RAPID COMMUNICATIONS

PHYSICAL REVIEW D VOLUME 28, NUMBER 9 1 NOVEMBER 1983

U(1) gap equation and consistency condition in quantum chromodynamics

R. Delbourgo
Physics Department, University of Tasmania, Hobart, Australia 7001

M. D. Scadron
Physics Department, University of Arizona, Tucson, Arizona 85721
(Received 26 May 1983)

We unify most aspects of the "resolution" of the U(1) problem in QCD—nonperturbative as well as perturbative—by calculating one second-order gluon quark-annihilation or "diamond" diagram. This leads to a U(1) gap equation and consistency condition which is numerically satisfied, providing one exchanges eight color, spin-one, massless gluons, as required in QCD.

Many learned papers have been written on the *statement* of the "U(1) problem": Glashow[1] long ago noted that (in the absence of gluon anomalies) the SU(2) × SU(2) isoscalar pseudoscalar η_0 mass must vanish in the chiral limit; Weinberg[2] generalized the problem to three quark flavors and computed a (physically unacceptable) upper bound for m_{η_0}; Kogut and Susskind[3] attempted to link the U(1) problem to quark confinement; 't Hooft,[4] Crewther,[5] and Arnowitt and Nath[6] investigated the vacuum in quantum chromodynamics (QCD) and other subtleties associated with the U(1) problem and massless gluons—semiclassical "instantons," "topological charge," etc.

In a parallel vein, there have been many important partial *resolutions* of some of the above facts of the U(1) problem and other seemingly unrelated isoscalar dynamical puzzles: Fritzsch, Gell-Mann, and Leutwyler[7] noted the analogy between the anomaly in spinor electrodynamics[8] involving two-photon emission and the U(1) anomalous axial-vector divergence involving two-gluon emission; Isgur[9] clarified the isoscalar η-η' mixing problem by working in the nonstrange-strange-quark basis; De Rújula, Georgi, and Glashow[10] contrasted C^+ two-gluon η-η' rediagonalization with C^- three-gluon ω-ϕ Okubo mixing; Jones and Scadron[11] and independently Genz[12] then employed the factorized[13] structure of the quark-annihilation diagrams of Ref. 10 to calculate the *one* off-diagonal strength $\beta_P \approx 14.7 m_\pi^2$ by rediagonalization, which thereafter predicts the *two* isoscalar mixing masses m_η^2 and $m_{\eta'}^2$; Witten[14] demonstrated that the U(1) anomalous divergence vanishes for large color number N_c, thus guaranteeing that m_{η_0} vanishes in the chiral $N_c \to \infty$ limit; Patrascioiu and Scadron[15] showed that the dressing equation = pseudoscalar-binding equation condition for spontaneous breakdown of chiral symmetry[16-18] *cannot* hold for the U(1) axial-vector current, again suggesting that m_{η_0} is *not* a Nambu-Goldstone boson (for finite N_c) even though the U(1) anomalous divergence contains a derivative coupling and is therefore "soft"[4]; Novikov *et al.*[19] related this nonperturbative η_0 mass to the nonperturbative gluon condensate; likewise m_{η_0} was approximately computed nonperturbatively by use of QCD sum rules.[20]

In this paper we rely on the insights provided in these papers to compute the *perturbative* two-gluon "diamond" diagram of Fig. 1 which represents the minimum quark-annihilation strength β_P of Refs. 9–13 and Fig. 2(a) on the one hand and also the U(1) anomalous divergence matrix element $\langle 0|\partial A^{(0)}|\eta_0\rangle$ of Refs. 4–7 and 14–17 and Fig. 2(b) on the other hand. The darkened quark propagators in Figs. 1 and 2 correspond to *nonperturbative* dynamically generated[21] quark mass $m_{\rm dyn}$ which is the chiral limit of the nonstrange and strange constituent quark masses $\hat{m}_{\rm con} \approx 340$ MeV, $m_{s,\rm con} \approx 510$ MeV. For the sake of simplicity we shall take[18] $m_{\rm dyn} \approx 315$ MeV as momentum independent because Fig. 1 converges in the ultraviolet and infrared region. Likewise we bind the $q\bar{q}$ quarks together with a (momentum-independent) U(1) pseudoscalar coupling $g_{\eta_0 qq}\bar{q}\gamma_5\lambda_0 q$ with $\lambda_0 = (\frac{2}{3})^{1/2} I$ such that the Witten $N_c \to \infty$ condition[14] ensures the validity of the Goldberger-Treiman relation at the quark level, with

$$g_{\eta_0 qq} = m_{\rm dyn}/f_{\eta_0} \approx 3.5 \qquad (1)$$

for the chiral-limiting value[18,22] $f_{\eta_0} \approx f_\pi \approx 90$ MeV. We expect (1) to hold even for finite $N_c = 3$.

