PHYSICAL REVIEW D VOLUME 28, NUMBER 10 15 NOVEMBER 1983

Gravitational radiation reaction in the Newtonian limit

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(Received 18 March 1983)

The asymptotic approximation scheme based on the theory of the Newtonian limit developed in the preceding paper is applied to the gravitational radiation-reaction problem. All divergences encountered in previous approaches disappear: For any ϵ the asymptotic approximation is finite to all orders we have calculated, even beyond radiation-reaction order. This is because the divergent terms in previous work were misordered, and make finite contributions to coefficients of lower-order terms in the asymptotic expansion. The logarithmic divergences, in particular, turn up as an $\epsilon^{10}\ln\epsilon$ term in the asymptotic expansion (i.e., between 2.5 and 3 post-Newtonian order) which shows that the relativistic sequence is not C^∞ at $\epsilon=0$. This does not, however, affect the asymptotic convergence of the approximation. The radiation-reaction terms are used to calculate the period shortening of a nearly-Newtonian binary system directly from the equations of motion, avoiding the well-known difficulties associated with energy in general relativity. It is proved that the prediction derived from the standard quadrupole formula applies in the Newtonian limit. It is also shown that random data for the initial gravitational wave field do not affect the calculation of radiation reaction, even if their amplitude is of first post-Newtonian order.

I. INTRODUCTION

This paper is concerned with gravitational radiation reaction in the Newtonian limit developed in the preceding paper by myself and Schutz[1] which will be referred to as paper I. The radiation-reaction problem has become very important recently because of the discovery of the binary pulsar PSR 1913+16. The observed period shortening is believed to be entirely the result of energy loss due to the emission of gravitational radiation. The observations[2] support with very good accuracy the so-called standard quadrupole formula, which was first derived by Einstein[3] within the linearized version of general relativity. The formula was extended to the nonlinear case by Chandrasekhar and Esposito[4] by using the formal slow-motion approximation for a perfect-fluid source, and by others.[5] However, as emphasized by Ehlers *et al.*,[6] and Ehlers,[7] the standard formula has not yet been derived fully rigorously. Walker and Will[8] have critically reviewed the previous derivations of the formula. The most successful derivation to date within the formal slow-motion approximation is that by Kerlick.[9] He was able to construct a divergent-free equation of motion up to the radiation-reaction order, i.e., 2.5 post-Newtonian (PN) order but found new divergent integrals at the next and higher orders. These divergences are uncontrollable in the sense that there is no guarantee at all in his work that they may not influence the results of his calculations at radiation-reaction order, because of the lack of an error estimation. On the other hand, Christodoulou and Schmidt[10] have given an argument which indicates that the fast-motion iteration method gives an asymptotic approximation to exact solutions of general relativity. Recently the fast-motion approximation has been iterated by Damour and Deruelle[11] to a high-enough order to calculate the radiation-reaction effect for pointlike particles treated by means of the Riesz-regularization method. However, it is not clear how their method could be rigorously justified. In paper I we have developed a new asymptotic approximation to general relativity by studying a C^∞ sequence of solutions to Einstein's equation that are defined by initial data having the Newtonian scaling property; $v^i\sim\epsilon$, $\rho\sim\epsilon^2$, $p\sim\epsilon^4$, where ϵ is the parameter along the sequence. To obtain an asymptotic approximation to this sequence at $\epsilon=0$ we have defined a map from one solution in the sequence to another by identifying them at constant spatial position x^i and Newtonian dynamical time $\tau=\epsilon t$. This mapping defines a congruence parametrized by ϵ on the sequence. Along this congruence one can follow the physically same event: A system which has completed one orbit is joined by the congruence to one which has also completed one orbit at least in the limit $\epsilon\to 0$. The various (post-) Newtonian approximations are defined as derivatives of the relativistic solutions along this congruence at $\epsilon=0$. We shall apply this scheme to the radiation-reaction problem in a binary star system.

As explained in Sec. III, our scheme is free of the divergences found in previous work. Kerlick found terms of the form $g_n\epsilon^n$, where the integral expression for g_n diverged as its upper limit r went to infinity. We call these power-of-r or $\log r$ divergences, according to the character of the divergent integral. Power-of-r divergences simply make finite contributions to lower powers of ϵ in our scheme. The $\log r$ divergences introduce new, nonpolynomial terms into our asymptotic approximation beyond 2.5 PN order, of the form $\epsilon^n\ln\epsilon$. These do not affect the asymptotic convergence of the approximation, and permit one to calculate the approximate expression for any finite ϵ at any order, in contrast to Kerlick's result. Moreover, it will be found that the coefficient of the lowest $\epsilon^n\ln\epsilon$ term depends only on the fourth time derivative of the quadrupole moment of the material system. Thus, if there is no quadrupole radiation from the source, the sequence remains differentiable at $\epsilon=0$ until beyond octupole-radiation-reaction order.

The above improvements in the slow-motion approximation are the result of our essentially local point of view based on an initial-value problem. The local approach to the radiation reaction based on the initial-value problem was proposed by Schutz[12] in the linearized theory. In this approach the evolution of the system is determined by the initial data for both the field and the matter.[13] The dynamical degrees of freedom for gravitational fields are determined by assigning a probability distribution to the initial data for the field and by averaging over them rather than using an asymptotic radiation condition, such as purely outgoing radiation condition at $\mathscr{I}^+$, future null infinity,[14] or no incoming radiation condition at $\mathscr{I}^-$, past null infinity. Apart from the advantage of not using such time-asymmetric conditions, his formalism can avoid mathematical difficulties associated with such global conditions. Since these difficulties have been fully discussed by Ehlers,[7] I shall here point out the relation between the divergences in the formal slow-motion approximation and the asymptotic radiation condition. These asymptotic conditions on $\mathscr{I}^\pm$ are of course mathematically precise and covariant boundary conditions to pick up only the retarded field as the dynamical degrees of freedom for the gravitational field. However, at the same time the conditions lead to the retarded integral over infinite volume. The slow-motion expansion for such retarded integral (i.e., Taylor expansion in retarded time) immediately leads to uncontrollable divergent integrals. Therefore, the asymptotic radiation condition, though mathematically well defined, is not a useful condition at least in the slow-motion approximation. One is forced to specify the boundary condition not at $\mathscr{I}^\pm$ in order to have controllable divergence. In fact, as already pointed out in paper I or as will be shown in Sec. III, our initial-value-problem approach eliminates divergences and gives controlled errors.

In paper I, zero data for the free initial gravitational field was chosen for the sake of clarification of the concept of the Newtonian limit. This is obviously a mathematical idealization and may not correspond to reality. There always exists background gravitational radiation. The only thing we can expect is that the emission of radiation will be accompanied by a damping of the motion of the source if the initial radiation field is not in resonance with a mode of the system. We shall take this into account in Sec. II and try to find the weakest condition on the free radiation data that does not disturb the radiation reaction.

Finally in Sec. IV, we shall calculate the observable effect on the equation of an almost-Newtonian binary system directly, i.e., the rate of change of the orbital period of the system. Much attention has previously been paid to the calculation of the energy-loss rate itself. These calculations use either the energy of a gravitational bound system, which is not well defined, or an energy balance between that in radiation seen at $\mathscr{I}^+$ and that in the material source, a balance which has not been established within general relativity and essentially depends on the existence of a well-behaved future null infinity. A well-behaved future and/or past null infinity is not expected in general spacetimes. By formulating the theory in terms of the initial-value problem we are able to avoid difficulties associated with an asymptotic radiation condition. It seems best to calculate everything without relying too much on the asymptotic structure of spacetime. (We shall, however, assume the existence of an asymptotically flat initial hypersurface.) I shall calculate only the reaction effects in the source and show that only the Newtonian functional form of the period is necessary to calculate the lowest reaction effect. We can avoid the well-known difficulties associated with energy in general relativity. We shall find that the prediction derived from the standard quadrupole formula is valid in the Newtonian limit. The formula is asymptotically exact.

