2+\frac{\pi c\omega}{2}\sum_j m_j{}^4\int_{-\infty}^{\infty}\frac{H_j}{\mu_j}dv_{||}\right] . \qquad (51)$$

Since the integral in (51) is singular, we have to analytically continue the integrand H_j/μ_j into the region where the poles exist in order to employ contour integration.[11]

Equivalently we can use the following representation of the integrand (see Asseo *et al.*[3]):

$$\frac{H_j}{\mu_j} = P\frac{H_j}{\mu_j} + i\pi\delta(\mu_j) , \qquad (52)$$

where P denotes the principal part. The integral in (51) then becomes

$$\int_{-\infty}^{\infty}\frac{H_j}{\mu_j}dv_{||} = P\int_{-\infty}^{\infty}\frac{H_j(v_{||})}{km_j(v_{||}-u+2u_{Lj})}dv_{||} + i\pi H_j(u-2u_{Lj}) . \qquad (53)$$

The Hilbert transform of $H_{0j}(v_{||})$ is defined by

$$H_j^*(w) = \frac{1}{i\pi}P\int_{-\infty}^{\infty}\frac{H_j(v_{||})}{v_{||}-w}dv_{||} \qquad (54)$$

and the dispersion relation (51) can then be written as

$$c^2-u^2=\chi\left\{\tfrac{3}{2}\frac{c^4B_0{}^2}{k^2}+\sum_j\frac{i\pi^2cm_j{}^3}{2k^2}u[H_j^*(u-2u_{Lj})+H_j(u-2u_{Lj})]\right\} . \qquad (55)$$

We assume that the modes are slowly decaying so that

$$u=u_r+iu_i \qquad (56)$$

with $u_i \ll u_r$. The dispersion relation can then be written in the approximate form

$$c^2-u_r{}^2-2iu_iu_r=\chi\left\{\tfrac{3}{2}\frac{c^4B_0}{k^2}+\sum_j\frac{i\pi^2cm_j{}^3}{2k^2}u_r[H_j^*(u_r-2u_{Lj})+H_j(u_r-2u_{Lj})]\right\} \qquad (57)$$

and from the real and imaginary parts we get for the right circularly polarized mode

$$c^2-u_r{}^2=\frac{4\pi G}{k^2}\left[3B_0{}^2+\frac{i\pi}{c^3}u_r\sum_j m_j{}^3H_j^*(u_r-2u_{Lj})\right] \qquad (58)$$

and

$$u_i=-\frac{2\pi^2G}{k^2c^3}\sum_j m_j{}^3H_j(u_r-2u_{Lj}) . \qquad (59)$$

For the left circularly polarized mode, A_l, the same procedure yields

$$c^2-u_r{}^2=\frac{4\pi G}{k^2}\left[3B_0{}^2+\frac{i\pi}{c^3}u_r\sum_j m_j^3H_j^*(u_r+2u_{Lj})\right], \tag{60}$$

$$u_i=-\frac{2\pi^2 G}{k^2c^3}\sum_j m_j{}^3H_j(u_r+2u_{Lj}) . \tag{61}$$

We find therefore that the normal modes (which in a vacuum were A_+ and $A_\times$) have been changed due to the presence of the magnetic field into the circularly polarized modes A_r and A_l, propagating at slightly different phase velocities [compare (58) and (60)], and both slightly damped.

From Eqs. (59) and (61) we can calculate the strength of the damping. Evaluating the integral (50) using (46), we obtain

$$H_j(u_r\pm 2u_{Lj}) =-\frac{nc}{\tau}\left[\frac{2\tau}{\pi m_j}\right]^{3/2}\exp\left[-\frac{m_j(u_r\pm 2u_{Lj})^2}{2\tau}\right] . \tag{62}$$