To proceed with the calculation, we work specifically within the context of QCD, with the quark-gluon coupling $g_s \frac{1}{2}\bar{q}\lambda_a B^a_\mu \gamma^\mu q$ for *eight, spin-1, massless* color gluons and take the QCD coupling constant

$$\alpha_s = g_s^2/4\pi \approx 0.5 \quad , \qquad (2)$$

as expected[10] in the η_0 mass region of $q^2 \sim 1\ \mathrm{GeV}^2$. Then the U(1) off-shell two-gluon anomaly of Fig. 3 is

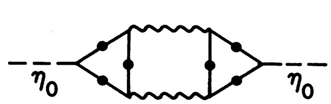

FIG. 1. Quark diamond diagram for isoscalar pseudoscalar $\bar{q}q$ mesons with only two gluons exchanged.

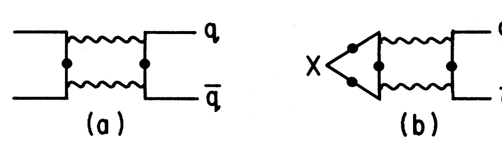

FIG. 2. Quark-annihilation diagram (a) with strengths β_P and axial-vector divergence diagram (b) with strength $\langle 0|\partial \cdot A^{(0)}|\eta_0\rangle$. Both graphs have at least two gluons exchanged.

$$\langle g_\mu^a g_\nu^b | \eta_0(q) \rangle = n_f(\tfrac{2}{3})^{1/2}\left[\mathrm{Tr}\frac{\lambda_a}{2}\frac{\lambda_b}{2}\right] g_{\eta_0 qq} m_{\mathrm{dyn}} \frac{2\alpha_s}{\pi}\int_0^1 \frac{x\,dx}{k^2 x(1-x) - m_{\mathrm{dyn}}^2}\epsilon_{\mu\nu}(kq) \ , \tag{3}$$

where n_f is the quark-flavor number and $\epsilon_{\mu\nu}(kq) = \epsilon_{\mu\nu\alpha\beta}k^\alpha q^\beta$. The structure of (3) recovers the $\pi^0\gamma\gamma$ anomaly[8] $-(\alpha/\pi f_\pi)\epsilon_{\mu\nu}(kq)$ for $k^2 = q^2 = 0$ and $\lambda_0 \to \lambda_3$ in flavor space for color-singlet photons. Combining (3) with the Feynman rules for Fig. 1, we are led to the chiral-limiting quark-annihilation strength [of Figs. 1 or 2(a)]

$$\beta_P = i\int \frac{d^4k}{(2\pi)^4}\frac{\langle \eta_0(q)|g_\mu^a g_\nu^b\rangle\langle g_\mu^a g_\nu^b|\eta_0(q)\rangle}{(k+\frac{1}{2}q)^2(k-\frac{1}{2}q)^2} \ , \tag{4a}$$

$$= \left[\frac{n_f\alpha_s g_{\eta_0 qq} m_{\mathrm{dyn}}}{\pi}\right]^2 \frac{2I}{3\pi^4} \ , \tag{4b}$$

where $\mathrm{Tr}\lambda_a\lambda_b = 2\delta_{ab}$, with $\delta_{aa} = 8$ for eight colored gluons, and $\epsilon_{\mu\nu}(kq)\epsilon^{\nu\mu}(kq) = 2[k^2q^2 - (k\cdot q)^2]$ means that the integral I in (4b) is

$$I = i\int \frac{d^4k[\alpha_s(k^2)/\alpha_s]^2[k^2q^2-(k\cdot q)^2]}{k^2(k^2-4m_{\mathrm{dyn}}^2)(k+\frac{1}{2}q)^2(k-\frac{1}{2}q)^2}\left[\ln\frac{\sqrt{\zeta+4}+\sqrt{\zeta}}{\sqrt{\zeta+4}-\sqrt{\zeta}}\right]^2 \ , \tag{5a}$$

with $\zeta = -k^2/m_{\mathrm{dyn}}^2$. Next we combine the four denominators in (5a) and, noting that the numerator $k^2q^2-(k\cdot q)^2$ is invariant under the change of variables $k \to k + (xyz/2)q$, we extract the q^2 dependence from (5a) to write

$$I = \frac{3}{4}\times 3q^2 i\int_{-1}^1 dx\int_0^1 y\,dy\int_0^1 z^2dz\int \frac{d^4k\,k^2[\alpha_s(k^2)/\alpha_s]^2[\ln\cdots]^2}{[k^2-4m_{\mathrm{dyn}}^2(1-z)+q^2yz(1-x^2yz)/4]^4} \ . \tag{5b}$$