The notation in this paper is the same as in paper I. The basic equations are (I.3.3) and its integral form (I.4.3) except for the definition of the stress-energy tensor for an isentropic perfect-fluid source. Here I shall adopt the form

$$T^{\alpha\beta}=\rho[1+\Pi(\rho)+p(\rho)/\rho]u^\alpha u^\beta+p(\rho)g^{\alpha\beta}\ , \qquad (1.1)$$

$$p=\rho^2\Pi'(\rho)\ , \qquad (1.2)$$

where ρ and Π are the rest-mass density and the specific internal energy of the fluid, respectively. The conservation of the rest-mass density then can be expressed as follows:

$$(\rho u^\mu\sqrt{-g})_{,\mu}=0\ . \qquad (1.3)$$

In this paper we call ϵ^{4+2N} order in $\bar{h}^{\tau\tau}$, ϵ^{2+2N} order in $\bar{h}^{i\mu}$ as the Nth PN order, and ϵ^{5+2N} order in $\bar{h}^{\tau\tau}$, ϵ^{3+2N} order in $\bar{h}^{i\mu}$ as the $N+\frac{1}{2}$ PN order (see Table I). Therefore, Eq. (I.4.26) is 1st PN order in $\bar{h}^{\tau\tau}$ and 2nd PN order in $\bar{h}^{\tau\mu}$. This notation is motivated by the fact that the Nth PN terms contribute to the Nth PN equations of motion.

II. STATISTICAL APPROACH TO THE PROBLEM

We shall consider here nontrivial free data for the initial gravitational field and find the weakest constraint on the data that does not affect the radiation reaction. We shall interpret the free data for the field as background radiation on the initial hypersurface. As pointed out in Sec. I, what we can expect for the background radiation is that it is uncorrelated with the motion of the source. In order to take this into account, I shall adopt the statistical definition of the radiation given by Schutz,[12] and solve the following initial-value problem in the Newtonian limit. We shall consider a perfect-fluid source with compact support

TABLE I. The name of the order in $\bar{h}^{\mu\nu}$, where N = Newtonian order, NPN = Nth post-Newtonian order, and RR = radiation-reaction order.

Orders	$\bar{h}^{\tau\tau}$	$\bar{h}^{\tau i}$	$\bar{h}^{ij}$
ϵ^4	N	1 PN	1 PN
ϵ^5	0	0	1.5 PN
ϵ^6	1 PN	2 PN	2 PN
ϵ^7	1.5 PN	RR	RR
ϵ^8	2 PN	3 PN	3 PN
ϵ^9	RR	3.5 PN	3.5 PN

on an asymptotically flat spacelike hypersurface, $\tau=0$, and choose the free data for the field at random from a statistical ensemble with some properties specified below. Then the expected evolution of the system is calculated by averaging over the random data, keeping the data for the fluid fixed. The averaging over the random data is supposed to account not only for the above condition of uncorrelatedness between background radiation and the motion of the system, but also for the fact that the initial radiation is not observable in general. Therefore, we shall require that the mean-free data are zero. We shall also impose a gauge condition on the free data.

For the fluid, the same initial conditions as in paper I will be imposed: (I.3.13). For the specific internal energy, Newtonian scaling of the initial data requires that

$$\Pi(\rho,\epsilon)=\epsilon^2 f(\epsilon^{-2}\rho) . \tag{2.1}$$

Then, since $p=\rho^2\Pi'(\rho)$ (for fixed ϵ),

$$\rho(\tau=0,x^j,\epsilon)=\epsilon^2 a(x^j)$$

implies

$$p(\tau=0,x^j,\epsilon)=\epsilon^4[a(x^j)]^2 f'(a(x^j)) =\epsilon^4 b(x^j)$$

as desired.[15]

For the gravitational field, we have the constraint equations (I.3.13)

$$\Delta \bar{h}^{\mu\tau}+16\pi\Lambda^{\mu\tau}-\bar{h}^{\mu i}{}_{,i}{}^{\tau}=0 . \tag{2.2}$$

I shall choose $(\bar{h}^{ij},\bar{h}^{ij}{}_{,\tau})$ as the free data from among the 20 unknowns $(\bar{h}^{\mu\nu},\bar{h}^{\mu\nu}{}_{,\tau})$ for this system.[16] I shall impose the transverse gauge condition on them:

$$\bar{h}^{ij}{}_{,j}=0 , \quad \bar{h}^{ij}{}_{,\tau j}=0 \tag{2.3}$$

and assume the following form for them at $\tau=0$:

$$\begin{aligned}\bar{h}^{ij}(\tau=0,x^j;\epsilon)&=\epsilon^4{}_4\bar{h}^{ij}(\tau=0,x^j) ,\\ \bar{h}^{ij}{}_{,\tau}(\tau=0,x^j;\epsilon)&=\epsilon^4{}_4\bar{h}^{ij}{}_{,\tau}(\tau=0,x^j) .\end{aligned} \tag{2.4}$$

The functions ${}_4\bar{h}^{ij}$ and ${}_4\bar{h}^{ij}{}_{,\tau}$ at $\tau=0$ are chosen from the statistical ensemble with the above conditions: the mean values of them vanish and (2.3) hold. Further I have to assume the following restrictions for them as $r\equiv|x^j|\to\infty$:

$$\begin{aligned}&{}_4\bar{h}^{ij}(\tau=0,x^j)=O(r^{-1}) , \quad {}_4\bar{h}^{ij}{}_{,k}=O(r^{-2}) ,\\ &{}_4\bar{h}^{ij}{}_{,\tau}(\tau=0,x^j)=O(r^{-2}) , \quad {}_4\bar{h}^{ij}{}_{,\tau k}=O(r^{-3}) .\end{aligned} \tag{2.5}$$

These conditions are sufficient to ensure the unique existence of solutions to the constraint equation (2.2),[17] and are very weak restrictions on the free radiation data, considering that the 1st PN contribution to $\bar{h}^{ij}$ (gravitational stress) is also of order ϵ^4 in our gauge.

The remaining initial data $(\bar{h}^{\mu\tau},\bar{h}^{\mu\tau}{}_{,\tau})$ are expressed as follows:

$$\begin{aligned}&\bar{h}^{\tau\tau}=-16\pi\nabla^{-2}\Lambda^{\tau\tau} ,\\ &\bar{h}^{\tau i}=-16\pi\nabla^{-2}\Lambda^{\tau i} ,\\ &\bar{h}^{\tau\tau}{}_{,\tau}=-\bar{h}^{\tau i}{}_{,i}=16\pi\nabla^{-2}\Lambda^{\tau i}{}_{,j} ,\\ &\bar{h}^{\tau i}{}_{,\tau}=-\bar{h}^{ij}{}_{,j}=0 ,\end{aligned} \tag{2.6}$$

where ∇^{-2} is the inverse flat-space Laplacian which vanishes as $r\to\infty$. These are, of course, implicit expressions, since $\bar{h}^{\tau\mu}$ and $\bar{h}^{\tau\mu}{}_{,\tau}$ contribute to $\Lambda^{\tau\mu}$. These initial data for the field and the fluid uniquely determine the homogeneous solution at $\tau=0$ via Kirchhoff's formula (I.4.3), i.e.,

$$\bar{h}^{\mu\nu}_{H}(\tau,x^j;\epsilon)=\frac{\tau}{4\pi}\oint_{s(\tau,x^j,\epsilon)}\bar{h}^{\mu\nu}{}_{,\tau}(\tau=0,y^j;\epsilon)d\Omega_y+\frac{1}{4\pi}\frac{\partial}{\partial\tau}\left[\tau\oint_{s(\tau,x^j,\epsilon)}\bar{h}^{\mu\nu}(\tau=0,y^j;\epsilon)d\Omega_y\right] . \tag{2.7}$$

The contributions from $T^{\mu\nu}$ to the initial data have compact support, so the terms they generate in the homogeneous solutions vanish at fixed x^j for sufficiently small ϵ.[12] Moreover, since we average over the free data for the field, the mean value of the homogeneous solution will depend only on terms having an even number of factors $\bar{h}^{ij},\bar{h}^{ij}{}_{,\tau}$.