To obtain resonant damping we require $u_r=\mp 2u_{\perp j}$ in order that the exponential in (62) is not very small, i.e., resonant damping for left and right circular polarizations occurs only for antiparallel magnetic field directions. In both cases the resonant damping factor is given by

$$u_i\simeq -7.8\frac{Gn\tau^{1/2}m_j{}^{3/2}}{\omega_{Lj}{}^2}c^2$$

or, in CGS units we obtain the following damping time for the waves:

$$t_D\sim\frac{c}{\omega_{Lj}u_i}\sim 7\times 10^4\frac{\omega_{Lj}}{nT^{1/2}m_j{}^{3/2}}\ \text{sec} .$$

The damping is almost negligible; for instance, in the interstellar medium ($B_0=10^{-6}$ G, $n=10^4$ cm^{-3}), the damping time will be of the order of 10^{21} sec for waves with frequency near the ion resonance, and 10^{34} sec for waves near the electron resonance.

IV. CASE OF PERPENDICULAR PROPAGATION

In the case when the wave perturbations travel in a direction perpendicular to the unperturbed magnetic field $\vec{B}_0(\theta=\pi/2)$, Eqs. (19)—(27) become

$$A_+=\alpha''\widetilde{E}_y+\alpha'\sum_j\frac{1}{m_j}\int_0^{2\pi}\int_0^\infty\int_{-\infty}^\infty p_{\|}{}^2p_\perp g_j^*\,dp_{\|}dp_\perp d\phi , \tag{63}$$

$$A_\times=-\delta''\widetilde{E}_x+\delta'\sum_j\frac{1}{m_j}\int_0^{2\pi}\int_0^\infty\int_{-\infty}^\infty p_{\|}p_\perp{}^2\sin\phi\, g_j^*\,dp_{\|}dp_\perp d\phi , \tag{64}$$

$$\widetilde{E}_x=\beta''A_\times-i\beta\sum_j\frac{e_j}{cm_j}\int_0^{2\pi}\int_0^\infty\int_{-\infty}^\infty p_{\|}p_\perp g_j^*\,dp_{\|}dp_\perp d\phi , \tag{65}$$

$$\widetilde{E}_\gamma=-\beta''A_+-i\beta\sum_j\frac{e_j}{cm_j}\int_0^{2\pi}\int_0^\infty\int_{-\infty}^\infty p_\perp{}^2\sin\phi\, g_j^*\,dp_{\|}dp_\perp d\phi , \tag{66}$$

$$\widetilde{E}_z=i\gamma\sum_j\frac{e_j}{cm_j}\int_0^{2\pi}\int_0^\infty\int_{-\infty}^\infty p_\perp{}^2\cos\phi\, g_j^*\,dp_{\|}dp_\perp d\phi , \tag{67}$$

with

$$\alpha'=\frac{\chi}{(k_0)^2-(k_3)^2},\quad \delta'=\frac{\chi}{(k_0)^2-(k_3)^2+\frac{1}{2}\chi c^2B_0{}^2},\quad \alpha''=\frac{ck_3B_0}{k_0}\alpha',\quad \delta''=\frac{ck_3B_0}{k_0}\delta',\quad \beta''=ck_3B_0\beta , \tag{68}$$

with g_j given by

$$g_j=\frac{\partial f_{0j}}{\partial p^0}\left[\frac{1}{mc}\left\{\frac{e}{\omega}\left[-p_{\|}i\widetilde{E}_x+\frac{kp_\perp{}^2}{2m(\omega^2-\omega_L^2)}(\omega_L\widetilde{E}_y-\omega i\widetilde{E}_z)\right]-\frac{1}{2}\left[\left[p_{\|}{}^2-\frac{p_\perp{}^2}{2}\right]A_++\frac{kp_\perp{}^2p_{\|}\omega_L}{m(\omega^2-\omega_L{}^2)}iA_\times\right]\right\}\right.$$

$$+\frac{1}{mc}\left\{\frac{2}{2}\left[\frac{kp_{\|}p_\perp}{m\omega(\omega-\omega_L)}i\widetilde{E}_x-\frac{p_\perp}{\omega-\omega_L}(\widetilde{E}_y-i\widetilde{E}_z)\right]\right.$$