Note that the denominator in (5b) vanishes for a finite positive value of q^2. We invoke color confinement to ignore this mild nonanalytic behavior while extrapolating q^2 to zero *within* the integrals in (5b), as otherwise the gluons could materialize as unconfined particles. Then (5b) takes the form

$$I \approx (q^2/m_{\mathrm{dyn}}^4)(3\pi^2J/4) \ , \tag{6}$$

where the finite dimensionless integral J can be evaluated numerically once we rotate to a Euclidean metric with $d^4k \to i\pi^2 m_{\mathrm{dyn}}^4\zeta\,d\zeta$:

$$J = \int_0^\infty [\alpha_s(\zeta)/\alpha_s]^2\left[\frac{1}{\zeta(\zeta+4)}\right]\left[\ln\frac{\sqrt{\zeta+4}+\sqrt{\zeta}}{\sqrt{\zeta+4}-\sqrt{\zeta}}\right]^2 \tag{7a}$$

$$\approx 2.10 \tag{7b}$$

for $\alpha_s(\zeta)$ taken as constant, independent of $\zeta = -k^2/m_{\mathrm{dyn}} \geq 0$.

In QCD, however, it is well known[23] that α_s varies for $-k^2 \gtrsim 1\ \mathrm{GeV}^2$ according to the asymptotic-freedom dependence $\alpha_s(k^2) = \pi d/\ln(-k^2/\Lambda^2)$. For three quark flavors,[24] $\Lambda_3 \approx 250$ MeV and $d = 12(33-2n_f)^{-1} \approx 0.444$; we recover the expected value (2) at $|k^2| = 1\ \mathrm{GeV}^2$. Substituting this QCD $\alpha_s(k^2)$ into (7a) with $\alpha_s \approx 0.5$ frozen out[25] for $|k^2| \lesssim 1\ \mathrm{GeV}^2$, the numerical estimate for J in (7b) is increased to

$$J_{\mathrm{QCD}} \approx 1.48 \ . \tag{7c}$$

In order that the calculation (4)–(7) provide a physically interesting result, we must first identify the quark-annihilation strength β_P with the *nonperturbative* η_0 mass. Since $|\eta_0\rangle = |\bar{u}u + \bar{d}d + \bar{s}s\rangle/\sqrt{3}$, this link via the meson mass matrix is

$$\beta_P = m_{\eta_0}^2/3 \ , \tag{8}$$

which recovers the expected estimates $m_{\eta_0} \approx 915$ MeV $\sim m_{\eta'}$ for the chiral-broken rediagonalization value[11]

$$\beta_P = \frac{(m_{\eta'}^2 - m_\pi^2)(m_\eta^2 - m_\pi^2)}{4(m_K^2 - m_\pi^2)} \approx 14.66 m_\pi^2 \ . \tag{9}$$

Combining (8) with (4)–(7) for $q^2 = m_{\eta_0}^2$, we note that the *nonperturbative* mass scale $m_{\eta_0}^2$ *cancels out* (because $m_{\eta_0} \neq 0$ and η_0 is *not* a Nambu-Goldstone boson) of this *perturbative* second-order calculation for β_P. This is a characteristic behavior of spontaneous-symmetry-breakdown dynamics as in the calculation of m_{dyn} in a four-fermion model,[16] of the gap energy Δ in superconductivity,[26] or even for the mean-field calculation of the magnetization $\langle M\rangle$ in ferromagnets.

Given this cancellation, our perturbative calculation (4)–(7) provides a *U(1) gap equation and consistency condition* for QCD:

$$1 = \left[\frac{n_f\alpha_s}{\pi^2}g_{\eta_0 qq}\right]^2\left[\frac{3J_{\mathrm{QCD}}}{2}\right] \ . \tag{10}$$

Equation (10) is a gap equation in the spirit of Ref. 16, because the integral (5b) means that (10) could, in principle, be solved for the nonperturbative mass m_{η_0}. Equation (10) is also a consistency condition because it should be numeri-

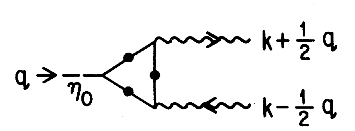

FIG. 3. Triangle diagram for $\eta_0 \to 2$ gluons.

cally approximated even to order $O(\alpha_s^2)$. In particular, substituting $n_f=3$ with (1), (2), and (7c) into (10), we see that (10) is reasonably well satisfied. Put another way, using (10) to solve for $g_{\eta_0 qq}$, we find $g_{\eta_0 qq}\approx 4.4$, which is close to the Goldberger-Treiman value (1).