From the above initial data the first nonzero homogeneous solutions which are not involved in averaging come from the following source terms:

$${}_6\Lambda^{\tau\tau}(\tau=0,x^j)={}_6[(-g)t^{\tau\tau}_{\rm LL}](\tau=0,x^j)=-\frac{7}{128\pi}(\nabla{}_4\bar{h}^{\tau\tau})^2=-\frac{7}{256\pi}\Delta({}_4\bar{h}^{\tau\tau})^2-\tfrac{7}{16}\,{}_2\rho\,{}_4\bar{h}^{\tau\tau} , \tag{2.8}$$

$${}_6\Lambda^{\tau i}(\tau=0,x^j)={}_6[(-g)t^{\tau i}_{\rm LL}](\tau=0,x^j)=\frac{1}{64\pi}[-3\,{}_4\bar{h}^{\tau\tau}{}_{,i}\,{}_4\bar{h}^{\tau j}{}_{,j}+4\,{}_4\bar{h}^{\tau\tau}{}_{,j}({}_4\bar{h}^{\tau j}{}_{,i}-{}_4\bar{h}^{\tau}{}_{,i}{}^{,j})] , \tag{2.9}$$

so

$$\begin{aligned}{}_6\Lambda^{\tau i}{}_{,i}(\tau=0,x^j)&=\frac{7}{128\pi}\frac{\partial}{\partial\tau}(\nabla{}_4\bar{h}^{\tau\tau})^2-\tfrac{3}{4}\,{}_2\rho_4\bar{h}^{\tau\tau}{}_{,\tau}-{}_2\rho_1 v^i{}_4\bar{h}^{\tau\tau}{}_{,i}\\ &=-\frac{7}{128\pi}\Delta({}_4\bar{h}^{\tau\tau}{}_4\bar{h}^{\tau j}{}_{,j})+\frac{7}{16}\frac{\partial}{\partial\tau}({}_2\rho_4\bar{h}^{\tau\tau})-\tfrac{3}{4}\,{}_2\rho_4\bar{h}^{\tau\tau}{}_{,\tau}-{}_2\rho_{,1}v^1{}_4\bar{h}^{\tau\tau}{}_{,i} ,\end{aligned} \tag{2.10}$$

where

$${}_4\bar{h}^{\tau\tau}(\tau=0,x^j)=-16\pi(\nabla^{-2}{}_4\Lambda^{\tau\tau})(\tau=0,x^j)=4\int d^3y^k\frac{{}_2\rho(\tau=0,y^k)}{|x^k-y^k|} , \tag{2.11}$$

$${}_4\bar{h}^{\tau i}(\tau=0,x^j)=-16\pi(\nabla^{-2}{}_4\Lambda^{\tau i})(\tau=0,x^j)=4\int d^3y^k\frac{({}_2\rho_1 v^i)(\tau=0,y^k)}{|x^k-y^k|} , \tag{2.12}$$

and we have used the identity

$$(\nabla f)^2 = -f\Delta f + \tfrac{1}{2}\Delta(f)^2 \tag{2.13}$$

for an arbitrary function f. It can be shown that the term with compact support in ${}_6\Lambda^{\tau\tau}$ vanishes and the terms with compact support in ${}_6\Lambda^{\tau i}{}_{,i}$ cancel each other after the surface integration in (2.7). Noncompact support terms give nonzero contributions of $O(\epsilon^8)$ in $\bar{h}^{\tau\tau}_H$. This can be seen from Lemma 1 of paper I by taking $N=6$ and $M\leq -2$. In order to calculate these contributions explicitly, one needs the multipole expansions for ${}_4\bar{h}^{\tau\tau}(\tau=0,x^j)$ and ${}_4\bar{h}^{\tau i}(\tau=0,x^j)$:

$${}_4\bar{h}^{\tau\tau}(\tau=0,x^j)=4\frac{M}{|x^j|}+6\mathcal{I}_{ij}\frac{x^ix^j}{|x^j|^5}+O(|x^j|^{-4}),\quad {}_4\bar{h}^{\tau i}(\tau=0,x^j)=4\frac{P^i}{|x^j|}+O(|x^j|^{-2}), \tag{2.14}$$

where $M\equiv\int d^3y\,{}_2\rho(\tau=0,y^j)$, $P^i\equiv\int d^3y[{}_2\rho\,,v^i](\tau=0,y^j)$, and $\mathcal{I}_{ij}$ is the reduced quadrupole moment defined by

$$\mathcal{I}_{ij}=I_{ij}-\tfrac{1}{3}\delta_{ij}I^l{}_l\,,\quad I_{ij}(\tau)\equiv\int d^3y\,{}_2\rho(\tau,y^k)y_iy_j\,. \tag{2.15}$$

Then one has the following contribution to $\bar{h}^{\tau\tau}$ from the monopole (MP) part:

$$\begin{aligned}\bar{h}^{\tau\tau}_{\rm MP}(\tau,x^j,\epsilon)&=-\frac{7}{2\pi}\epsilon^6\tau\oint_{(\tau,x^j,\epsilon)}\frac{{\rm MP}^jy_j}{|y^k|^4}d\Omega_y+\frac{7}{4\pi}\epsilon^6\frac{\partial}{\partial\tau}\left[\tau\oint_{(\tau,x^j,\epsilon)}-\frac{M^2}{|y^k|^2}d\Omega_y\right]\\&=-\frac{7}{2\pi}\epsilon^6\tau{\rm MP}^j\oint_{r=\tau/\epsilon}\left[\frac{n_j}{r^3}+O(r^{-4})\right]d\Omega+\frac{7}{4\pi}\epsilon^6\frac{\partial}{\partial\tau}\left[\tau M^2\oint_{r=\tau/\epsilon}\left[\frac{1}{r^2}+2\frac{n_kx_k}{r^3}+O(r^{-4})\right]d\Omega\right]\\&=-7\epsilon^8\frac{M^2}{\tau^2}+O(\epsilon^{10})\,,\end{aligned} \tag{2.16}$$

where $y^k=r^k+x^k$, $|r^k|=r$, and $n^k=r^k/r$. Thus the first nonzero contribution appears at 2 PN order (i.e., ϵ^8 order in $\bar{h}^{\tau\tau}$), and is a function of τ alone. The contribution at the radiation-reaction order, i.e., ϵ^9 order in $\bar{h}^{\tau\tau}$, vanishes by the angular integration. Higher multiple parts of ${}_8\bar{h}^{\tau\tau}$ are at most of order ϵ^{10}. The appearance of the nonzero homogeneous term at 2 PN order is in clear contrast with older approaches. This term, however, does not contribute to the equations of motion in the 2 PN order because the equations contain ${}_8\bar{h}^{\tau\tau}$ only with spatial derivatives. This term affects the conservation of mass at 3 PN order.

On the other hand, the first nonzero contribution to $\langle\bar{h}^{\tau\tau}_H(\tau,x^j)\rangle$ (the angular brackets express the averaging over the random free data) from the quadratic term in the initial free data for the field first appears at ϵ^{11} order. This is generated by the following terms in the Landau-Lifschitz pseudotensor:

$${}_{10}[(-g)t^{\tau\tau}_{\rm LL}]=-\tfrac{1}{2}{}_4\bar{h}^{ij}{}_{,k}\,{}_4\bar{h}_i{}^k{}_{,j}+\tfrac{1}{4}\,{}_4\bar{h}^{ij,k}\,{}_4\bar{h}_{ij,k}-\tfrac{1}{8}\,{}_4\bar{h}_i{}^{i,k}\,{}_4\bar{h}^j{}_{j,k}\,.$$

(The reason that the product of two "fourth-order" terms on the right-hand side gives a "tenth-order" term on the left-hand side is that the metric term $g^{\tau\tau}$ originally on the right-hand side is just ϵ^2.) It can be easily seen from Lemma 1 in paper I by taking $N=10$, $M=-1$ that these terms lead to ϵ^{11}-order contributions.