$$\left.+\frac{\omega}{2}\left[\frac{kp_\perp}{2m(\omega-\omega_L)}\left[\frac{p_{\|}{}^2-p_\perp^2/2}{\omega}+\frac{p_\perp{}^2}{4(\omega-2\omega_L)}\right]A_++\frac{p_{\|}p_\perp}{\omega-\omega_L}iA_\times\right]\right\}e^{i\phi}$$

$$+\frac{1}{mc}\left\{\frac{e}{2}\left[-\frac{kp_{\|}p_\perp}{m\omega(\omega+\omega_L)}i\widetilde{E}_x-\frac{p_\perp}{\omega+\omega_L}(\widetilde{E}_y+i\widetilde{E}_z)\right]\right.$$

$$+\frac{\omega}{2}\left[\frac{kp_\perp}{2m(\omega+\omega_L)}\left[-\frac{p_{\|}^2-p_\perp^2/2}{\omega}+\frac{p_\perp^2}{4(\omega+2\omega_L)}\right]A_+ +\frac{p_{\|}p_\perp}{\omega+\omega_L}iA_\times\right]\Bigg\}e^{-i\phi}$$

$$+C_4e^{2i\phi}+C_5e^{-2i\phi}+C_6e^{3i\phi}+C_7e^{-3i\phi}\Bigg] , \tag{69}$$

where c_4 to c_7 are functions of $p_{\|}$, $p_\perp$, k, ω, A_+, $A_\times$, $\tilde{E}_x$, $\tilde{E}_y$, and $\tilde{E}_z$. If we now define the quantities $I_{mn(j)}$ as being the integrals

$$I_{mn(j)}=\int_{-\infty}^{\infty}\int_0^{\infty} v_{\|}^m v_\perp^n F_{0j}dv_{\|}dv_\perp , \tag{70}$$

with

$$F_{0j}=\frac{\partial f_{0j}}{\partial p^0} , \tag{71}$$

then, since F_{0j} are even functions on the variables $v_\perp$ and $v_{\|}$, $I_{ij(k)}$ will be zero when n is odd:

$$I_{2n+1m(j)}=0 ,$$
$$I_{2nm(j)}\neq 0,\quad m,n \text{ integers} . \tag{72}$$

If we substitute (69) into Eqs. (63)—(68) taking this fact into account, then the system of equations becomes

$$A_+=\alpha''\tilde{E}_y+2\pi\alpha'\sum_j\frac{m_j}{c}\left[-m_jI_{23j}A_+ +\frac{ek\omega_{Lj}}{\omega(\omega^2-\omega_{Lj}{}^2)}I_{23j}\tilde{E}_y-\frac{iek}{\omega^2-\omega_{Lj}{}^2}I_{23j}\tilde{E}_z\right] , \tag{73}$$

$$A_\times=-\delta''E_x-\pi\delta\sum_j\frac{m_j}{c}\left[m_j\omega^2I_{23j}A_\times-lk\frac{\omega_{Lj}}{\omega}I_{23j}\tilde{E}_x\right] , \tag{74}$$

$$\tilde{E}_x=\beta''A_\times-2\pi\beta\sum_j\frac{e_jm_j}{c^2}\left[\frac{k\omega_{Lj}}{\omega^2-\omega_{Lj}{}^2}I_{23j}A_\times+\frac{e_j}{m_j\omega}I_{21j}\tilde{E}_x\right] , \tag{75}$$

$$\tilde{E}_y=-\beta''A_+ +\pi\beta\sum_j\frac{e_jm_j}{c^2(\omega^2-\omega_{Lj}{}^2)}\left[\frac{k\omega_{Lj}(\omega^2+2\omega_{Lj}{}^2)}{\omega^2-4\omega_{Lj}{}^2}I_{23j}A_+ -\frac{2e_j}{m_j}I_{21j}(\omega\tilde{E}_y-i\omega_{Lj}\tilde{E}_z)\right] , \tag{76}$$

$$\tilde{E}_z=-i\pi\gamma\sum_j\frac{e_jm_j}{c^2(\omega^2-\omega_{Lj}{}^2)}\left[-\frac{\omega\omega_{Lj}{}^2k}{\omega^2-4\omega_{Lj}{}^2}I_{23j}A_+ +\frac{2e_j}{m_j}I_{21j}(\omega_{Lj}\tilde{E}_y-i\omega\tilde{E}_z)\right] , \tag{77}$$

with

$$I_{21j}=-\frac{nc}{2\pi m_j} ,$$
$$I_{23j}=-\frac{nc\tau}{\pi m_j{}^2} . \tag{78}$$

In this case we can write the matrix equation (33) in a different form by exchanging rows and columns in matrix D in such a way that the system can be written as

$$D''=\begin{bmatrix}A_\times\\ \tilde{E}_x\\ A_+\\ \tilde{E}_y\\ \tilde{E}_z\end{bmatrix}=0 \tag{79}$$

and in this case the matrix D'' is quasidiagonal of the form

$$D''=\begin{bmatrix}D''_{11} & D''_{12} & 0 & 0 & 0\\ D''_{21} & D''_{22} & 0 & 0 & 0\\ 0 & 0 & D''_{33} & D''_{34} & D''_{35}\\ 0 & 0 & D''_{43} & D''_{44} & D''_{45}\\ 0 & 0 & D''_{53} & D''_{54} & D''_{55}\end{bmatrix} . \tag{80}$$

From the theory of the propagation of electromagnetic waves in a magnetized plasma, we know that for waves propagating in a perpendicular direction to the unperturbed magnetic field, there are two wave modes: the ordinary wave which is a purely transverse mode linearly polarized with the wave electric field in the direction of the unperturbed magnetic field and the extraordinary wave which is an elliptically polarized mode with the plane of the electric field ellipse perpendicular to the direction of the unperturbed magnetic field.

The structure of the matrix D'' shows that there is a coupling between the $A_\times$ gravitational mode and the ordinary electromagnetic wave, and also between the A_+ mode and the extraordinary electromagnetic wave.

The two new types of coupled modes will be referred to as the gravi-electro ordinary mode and gravi-electro extraordinary mode, respectively.

Physically, the existence of each coupling is connected with the fact that the anisotropic perturbations of the distribution function induce drifts in the electrons and ions, which in turn generate an electric current and hence a perturbation in the electromagnetic field. In addition, as mentioned by Gerlach,[6] the existence of an unperturbed magnetic field on its own couples the two fields.

We can see from Eqs. (73)–(79) that for high-frequency waves ($\omega \gg \omega_{Lj}$) the contribution of the ions to the dispersion relations is negligible compared with the contribution of the electrons, since the terms due to the ions are much smaller than those due to the electrons in view of $m_i \gg m_e$.

A. The gravi-electro ordinary (GEO) modes

The dispersion relation for these modes is obtained from the condition for nontriviality of the first subsystem of Eqs. (74) and (75), i.e.,

$$\det \begin{bmatrix} D''_{11} & D''_{12} \\ D''_{21} & D''_{22} \end{bmatrix} = 0 \tag{81}$$

from now on, we will be considering only the propagation of high-frequency waves, and therefore we will use ω_L, ω_p instead of ω_{Le} and ω_{pe}.