Therefore, our choice of the free data for the field, (2.5), does not affect the lowest radiation-reaction terms in $\bar{h}^{\tau\tau}$.

Similarly one can show that the homogeneous part of the solution in $\bar{h}^{\tau i}$ does not affect the radiation-reaction order here, either.

III. HIGHER-ORDER POST-NEWTONIAN APPROXIMATION

In paper I, we have calculated up to 1 PN approximation in $\bar{h}^{\tau\tau}$ and up to 2 PN in $\bar{h}^{i\mu}$. Here we shall continue past radiation-reaction order.

Since one can neglect the homogeneous part of the solution as explained in Sec. II, we shall concentrate on the retarded solution. In the earlier approaches it was always assumed that the slow-motion assumption enabled one to Taylor expand the retarded integrals in retarded time and assign the higher terms to higher orders. For example, if we make use of this method in our scheme, we may get

$$\bar{h}^{\tau\tau}(\tau,x^k,\epsilon)=4\epsilon^4\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\epsilon^n\,{}_{(n-1)}J(\tau,x^k,\epsilon)\,, \tag{3.1}$$

where

$${}_{(n-1)}J(\tau,x^k,\epsilon)=\epsilon^{-4}\int_{c(\tau,x^k,\epsilon)}d^3y^k\,|x^k-y^k|^{n-1}\frac{\partial^n}{\partial\tau^n}\Lambda^{\tau\tau}(\tau,y,\epsilon)\,. \tag{3.2}$$

There is still explicit ϵ dependence in J, coming from ϵ dependence of the integral region $c(\tau,x^k,\epsilon)$ as well as the nonlinearity of the field equations. Therefore, a higher ϵ dependence in front of J does not necessarily imply the real smallness of these terms compared with a term with a lower ϵ dependence in front. One cannot always ignore the retardation effects even in the limit as $\epsilon\to 0$. This fact has already been explained in Paper I. In this paper I shall use the following criterion for neglecting retardation which was explained in the appendix in paper I.

Criterion. Consider the following retarded integral:

$$\int_{c(\tau,x^k,\epsilon)} h(\tau-\epsilon r\ ,y^k\ ,\epsilon)d^3y\ . \tag{3.3}$$

Then the condition for neglecting retardation is

$$\begin{aligned} &|\, rh_{,\tau}(\tau,r)\,| < A\ ,\\ &h(\tau,r)=N(\tau)/r+O(r^{-2}) \ \text{ as } r\to\infty\ ,\\ &h(r) \text{ is regular at } r=0\ , \end{aligned} \tag{3.4}$$

where A is a constant and $N(\tau)$ is an arbitrary bounded function of τ. It turns out below that one can always ignore the retardation up to and including the radiation-reaction order according to the above criterion and one finds that the metric obtained up to and including that order is essentially the same with that obtained by Kerlick by using the formal slow-motion approximation. Therefore, we find the notations defined in (3.2) useful, even though we do not claim that the series in (3.1) are convergent or even asymptotic. Anyway I shall not use the Taylor-expansion method, rather I shall use the ϵ-derivative method developed in paper I.

We have seen in paper I that the 1.5 PN order in $\bar{h}^{\mu i}$ vanishes. Since ${}_7\Lambda^{\tau\tau}=0$, one finds for the 1.5 PN order in $\bar{h}^{\tau\tau}$,

$${}_7\bar{h}^{\tau\tau}(\tau\ ,x^k)=\lim_{\epsilon\to 0}\left[-4\int_{c(\tau\ ,x^k\ ,\epsilon)} {}^{(1)}_{6}\Lambda^{\tau\tau}(\tau\ ,y^k)d^3y-\tfrac{2}{3}\int_{c(\tau,x^k,\epsilon)} {}^{(3)}_{4}\Lambda^{\tau\tau}(\tau,y^k)d^3y\right]\ , \tag{3.5}$$

where the integral is not retarded because the integrand satisfies the above criterion [as well as the condition (ii) of Lemma 2 in paper I] as easily seen from the expression of ${}_6\Lambda^{\tau\tau}$, (I.4.27). Since ${}_6\Lambda^{\tau\tau}{}_{,\tau}=-{}_6\Lambda^{\tau i}{}_{,i}$, the first integral in the right-hand side of (3.5) becomes the surface integral of ${}_6\Lambda^{\tau i}$ over $S(\tau,x^k,\epsilon)$. This surface integral goes to zero in the limit $\epsilon\to 0$ because ${}_6\Lambda^{\tau i}=O(r^{-4})$. So we get the usual 1.5 PN term in $\bar{h}^{\tau\tau}$:

$${}_7\bar{h}^{\tau\tau}(\tau\ ,x^k)=-\tfrac{2}{3}\int {}^{(3)}_{4}\Lambda^{\tau\tau}(\tau\ ,y^k)r^2d^3y=-\tfrac{2}{3}{}^{(3)}I^l{}_l(\tau)\ , \tag{3.6}$$

where $I^l{}_l=\eta^{ij}I_{ij}$.

At the 2 PN order one finds that the condition (ii) of Lemma 2 in paper I breaks down because of the appearance of the terms in ${}_8\Lambda^{\tau\tau}$ and ${}_8\Lambda^{ij}$ which contain ${}_4\bar{h}^{ij}$. The time derivative of ${}_4\bar{h}^{ij}$ does not change to asymptotic falloff behavior of ${}_4\bar{h}^{ij}$, so one has to appeal the above criterion. Let us examine the leading asymptotic falloff behavior of ${}_8\Lambda^{\tau\tau}$ and ${}_6\Lambda^{ij}$. The expression for ${}_8\Lambda^{\tau\tau}$ can be obtained from the knowledge of the lower-order terms as follows:

$$\begin{aligned} {}_8\Lambda^{\tau\tau}=&(16\pi)^{-1}[{}_4\bar{h}^{\tau i}{}_4\bar{h}^{\tau j}-{}_4\bar{h}^{\tau\tau}{}_4\bar{h}^{ij}]_{,ij}\\ &+(16\pi)^{-1}[-\tfrac{11}{8}({}_4\bar{h}^{\tau l}{}_{,l})^2+\tfrac{1}{4}\,{}_4\bar{h}^{\tau i}{}_{,j}({}_4\bar{h}^{\tau j}{}_{,i}+{}_4\bar{h}^{\tau}{}_{i}{}^{,j})+{}_4\bar{h}^{\tau\tau}{}_{,i}\,{}_4\bar{h}^{ij}{}_{,j}+\tfrac{1}{4}\,{}_4\bar{h}^{\tau\tau,i}{}_4\bar{h}^{j}{}_{j,i}-\tfrac{7}{4}\,{}_4\bar{h}^{\tau\tau,i}{}_4\bar{h}^{\tau\tau}{}_{,i}]\\ &+{}_6\rho+{}_4\rho({}_1v^2+\tfrac{3}{2}\,{}_4\bar{h}^{\tau\tau})+{}_2\rho(\tfrac{1}{2}\,{}_6\bar{h}^{\tau\tau}-\tfrac{1}{2}\,{}_4\bar{h}^{i}{}_{i}+\tfrac{3}{8}\,{}_4\bar{h}^{\tau\tau}{}_4\bar{h}^{\tau\tau}-2\,{}_4\bar{h}^{\tau i}{}_1v^i+\tfrac{5}{2}\,{}_4\bar{h}^{\tau\tau}{}_1v^2+{}_1v^4+2\,{}_1v^k{}_3v_k)\ . \end{aligned} \tag{3.7}$$

For ${}_6\Lambda^{ij}$, see (I.4.27). The multipole expansion for ${}_4\bar{h}^{\tau\tau}$ and ${}_4\bar{h}^{ij}$ is given by

$${}_4\bar{h}^{\tau\tau}=\frac{4M}{r}+6I^{ij}\frac{n_in_j}{r^3}+O(r^{-4})\ ,\quad {}_4\bar{h}^{ij}=\frac{2^{(2)}I^{ij}}{r}+O(r^{-2})\ . \tag{3.8}$$