Now, substituting for D''_{11}, D''_{12}, D''_{21}, and D''_{22} from Eqs. (74) and (75) and calculating the determinant we obtain the dispersion relation

$$(\omega^2-\omega_p{}^2-c^2k^2)(\omega^2-c^2k^2+\tfrac{1}{2}c_1\omega_L{}^2)(\omega^2-\omega_L{}^2) - \frac{c_1}{c_2}\left[(\omega^2-\omega_p{}^2-c^2k^2)\omega^2\omega_p{}^2+c_2c^2k^2(\omega^2-\omega_L{}^2)\omega_L{}^2-c^2k^2\omega_p{}^2\omega_L{}^2+\frac{2}{c_2}\frac{c^2k^2\omega_L{}^2\omega_p{}^2}{\omega^2-\omega_L{}^2}\right]=0\ , \tag{82}$$

where (ω_p = plasma frequency)

$$c_1 = 4\pi G\frac{m^2}{e^2},\quad c_2=\frac{mc^2}{\tau}\ .$$

We can obtain the ratio of the amplitudes of the perturbations $A_\times$ and $\tilde{E}_x$ from Eq. (75), viz.,

$$\frac{A_\times}{\tilde{E}_x} = \frac{e(\omega^2-\omega_p{}^2-k^2c^2)}{\tau k\omega\omega_L[c_2+2\omega_p{}^2/(\omega^2-\omega_L{}^2)]}\ . \tag{83}$$

From this we obtain the ratios of energy fluxes of the perturbed gravitational and electromagnetic fields (Landau and Lifshitz[14])

$$\frac{w_g}{w_e} = \frac{\omega^2c^2}{4G}\left[\frac{A_\times}{\tilde{E}_x}\right]^2 . \tag{84}$$

c_1 is a dimensionless number of the order of 10^{-42}, so, in order to understand better the behavior of these modes, we will expand the dispersion relation as a power series in c_1. To first order in c_1 we have two solutions for k^2 corresponding to the two different types of wave modes.

1. Quasielectromagnetic mode. To first order in c_1 its dispersion relation is given by

$$c^2k^2 = (\omega^2-\omega_p{}^2)\left\{1-c_1\frac{\omega_L{}^2}{\omega_p{}^2(\omega^2-\omega_L{}^2)}\left[(\omega^2-\omega_L{}^2)-\frac{\omega_p{}^2}{c_2}+\frac{2}{c_2{}^2}\frac{\omega_p{}^4}{\omega^2-\omega_L{}^2}\right]\right\} . \tag{85}$$

We can see from this dispersion relation that this mode behaves almost as an electromagnetic mode due to the smallness of c_1, except for the case when $\omega \sim \omega_L$ in which case the electromagnetic-gravitational coupling becomes stronger and the expansion in powers of c_1 ceases to be valid.

Using Eqs. (85) and (83) in (84) for the case of ω away from the near vicinity of ω_L and provided $\omega \gg \omega_p$, we obtain

$$\frac{w_g}{w_e} \simeq c_1\frac{\omega_L{}^2(\omega^2-\omega_p{}^2)[\omega^2-\omega_L{}^2-\omega_p{}^2/c^2-\omega_p{}^4/c_2{}^2(\omega^2-\omega_L{}^2)]^2}{\omega_p{}^4(\omega^2-\omega_L{}^2)^2[1+(2/c_2)\omega_p{}^2/(\omega^2-\omega_L{}^2)]^2}\ . \tag{86}$$

We can see from (86) that the relative energy flux of the perturbation in the gravitational field compared with the energy carried in the electromagnetic field grows when ω approaches ω_L and goes to zero when ω approaches ω_p.

Defining the coupling width (CW) for a wave mode as the width of the frequency band $\delta\omega$ for which

$$w_g/w_{\rm em} = 1 \tag{87}$$

at $\omega = \omega_{\rm Res} \pm \delta\omega/2$ (where $\omega_{\rm Res}$ is the resonance frequency) then, the coupling width for this mode is obtained from Eq. (86) and is

$$\delta\omega \sim \frac{\sqrt{c_1}}{c_2}\omega_L\ . \tag{88}$$

For example, in the interstellar medium $\omega_L \sim 10$ Hz and $C_2 \sim 10^6$, so we obtain for $\delta\omega$ a value of 10^{-25} Hz. This frequency band is too narrow for this coupling to be physically significant.

2. Quasigravitational modes. To first order in c_1, the dispersion relation for this mode is

$$k^2c^2=\omega^2+\frac{c_1}{\omega_p{}^2(\omega^2-\omega_L{}^2)}\left[\omega_L{}^2(\omega^2-\omega_L{}^2)\left[\omega^2+\frac{\omega_p{}^2}{2}\right]-\frac{\omega^2\omega_p{}^2}{c_2}(\omega_L{}^2+\omega_p{}^2)+\frac{2}{c_2{}^2}\frac{\omega^2\omega_L{}^2\omega_p{}^2}{\omega^2-\omega_L{}^2}\right]. \quad (89)$$

This mode has a dispersion relation which nearly corresponds to the dispersion relation for a gravitational wave propagating in a nondispersive medium. In general they behave almost like gravitational waves in a nondispersive medium except in a very narrow frequency band near the Larmor frequency ω_L, where the coupling is stronger.

Using (89) in (83) we obtain for this type of mode, away from the near vicinity of ω_L,

$$\frac{w_g}{w_e}=\frac{e^2c^4}{4G\tau^2}\frac{\omega_p{}^4}{\omega^2\omega_L{}^2[c_2+2\omega_p{}^2/(\omega^2-\omega_L{}^2)]^2}. \quad (90)$$

We can see from (90) that in this type of mode, the coupling is stronger where the temperature is higher. It also increases when ω approaches ω_L in the sense that the relative amplitude and therefore the relative flux of energy in the perturbation of the gravitational field decreases when ω approaches ω_L. The CW for this mode can be calculated using (90) and is the same as for the previous mode and given therefore by (88). This effect is therefore also negligible in the interstellar plasma.

B. The gravi-electro extraordinary modes

The dispersion relation for these modes is obtained from the nontriviality condition of the second subsystem of Eqs. (73), (76), and (77), viz.,

$$\det\begin{bmatrix} D''_{33} & D''_{34} & D''_{35} \\ D''_{43} & D''_{44} & D''_{45} \\ D''_{53} & D''_{54} & D''_{55}\end{bmatrix}=0. \quad (91)$$

If we now substitute for D''_{33}, D''_{34}, D''_{35}, D''_{43}, D''_{44}, D''_{45}, D''_{53}, D''_{54}, and D''_{55} from Eqs. (73), (76), and (77), and calculate the determinant we obtain the following dispersion relation for these modes:

$$(c^2k^2-\Omega)\left[\frac{\omega^2-k^2c^2}{\omega_p{}^2}-4\frac{c_1}{c_2}\right](\omega^2-\omega_L{}^2-\omega_p{}^2)(\omega^2-\omega_L{}^2)+2c^2k^2\omega_L{}^2\omega_p{}^2)\left[\frac{c_1}{c_2}+\frac{c_1}{c_2}\frac{3(\omega^2-c^2k^2)+\omega_p{}^2}{\omega^2-4\omega_L{}^2}\right]$$
$$+c^2k^2\omega_L{}^2\left[c_1(\omega^2-\omega_L{}^2)+2\frac{c_1}{c_2}\omega_p{}^2\right]\left[\frac{\omega^2-\omega_L{}^2-\omega_p{}^2}{\omega_p{}^2}+\frac{1}{c_2}\frac{\omega^2+2\omega_L{}^2-\omega_p{}^2}{\omega^2-4\omega_L{}^2}\right]=0, \quad (92)$$

where

$$\Omega=\omega^2\left[\left[1-\frac{\omega_p{}^2}{\omega^2}\right]^2-\frac{\omega_L{}^2}{\omega^2}\right]\Big/\left[1-\frac{\omega_p{}^2}{\omega^2}-\frac{\omega_L{}^2}{\omega^2}\right].$$