Therefore, the leading asymptotic behavior of ${}_8\Lambda^{\tau\tau}$ and ${}_6\Lambda^{ij}$ is proportional to $M^{(2)}I^l{}_l\,/r^4$ and $M^{(2)}I^{ij}/r^4$, respectively, and the above criterion can be satisfied provided ${}^{(3)}I_{ij}$ is a bounded function of τ. We can ignore the retardation in the limit as $\epsilon\to 0$. Thus one gets the same expression for the 2 PN metric with that given by Kerlick except the higher-order contribution of ρ and v^i, and the surface integral term in Lemma 3 of paper I. In fact, the surface integral first appears at 2 PN order. For the explicit calculation of the surface integral we shall again make use of the multipole expansion (3.8) for ${}_4\bar{h}^{\tau\tau}$. Since

$${}_6\Lambda^{\tau\tau}(\tau\ ,x^k)=-\frac{7}{8\pi}\frac{M^2}{|x^k|^2}+\text{ terms of compact support}\ , \tag{3.9}$$

one finds the following expression for the surface integral $\bar{h}^{\tau\tau}_s$:

$$\begin{aligned} {}_8\bar{h}^{\tau\tau}_s(\tau,x^k)&=\lim_{\epsilon\to 0}\frac{1}{8!}\frac{\partial^7}{\partial\epsilon^7}\left[-4\frac{\tau}{\epsilon^2}\int_{|x-y|=\tau/\epsilon}|x^k-y^k|\,\Lambda^{\tau\tau}(0\ ,y^k\ ,\epsilon)d\Omega_y\right]\\ &=\lim_{\epsilon\to 0}\frac{1}{8!}\frac{\partial^7}{\partial\epsilon^7}\left[4\frac{7}{8\pi}\tau\epsilon^4\int_{r=\tau/\epsilon}\frac{M^2}{r^3}\left[1-\frac{x^kn_k}{r}+O(r^{-2})\right]d\Omega+\epsilon^6\int_{r=\tau/\epsilon}O(r^{-3})d\Omega\right]\\ &=\frac{7}{4}\frac{M^2}{\tau^2}\ . \end{aligned} \tag{3.10}$$

This term can be combined with the homogeneous solution (2.16):

$$H(\tau)\equiv{}_8\bar{h}\,^{\tau\tau}_{H}(\tau,x^k)+{}_8\bar{h}\,^{\tau\tau}_{S}(\tau,x^k)=-\tfrac{21}{4}(M^2/\tau^2)\ . \tag{3.11}$$

As explained in Sec. II, this term does not affect the equations of motion at 2 PN order. Finally we get the following expression for 2 PN order:

$${}_8\bar{h}\,^{\tau\tau}(\tau,x^k)=4\int {}_8\Lambda^{\tau\tau}(\tau,y)r^{-1}d^3y+2\int {}^{(2)}_{6}\Lambda^{\tau\tau}(\tau,y)r\,d^3y+\tfrac{1}{6}\int {}^{(4)}_{4}\Lambda^{\tau\tau}(\tau,y^k)r^3d^3y+H(\tau)\ . \tag{3.12}$$

For ${}_6\bar{h}\,^{i\mu}$, see (I.4.26).

At 2.5 PN we find that

$$\begin{aligned}
{}_9\bar{h}\,^{\tau\tau}(\tau,x^k)&=4\int {}_9\Lambda^{\tau\tau}(\tau,y^k)r^{-1}d^3y-4\int {}^{(1)}_{8}\Lambda^{\tau\tau}(\tau,y^k)d^3y-\tfrac{1}{3}\int {}^{(3)}_{6}\Lambda^{\tau\tau}(\tau,y^k)r^2d^3y-\tfrac{1}{30}\int {}^{(5)}_{4}\Lambda^{\tau\tau}(\tau,y^k)r^4d^3y\ ,\\
{}_7\bar{h}\,^{\tau i}(\tau,x^k)&=-\tfrac{2}{3}\int {}^{(3)}_{4}\Lambda^{\tau i}(\tau,y^k)r^2d^3y\ ,\\
{}_7\bar{h}\,^{ij}(\tau,x^k)&=-4\int {}^{(1)}_{6}\Lambda^{ij}(\tau,y^k)d^3y-\tfrac{2}{3}\int {}^{(3)}_{4}\Lambda^{ij}(\tau,y^k)r^2d^3y\ ,
\end{aligned} \tag{3.13}$$

where

$${}_9\Lambda^{\tau\tau}=(16\pi)^{-1}({}_4\bar{h}\,^{\tau\tau}\,{}_5\bar{h}\,^{ij})_{,ij}\ . \tag{3.14}$$

All integrals appearing in (3.10) are not retarded because each term satisfies the above criterion: ${}_9\Lambda^{\tau\tau}=O(r^{-3})$, ${}^{(1)}_{8}\Lambda^{\tau\tau}=O(r^{-4})$, ${}^{(3)}_{6}\Lambda^{\tau\tau}=O(r^{-4})$, $|{}^{(1)}_{4}\Lambda^{\tau\tau}|<MRr^{-2}$, $|{}^{(2)}_{8}\Lambda^{\tau\tau}|<MBr^{-4}$, $|{}^{(4)}_{6}\Lambda^{\tau\tau}|<MBr^{-4}$ where $|{}^{(4)}I_{ij}|<B$ with a constant B. There is no surface integral term in this order. This expression for the radiation-reaction order is exactly equal to that given by Kerlick except again the appearance of the higher orders in ρ and v^i.

Next we shall consider the 3 PN order. The study of this order has crucial importance for the proof of the asymptotic convergence of the approximation up to and including 2.5 PN order. Since it is not difficult to see that the homogeneous solution (more generally terms expressed by the surface integral) does not have peculiar behavior in ϵ at this order, we shall omit these terms. One might think that $\bar{h}\,^{\mu\nu}$ at this order is given by the following expression:

$$\begin{aligned}
{}_{10}\bar{h}\,^{\tau\tau}(\tau,x^k)&=4\int {}_{10}\Lambda^{\tau\tau}(\tau,y^k)r^{-1}d^3y-4\int {}^{(1)}_{9}\Lambda(\tau-\epsilon r,y^k)d^3y+2\int {}^{(2)}_{8}\Lambda^{\tau\tau}(\tau-\epsilon r,y^k)r\,d^3y\\
&\quad+\tfrac{1}{6}\int {}^{(4)}_{6}\Lambda^{\tau\tau}(\tau-\epsilon r,y^k)r^3d^3y+\tfrac{1}{180}\int {}^{(6)}_{4}\Lambda^{\tau\tau}(\tau,y^k)r^5d^3y\ ,\\
{}_8\bar{h}\,^{\tau i}(\tau,x^k)&=4\int {}_8\Lambda^{\tau i}(\tau,r)r^{-1}d^3y+2\int {}^{(2)}_{6}\Lambda^{\tau i}(\tau,y^k)r\,d^3y+\tfrac{1}{6}\int {}^{(4)}_{4}\Lambda^{\tau i}(\tau,y^k)r^3d^3y\ ,\\
{}_8\bar{h}\,^{ij}(\tau,x^k)&=4\int {}_8\Lambda^{ij}(\tau-\epsilon r,y^k)r^{-1}d^3y+2\int {}^{(2)}_{6}\Lambda^{ij}(\tau-\epsilon r,y^k)r\,d^3y+\tfrac{1}{6}\int {}^{(4)}_{4}\Lambda^{ij}(\tau-\epsilon r,y^k)r^3d^3y\ ,
\end{aligned} \tag{3.15}$$