We obtain the ratios $\tilde{E}_y/A_+$ and $\tilde{E}_z/A_+$ from Eqs. (76) and (77), viz.,

$$\frac{\tilde{E}_y}{A_+}=\frac{mc^2\omega k\omega_L}{e(\omega^2-4\omega_L{}^2)}\frac{(3/c_2)\omega_L{}^2\omega_p{}^4+(\omega^2-\omega_L{}^2-\omega_p{}^2)[(\omega^2-\omega_L{}^2)(\omega^2-4\omega_L{}^2)+(1/c_2)(\omega^2+2\omega_L{}^2)\omega_p{}^2]}{\omega_L{}^2\omega_p{}^4+(\omega^2-\omega_L{}^2-\omega_p{}^2)[\omega^2\omega_p{}^2-(\omega^2-k^2c^2)(\omega^2-\omega_L{}^2)]} \quad (93)$$

and

$$\frac{\tilde{E}_z}{A_+}=i\frac{mc^2k\omega_L{}^2\omega_p{}^2}{e(\omega^2-4\omega_L{}^2)}\frac{3/c_2[(\omega^2-c^2k^2)(\omega^2-\omega_L{}^2)-\omega^2\omega_p{}^2]-[(\omega^2-\omega_L{}^2)(\omega^2-4\omega_L{}^2)+1/c_2(\omega^2+2\omega_L{}^2)\omega_p{}^2]}{\omega_L{}^2\omega_p{}^4+(\omega^2-\omega_L{}^2-\omega_p{}^2)[\omega^2\omega_p{}^2-(\omega^2-k^2c^2)(\omega^2-\omega_L{}^2)]}. \quad (94)$$

From these ratios, we can calculate the ratios of the energy fluxes in the gravitational and electromagnetic fields as before, viz.,

$$\frac{w_g}{w_e}=\frac{\omega^2c^2}{4G}\left[\frac{A_+{}^2}{\tilde{E}_y{}^2+\tilde{E}_z{}^2}\right]. \quad (95)$$

Now, for the modes whose dispersion relation is given by (92) we can use the same procedure as used for the GEO modes, i.e., expand in powers of c_1.

Again we have two types of modes.

1. Quasi-electromagnetic mode.

For this mode the expansion yields

$$c^2k^2=\Omega\left\{1-\frac{c_1\omega_L{}^2}{\omega^2-\omega_p{}^2}\left[\frac{\omega^2-\omega_L{}^2-\omega_p{}^2}{\omega_p{}^2}-\frac{1}{c_2}\left[2+\frac{\omega^2+2\omega_L{}^2-\omega_p{}^2}{\omega^2-4\omega_L{}^2}\right]\right.\right.$$
$$\left.\left.-\frac{2\omega_p{}^2}{c_2{}^2}\left[\frac{3\omega_p{}^2}{(\omega^2-\omega_L{}^2)(\omega^2-4\omega_L{}^2)}-\frac{\omega^2+2\omega_L{}^2-\omega_p{}^2}{(\omega^2-\omega_L{}^2-\omega_p{}^2)(\omega^2-4\omega_L{}^2)}\right]\right]\right\}. \tag{96}$$

We can see using (96) in (95) that the ratio of energy fluxes for this mode is

$$\frac{w_e}{w_g}\simeq\frac{\pi\omega_p{}^4(\omega^2-\omega_p{}^2)^2\Omega}{c_1\omega^2\omega_L{}^2(\omega^2-4\omega_L{}^2)^2[(\omega^2-\omega_p{}^2)^2-\omega^2\omega_L{}^2)]}$$
$$\times\frac{\left[(\omega^2-\omega_L{}^2-\omega_p{}^2)(\omega^2-4\omega_L{}^2)+\frac{\omega_p{}^2}{c_2}(\omega^2+2\omega_L{}^2-\omega_p{}^2)\right]^2\omega^2-\left[\omega_p{}^2(\omega^2-4\omega_L{}^2)+\frac{\omega_p{}^4}{c_2}\left[1-\frac{\omega_L{}^2+4\omega_p{}^2}{\omega^2-\omega_L{}^2-\omega_p{}^2}\right]\right]^2\omega_L{}^2}{\left[\omega^2-\omega_L{}^2-\omega_p{}^2-\frac{\omega_p{}^2}{c_2}\frac{3\omega^2-2\omega_L{}^2-\omega_p{}^2}{\omega^2-4\omega_L{}^2}-\frac{2\omega_p{}^4}{c_2{}^2}\left[\frac{3\omega_p{}^2}{(\omega^2-\omega_L{}^2)(\omega^2-4\omega_L{}^2)}-\frac{\omega^2+2\omega_L{}^2-\omega_p{}^2}{(\omega^2-\omega_L{}^2-\omega_p{}^2)(\omega^2-4\omega_L{}^2)}\right]\right]^2} \tag{97}$$

In general this mode behaves almost like a pure electromagnetic extraordinary mode except for frequencies in the near vicinity of ω_L and ω_p when the gravitational-electromagnetic coupling becomes stronger.

The CW for this mode is given by

$$\delta\omega=\frac{\sqrt{c_1}}{c_2}\frac{\omega_p{}^4}{\omega_L{}^3} \tag{98}$$

which is again negligible in the interstellar plasma.

2. *Quasi-gravitational mode.* For this mode, we obtain first order in c_1 the following dispersion relation:

$$c^2k^2=\omega^2\left\{1+\frac{c_1\omega_L{}^2}{\omega^2-\omega_p{}^2}\left[\frac{\omega^2-\omega_L{}^2-\omega_p{}^2}{\omega_p{}^2}+\frac{1}{c_2}\left[\frac{3(\omega^2-\omega_L{}^2)-\omega_p{}^2}{\omega^2-4\omega_L{}^2}-\frac{4\omega_p{}^2(\omega^2-\omega_p{}^2)}{\omega^2\omega_L{}^2}\right]-\frac{2}{c_2{}^2}\frac{\omega_p{}^2(\omega^2+2\omega_L{}^2)}{(\omega^2-\omega_L{}^2)(\omega^2-4\omega_L{}^2)}\right]\right\}. \tag{99}$$

As we can see from (99), this mode behaves as a pure gravitational wave in a nondispersive medium except for a narrow frequency band near $2\omega_L$ where the coupling again gets stronger. The energy flux is dominated by the gravitational field since substituting (99) in (95) (outside the near vicinity of ω_L, $2\omega_L$, and ω_p), we obtain

$$\frac{w_e}{w_g}\simeq\frac{\pi c_1\omega_L{}^2}{\omega_p{}^4(\omega^2-4\omega_L{}^2)^2(\omega^2-\omega_p{}^2)^2}$$
$$\times\left\{\omega^2\left[(\omega^2-\omega_L{}^2-\omega_p{}^2)(\omega^2-4\omega_L{}^2)+\frac{\omega_p{}^2}{c_2}(\omega^2+2\omega_L{}^2-\omega_p{}^2)\right]^2-\omega_L{}^2\omega_p{}^4\left[(\omega^2-4\omega_L{}^2)-\frac{2\omega_p{}^2}{c_2}\right]^2\right\}. \tag{100}$$

The CW for this mode is also given by (88) and this coupling is negligible in the interstellar plasma.

V. CONCLUSIONS

From what we have already mentioned, we conclude that gravitational waves are coupled to electromagnetic waves in a magnetized plasma and can be cyclotron damped, but the weakness of the damping as well as the narrowness of the frequency band over which the coupling is significant makes these effects physically negligible in the interstellar medium. One can therefore conclude that these effects put no constraints on the frequency bands which should be explored in order to detect gravitational waves.

However, the subject is far from being thoroughly studied since the case when the waves propagate at an arbitrary angle to the magnetic field was not considered. This, as well as the propagation of the coupled waves in weakly collisional plasmas, will be a subject for further study.

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*Present address: Departamento de Matemática, Universidad de Coimbra, Portugal.

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