where the last integrals in ${}_{10}\bar{h}\,^{\tau\tau}$ and ${}_8\bar{h}\,^{\tau i}$ are not retarded because ${}_4\Lambda^{\tau\mu}$ has compact support. Explicit expressions for ${}_{10}\Lambda^{\tau\tau}$ and ${}_8\Lambda^{i\mu}$ have been calculated by Kerlick.[9] According to his calculation, the leading asymptotic falloff terms in ${}_{10}\Lambda^{\tau\tau}$ and ${}_8\Lambda^{\tau i}$ contain ${}^{(3)}I_{ij}$ and ${}^{(2)}I_{ij}$, respectively, and fall off like r^{-4} and r^{-3}, respectively. Therefore, the first integrals in ${}_{10}\bar{h}\,^{\tau\tau}$ and ${}_8\bar{h}\,^{\tau i}$ are not retarded and give convergent integral provided that ${}^{(4)}I_{ij}$ and ${}^{(3)}I_{ij}$ are bounded functions, respectively. The second integrals in ${}_8\bar{h}\,^{\tau i}$ are also not retarded because ${}^{(2)}_{6}\Lambda^{i\tau}=O(r^{-5})$. Since ${}^{(1)}_{9}\Lambda^{\tau\tau}=O(r^{-3})$, ${}^{(2)}_{8}\Lambda^{\tau\tau}=O(r^{-4})$, ${}^{(4)}_{6}\Lambda^{\tau\tau}=O(r^{-6})$, ${}_8\Lambda^{ij}=O(r^{-2})$, ${}^{(2)}_{6}\Lambda^{ij}=O(r^{-4})$, and ${}^{(4)}_{4}\Lambda^{ij}=O(r^{-6})$. Simple ordering argument shows that the second, third, and fourth integrals in ${}_{10}\bar{h}\,^{\tau\tau}$ and the first, second, and third integrals in ${}_8\bar{h}\,^{ij}$ are logarithmically divergent. Let me examine these potentially logarithmically divergent terms in detail. First of all note that the leading asymptotically falloff terms of ${}^{(1)}_{9}\Lambda^{\tau\tau}$, ${}^{(2)}_{8}\Lambda^{\tau\tau}$, ${}^{(4)}_{6}\Lambda^{\tau\tau}$, ${}_8\Lambda^{ij}$, ${}^{(2)}_{6}\Lambda^{ij}$, and ${}^{(4)}_{4}\Lambda^{ij}$ are all proportional to ${}^{(4)}I_{ij}$. Therefore, according to the Appendix of paper I, the "instantaneous" part of the integrals gives divergences, and the "retarded" part gives convergent integrals at that order provided ${}^{(5)}I_{ij}$ is a bounded function of τ. I shall show further that some of these logarithmic divergences are killed by the angular integration. This fact was first pointed out by Schutz and Breuer and Rudolph.[18] In our scheme the angular integration always has to be carried out before we take the limit $\epsilon\rightarrow 0$, i.e., $r\rightarrow\infty$. Consider, for example, the following logarithmically divergent integral in the second term in ${}_{10}\bar{h}\,^{\tau\tau}$:

$$\begin{aligned}
I&=\int {}^{(1)}_{5}\bar{h}\,^{ij}\,{}_4\bar{h}\,^{\tau\tau}{}_{,ij}(\tau,y^k)d^3y\\
&={}^{(1)}_{5}\bar{h}\,^{ij}\int_0^{\tau/\epsilon}dr\,r^2\int_{4\pi}d\Omega_n\left[\frac{4M}{r^3}(3n_in_j-\delta_{ij})+O(r^{-5})\right]=O(\epsilon^2)\ .
\end{aligned} \tag{3.16}$$

The angular integration killed the potentially logarithmic divergences. Similarly one can show that the logarithmic divergence in the fourth term in ${}_{10}\bar{h}\,^{\tau\tau}$ and the third term in ${}_8\bar{h}\,^{ij}$ are killed by the angular integration. Thus the real logarithmic divergences are only the third term in ${}_{10}\bar{h}\,^{\tau\tau}$ and the fourth, and the second terms in ${}_8\bar{h}\,^{ij}$. Explicitly we have

$${}_{10}\bar{h}\,^{\tau\tau}=\tfrac{7}{3}M^{(4)}I^l{}_l\ln\infty+\text{ finite terms}\ ,\quad {}_8\bar{h}\,^{ij}=\tfrac{4}{3}M(-10^{(4)}I^{ij}+\delta^{ij(4)}I^l{}_l)\ln\infty+\text{ finite terms}\ , \tag{3.17}$$

where "$\ln\infty$" stands for $\lim_{\epsilon\rightarrow 0}(-\ln\epsilon)$. This divergence means that the sequence is not differentiable at the 3 PN order at $\epsilon=0$. This is not, however, a serious problem. Taylor's remainder theorem [paper I, Eqs. (4.1) and (4.2)] still guarantees that the series through 2.5 PN order is asymptotic. Moreover, it is clear from (3.17) that the next term in the asymptotic series for $\bar{h}\,^{\tau\tau}$ should be $\epsilon^{10}\ln\epsilon$ rather than simply ϵ^{10}.

It is also interesting to notice that the above logarithmic divergence disappears if the system does not emit quadrupole radiation. In such a case differentiability extends beyond 2.5 PN order to octupole-reaction order (3.5 PN).

Finally I shall point out the fact that there is another 3 PN term which is not in the expression (3.15). Consider the integral

$$J^{ij}(\tau, x^k) = \int_{c(\tau,x^k,\epsilon)} \epsilon^9 ({}_4\bar{h}^{\tau\tau}\,{}^{(2)}_{5}\bar{h}^{ij})(\tau - \epsilon r, y^k) r^{-1} d^3y \;, \tag{3.18}$$

where the integrand ${}_4\bar{h}^{\tau\tau}\,{}^{(2)}_{5}\bar{h}^{ij}$ is one of terms in ${}_{11}\Lambda^{ij}$. Since ${}_4\bar{h}^{\tau\tau}\,{}^{(2)}_{5}\bar{h}^{ij} = O(r^{-1})$, this integral diverges. In the formal slow-motion approximation, this gives linear divergence at ϵ^9 order, quadratic divergence at ϵ^{10} order, cubic divergence at ϵ^{11} order, and so on. In fact, if we Taylor expand the integrand in τ, we have

$$\begin{aligned} J^{ij} &\sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \epsilon^n \epsilon^9 \int_{\text{all space}} |x^k - y^k|^{n-1}\, {}^{(n+2)}_{5}\bar{h}^{ij}\, {}_4\bar{h}^{\tau\tau} \\ &\sim \sum_n \epsilon^{n+9} \int dr\, r^2 r^{n-1} \frac{1}{r} \sim \sum_n \epsilon^{n+9} \infty^{n+1} \;, \end{aligned} \tag{3.19}$$

where the asymptotic sign expresses the fact that we have picked up terms with the worst divergent behavior and "∞^{n+1}" expresses the way the integral diverges as the outer limit of r goes to ∞. On the other hand, these power-of-ϵ divergences never appear in our scheme. In fact the expression for J^{ij} becomes in our scheme

$$J^{ij}(\tau, x^k) = \int_0^{\tau/\epsilon} \epsilon^9 ({}_4\bar{h}^{\tau\tau}\,{}^{(2)}_{5}\bar{h}^{ij})(\tau, y^k) r\, dr - \int_0^{\tau/\epsilon} \int_0^{\epsilon r} \epsilon^9 ({}_4\bar{h}^{\tau\tau}\,{}^{(2)}_{5}\bar{h}^{ij})_{,\tau}(\tau - \eta, y^k) r\, dr\, d\eta \;, \tag{3.20}$$

where the angular integral is understood. By substituting the multiple expansion for ${}_4\bar{h}^{\tau\tau}$ and ${}_5\bar{h}^{ij} = -2\,{}^{(3)}I^{ij}(\tau)$, we have

$$J^{ij}(\tau, x^k) \sim \epsilon^9 (-8M\,{}^{(5)}I^{ij}) \int_0^{\tau/\epsilon} dr + \epsilon^{10} 8M \int_0^{\tau/\epsilon} {}^{(6)}I^{ij}(\tau - \eta_0) r\, dr \;, \quad 0 \le \eta_0 \le \epsilon r \tag{3.21}$$

where we have used the mean-value theorem as in the Appendix in paper I. If we assume ${}^{(6)}I^{ij}$ is a bounded function of τ, one sees that the "retarded" part as well as the "instantaneous" part is order of ϵ^8. *All* the retardations contribute to the same order (3 PN order in this case) and the retarded Taylor expansion is not valid at all: The integrand must be treated in its retarded form.

The above argument also shows that the power-of-ϵ divergence in the formal slow-motion approximation is simply the result of misordering and, moreover, this divergence reduces to a finite contribution at some lower order but still higher than 2.5 PN order. The last statement can be proved as follows. Suppose ∞^k divergence first appears at ϵ^n order in the formal slow-motion approximation. Then the divergence reduced to a finite term at ϵ^{N-k} order. Now we can write $k = 2 + p - q$, $N = n + p$ when the "∞^k" divergence appears in

$$\int dr\, r^2 r^{p-1}\, {}^{(p)}_{N}\Lambda^{\mu\nu}(\tau - \epsilon r, r) \;,$$

where ${}^{(p)}\Lambda^{\mu\nu} = O(r^{-q})$. So $N - k = n - 2 + q$. Since the worst asymptotic falloff at $\Lambda^{\mu\nu}$ is $O(1)$, $q \ge 0$ so $N - k \ge r - 2$. It is easy to see that such situation $q = 0$ does not occur at 3.5 PN order. Therefore, the "power-of-ϵ divergence" in the formal slow-motion expansion does not affect the result at 2.5 PN order.

In summary we have seen that the expression (3.13) for 2.5 PN order does describe the leading radiation reaction and our approximation scheme is asymptotic up to and including 2.5 PN order.

The metric can be calculated from the knowledge of $\bar{h}^{\mu\nu}$. I shall here write down only the radiation-reaction metric:

$$\begin{aligned} {}_3g_{\tau\tau} &= \tfrac{1}{2}({}_7\bar{h}^{\tau\tau} + {}_5\bar{h}^{l}{}_{l}) \;, \\ {}_5g_{\tau\tau} &= \tfrac{1}{2}({}_9\bar{h}^{\tau\tau} + {}_7\bar{h}^{l}{}_{l}) - \tfrac{3}{4}\,{}_4\bar{h}^{\tau\tau}\,{}_7\bar{h}^{\tau\tau} - \tfrac{1}{4}\,{}_4\bar{h}^{\tau\tau}\,{}_5\bar{h}^{l}{}_{l} \;, \\ {}_5g_{\tau i} &= -{}_7\bar{h}^{\tau i} \;, \\ {}_5g_{ij} &= {}_5\bar{h}^{ij} + \tfrac{1}{2}\delta_{ij}({}_7\bar{h}^{\tau\tau} - {}_5\bar{h}^{l}{}_{l}) \;. \end{aligned} \tag{3.22}$$

For later convenience we shall define $A_{\mu\nu}$ as follows:

$$\begin{aligned} &{}_3g_{\tau\tau} = A^{(7)}_{\tau\tau} \;, \quad {}_5g_{\tau\tau} = A^{(9)}_{\tau\tau} - \tfrac{3}{4}\,{}_4\bar{h}^{\tau\tau} A^{(7)}_{\tau\tau} \;, \\ &{}_5g_{\tau i} = A^{(7)}_{\tau i} \;, \quad {}_5g_{ij} = A^{(5)}_{ij} \;, \end{aligned} \tag{3.23}$$

then we find

$$A^{(7)}_{\tau\tau} = -\tfrac{4}{3}\,{}^{(3)}I^{l}{}_{l}(\tau) \;, \quad A^{(5)}_{ij} = -2\,{}^{(3)}I_{ij}(\tau) \;. \tag{3.24}$$

IV. THE PERIOD CHANGE IN A BINARY SYSTEM

The effect of the emission of gravitational radiation on a binary system is observed as the secular change of the orbital period. In the observation[2] the period is defined by fitting a post-Newtonian orbit to the data in short segments. (The period only has a meaning up to and including 2nd PN order because there is no damping up to and including that order.) Because of the well-known problems defining energy in general relativity, it seems safer to calculate the period change directly from the equations of motion rather than use an argument based on the energy-loss rate. However, one does not know the explicit functional term of the period in terms of the matter variables, ρ and v^i. We know that there is no net change of the orbital period up to and including 2nd PN order. This is because the dynamical degrees of freedom for the gravitational field do not appear until 2.5 PN order and, therefore, the material system has a conserved Hamiltonian and some action-angle variables which are adiabatically invariant up to and including 2nd PN order. Those quantities are functionals of the matter-dynamical variables,

$$\begin{aligned} \rho_2(\tau, x^j, \epsilon) &= \epsilon^2\, {}_2\rho(\tau, x^j) + \epsilon^4\, {}_4\rho(\tau, x^j) \\ &\quad + \epsilon^6\, {}_6\rho(\tau, x^j) \;, \\ v_2^i(\tau, x^j, \epsilon) &= \epsilon^1\, {}_1v^i(\tau, x^j) + \epsilon^3\, {}_3v^i(\tau, x^j) \\ &\quad + \epsilon^5\, {}_5v^i(\tau, x^j) \;, \end{aligned} \tag{4.1}$$

and can also be written as a sum of a Newtonian order terms and higher-order correction terms to it. Now the orbital period can be expressed in terms of one of the action-angle variables and the Hamiltonian.[20] Therefore, the orbital period itself is a functional of ρ_2 and v_2^i defined above and can be written as a sum of a Newtonian order term and its connection terms. Expressed as a τ time, the period is

$$P(\rho_2, v^i{}_2) = P(\rho_2, v^i{}_2)\,|_{\epsilon=0} + \frac{\epsilon^2}{2!}\frac{\partial^2}{\partial\epsilon^2}P(\rho_2, v^i{}_2)\Big|_{\epsilon=0} + \frac{\epsilon^4}{4!}\frac{\partial^4}{\partial\epsilon^4}P(\rho_2, v^i{}_2)\Big|_{\epsilon=0} . \quad (4.2)$$

We shall define the Newtonian period and the post-Newtonian and 2nd post-Newtonian corrections to it as follows:

$$P_N({}_2\rho, {}_1v^i) \equiv P(\rho_2, v^i{}_2)\,|_{\epsilon=0} ,$$

$$\Delta P_{\rm PN}({}_2\rho, {}_4\rho; {}_1v^i, {}_3v^i) \equiv \frac{1}{2!}\frac{\partial^2}{\partial\epsilon^2}P(\rho_2, v^i{}_2)\Big|_{\epsilon=0} , \quad (4.3)$$

$$\Delta P_{2\rm PN}({}_2\rho, {}_4\rho, {}_6\rho; {}_1v^i, {}_3v^i, {}_5v^i) \equiv \frac{1}{4!}\frac{\partial^4}{\partial\epsilon^4}P(\rho_2, v^i{}_2)\Big|_{\epsilon=0} .$$

Now we know that the form of P_N is the same as that of Newtonian mechanics:

$$P_N({}_2\rho, {}_1v^i) = k\,|E_N({}_2\rho, {}_1v^i)|^{-3/2} , \quad (4.4)$$

where k is a constant and E_N is the total Newtonian energy

$$E_N({}_2\rho, {}_1v^i) = \int d^3x\, {}_2\rho(\tfrac{1}{2}\,{}_1v^2 + {}_2\Pi - \tfrac{1}{2}\,{}_2U) . \quad (4.5)$$

[Here as in (1.1) we have ignored tidal and other possible Newtonian dissipative effects.] Since the period is a functional of $\rho_2, v^i{}_2$, we can also write P as

$$P(\rho_2, v^i{}_2) = P_N(\rho_2, v^i{}_2) + \Delta P(\rho_2, v^i{}_2) \quad (4.6)$$

in which the functional form of P_N is the same as (4.4), but now ρ_2 and $v^i{}_2$ are used instead of Newtonian quantities. It is obvious that ΔP is at least higher than the lowest order of P_N whatever the explicit functional form ΔP is. The τ-time derivative of P can be calculated in terms of τ-time derivatives of ρ and v^i as follows[21]:

$$\frac{d}{d\tau}P = \frac{\delta P}{\delta\rho_2}\left[\frac{\partial\rho_2}{\partial\tau}\right] + \frac{\delta P}{\delta v^i{}_2}\left[\frac{\partial v^i{}_2}{\partial\tau}\right] , \quad (4.7)$$

where $\delta P/\delta\rho_2$, $\delta P/\delta v^i{}_2$ are linear operators which represent the functional derivatives of P with respect to ρ_2, $v^i{}_2$, respectively. If $\partial\rho_2/\partial\tau$, $\partial v^i{}_2/\partial\tau$ were calculated from the 2nd PN equations of motion, we would obtain zero for $dP/d\tau$; the orbital period does not change up to and including 2nd PN order. This result does not change even if we shall consider P in the above formula (4.7) as a functional of the fully relativistic ρ and v^i as long as we use only the 2nd PN form of the equations of motion for $\partial\rho/\partial\tau$ and $\partial v^i/\partial\tau$. This is because the vanishing of $dP/d\tau$ means the vanishing of the integral on the right-hand side of (4.7) and this is a functional relationship between P and the equations of motion, and we do not change the functional form of P. Therefore, a nonzero contribution to $dP/d\tau$ is obtained only if we include 2.5 PN (radiation-reaction) terms in the equations of motion. Then it is clear that only the Newtonian functional form of the period $P_N(\rho, v^i)$ is necessary in this calculation, since its correction term $\Delta P(\rho, v^i)$ is at least ϵ^2 higher than P_N and, moreover, the lowest-order nonzero value (i.e., the lowest-order radiation-reaction effect) for the period change $dP/d\tau$ comes from the Newtonian order of the period $P_N({}_2\rho, {}_1v^i)$. In the end, the calculation of the period change is reduced to the calculation of the time derivative of the Newtonian functional part of the period and hence to the calculation of the time derivative of the Newtonian functional form of the energy $E_N(\rho, v^i)$ through the relation (4.4).[22]

The time derivative of $E_N(\rho, v^i)$ is expressed in terms of the time derivatives of ρ and v^i by the formula (4.7) for E_N. This time the fully relativistic ρ and v^i are used instead of ρ_2 and $v^i{}_2$. The time derivatives of ρ and v^i are given by (1.2) and $T_i{}^\nu{}_{;\nu}=0$, respectively. We shall write them as

$$\frac{\partial\rho}{\partial\tau} + \frac{\partial}{\partial x^j}(\rho v^i) = -{}_8(\rho u^\mu\sqrt{-g}\,)^R{}_{\mu} = 0 ,$$
$$\frac{\partial}{\partial\tau}(\rho v^i) + \frac{\partial}{\partial x^j}(\rho v^i v^j + p\delta^{ij}) - \rho U^i = -{}_aT_i^{R\nu}{}_{;\nu} , \quad (4.8)$$

where we have picked up only reaction terms in the right-hand side and

$$U(\tau, x^k, \epsilon) = \int d^3y^k\,|x^k - y^k|^{-1}\rho(\tau, x^k, \epsilon)$$

and ${}_8(\rho u^\mu\sqrt{-g}\,)^R{}_{,\mu}$, ${}_9T_i^{R\nu}{}_{;\nu}$ are defined from ${}_8(\rho u^\mu\sqrt{-g}\,)_{,\mu}$, ${}_9T_i{}^\nu{}_{;\nu}$ by subtracting ${}_8[\rho_{,\tau} + (\rho v^i)_{,i}]$, ${}_9[(\rho v^i)_{,\tau} + (\rho v^i v^j + p\delta^{ij})_{,j} - U^{,i}]$, respectively. We have also used the fact that ${}_8(\rho u^\mu\sqrt{-g}\,)^R{}_{,\mu}=0$ which can be easily shown. The time derivative of the internal energy is calculated from (1.3). By using the above equations, one finds the lowest-order nonvanishing contribution is

$$\left[\frac{dE_N}{d\tau}\right]_{10} = -\int d^3x\,{}_1v^i\,{}_9T_i^{R\nu}{}_{;\nu} . \quad (4.9)$$

Therefore, the "Newtonian energy" loss may be interpreted as the work done by the reaction force ${}_9T_i{}^{R\nu}{}_{;\nu}$:

$${}_9T_i{}^{R\nu}{}_{;\nu} = -\tfrac{1}{2}\,{}_2\rho(A^{(9)}_{\tau\tau,i} - 3\,{}_2U_{,i}A^{(7)}_{\tau\tau}) + {}_2\rho A^{(7)}_{\tau i,\tau} + \tfrac{1}{2}\,{}_2\rho_1 v^i A^{(7)}_{\tau\tau,\tau} + {}_2\rho\,{}_1v^i A^{(5)}_{ij,\tau} - {}_4P_{,i}A^{(7)}_{\tau\tau} - {}_4P_{,j}A^{(5)}_{ij} + 2\,{}_2U_{,j}A^{(5)}_{ij} , \quad (4.10)$$

where

$${}_2U(\tau, x^k) = \int d^3y\,|x^k - y^k|^{-1}\,{}_2\rho(\tau, y^k) .$$

This expression is essentially the same as the corresponding expression in Kerlick. So we shall obtain the standard quadrupole formula for E_N after time averaging:

$$\left\langle\frac{dE_N}{d\tau}\right\rangle = -\tfrac{1}{5}\langle {}^{(3)}\!I_{kl}\,{}^{(3)}\!I^{kl}\rangle . \quad (4.11)$$

The only assumption I have made for the system is the perfect-fluid form for the stress-energy tensor (1.1) and the relation (4.3). These assumptions are very reasonable, particularly for a nearly Newtonian binary system.

V. DISCUSSION

In this paper we have studied the radiation reaction for a nearly Newtonian (weak internal gravity) perfect-fluid system by applying the approximation scheme based on the theory of Newtonian limit. It was found that the standard quadrupole formula gives a genuinely asymptotic formula for the rate of period change of such a system. Obviously, the most important question about the quadrupole formula is whether the formula can be applied to the binary pulsar with strong internal gravity. We have assumed that the initial data for matter satisfy the Newtonian scaling properties; $v^i \sim \epsilon$, $\rho \sim \epsilon^2$. Therefore, if we fix the linear dimension of the material system L in the limit $\epsilon \to 0$, the internal gravity is getting weaker and weaker, $M/L \sim \epsilon^2$. However, in order to study the Newtonian-type orbital motion it will be enough to assume that the mass, not the density, scales like ϵ^2 in the limit. Strong gravity can be incorporated in the limit if we assume L also scales like ϵ^2. Then we can follow physically the same event in the limit $\epsilon \to 0$; M/L, which is the measure of the compactness of the source, remains constant as $\epsilon \to 0$. Therefore, we have to introduce another limit very near the body (body zone limit) as well as the limit we have introduced outside the body. D'Eath[23] has introduced this limit for the study of the interaction of a small black hole with the background metric. He showed that the black hole moves on the geodesic in the background metric. In our case the limit can be defined by requiring the initial density scales like ϵ^{-4} if we take the uniform density distribution initially and also introduce the body coordinate $X^i \equiv \epsilon^{-2} x^i$, which remains constant in the course of limiting process. Thus the integrals we have studied in this paper have additional ϵ dependence. Then one can in principle apply our ϵ-derivative method though the calculation is much more complicated. This study remains for the future.

Another study for the near future is an examination of the outgoing radiation in this sequence of solutions of general relativity. We have succeeded here in deriving radiation reaction without examining this radiation, but we believe general relativity is a conservative theory and that there will be a meaningful sense in which the energy lost locally does turn up far away. A study of the far-field limit of this sequence of solutions should elucidate this problem.

ACKNOWLEDGMENTS

I would like to thank Bernard Schutz for suggesting this problem, for several very important comments in the course of the study, and for his continuous encouragement. I wish to thank Thibaut Damour, John Friedman, Ronald Kates, Bernd Schmidt, Martin Walker, and especially Jürgen Ehlers for helpful comments. Part of this paper was written while I was a visitor at Max-Planck-Institut für Astrophysik, whose support I gratefully acknowledge.

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