The U(1) consistency condition (10) pertains to the diamond and quark-annihilation graphs of Figs. 1 and 2(a). For the U(1) axial-vector current of Fig. 2(b), we have instead the identity in the chiral limit

$$\langle 0|\partial\cdot A^{(0)}|\eta_0\rangle = f_{\eta_0} m_{\eta_0}{}^2 \ . \tag{11}$$

Again constructing the state $|\eta_0\rangle$ as the "pinched down" $\bar{q}q$ with pseudoscalar coupling $g_{\eta_0 qq}\bar{q}\lambda_0\gamma_5 q$, the left-hand side of (11) once more is proportional to q^2 as is β_P. For $q^2=m_{\eta_0}{}^2\neq 0$, this nonperturbative η_0 mass also cancels out from (11). Finally invoking the Goldberger-Treiman identity (1) to (11) reproduces the U(1) consistency condition (10) in the chiral limit for the minimum two gluons exchanged. A chiral-broken particle-mixing analysis of (11) can also be carried out.[27]

We conclude from this analysis the following.

(a) Essentially all of the statements and partial dynamical resolutions associated with the U(1) problem are compatible with the U(1) gap equation and consistency condition (10) except the confinement arguments of Ref. 3. To obtain (10) we have employed the standard gluon propagators $\sim k^{-2}$ rather than the *quark-confinement* form k^{-4} with a massless η_0 used in Ref. 3. Instead we evaluate these graphs at the nonperturbative U(1) mass $q^2=m_{\eta_0}{}^2$ in the numerator of (5b) (but with $q^2=0$ in the propagator denominator). This is then consistent with the dressing $\neq$U(1) binding result of Ref. 15 that η_0 is not massless in the chiral limit.

(b) Specifically with reference to perturbative QCD, two exchanged gluons suffice to saturate the diamond or quark-annihilation pseudoscalar-isoscalar graphs of Figs. 1 and 2; additional gluons serve to build up $m_{\rm dyn}$ in the quark loops and $\alpha_s(k^2)$ or give small $O(\alpha_s^3)$ corrections to β_P and to (10).

(c) The U(1) consistency condition offers one of the few physical examples where the QCD gluon color number (eight), spin value (one), and mass (zero) can be meaningfully tested.

[1]S. L. Glashow, in *Hadrons and Their Interactions* (Academic, New York, 1968), p. 83; also see S. L. Glashow, R. Jackiw, and S. S. Shei, Phys. Rev. 187, 1916 (1969); M. Gell-Mann, in *Proceedings of the Third Topical Conference on Particle Physics, Honolulu, 1969*, edited by W. A. Simmons and S. F. Tuan (Western Periodicals, Los Angeles, 1970), p. 1.

[2]S. Weinberg, Phys. Rev. D 11, 3583 (1975).

[3]J. Kogut and L. Susskind, Phys. Rev. D 10, 3468 (1974); 10, 3594 (1975).

[4]G. 't Hooft, Nucl. Phys. B72, 461 (1974).

[5]R. J. Crewther, Phys. Lett. 70B, 349 (1977); Nuovo Cimento 2, 63 (1979).

[6]R. Arnowitt and P. Nath, Phys. Rev. D 23, 473 (1981).

[7]H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973); also see R. Mohapatra and J. Pati, Phys. Rev. D 8, 4212 (1973).

[8]S. L. Adler, Phys. Rev. 177, 2426 (1969); J. S.Bell and R. Jackiw, Nuovo Cimento 60, 47 (1969).

[9]N. Isgur, Phys. Rev. D 12, 3770 (1975).

[10]A. De Rújula, H. Georgi, and S. Glashow, Phys. Rev. D 12, 147 (1975).

[11]H. F. Jones and M. D. Scadron, Nucl. Phys. B155, 409 (1979); also see M. D. Scadron, Phys. Rev. D 26, 239 (1982).

[12]H. Genz, Acta Phys. Austriaca Suppl. 21, 559 (1979).

[13]H. Fritzsch and J. D. Jackson, Phys. Lett. 66B, 365 (1977).

[14]E. Witten, Nucl. Phys. B156, 269 (1979).

[15]A. N. Patrascioiu and M. D. Scadron, Phys. Rev. D 22, 2054 (1980).

[16]Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).

[17]R. Delbourgo and M. D. Scadron, J. Phys. G 5, 162 (1979); Phys. Rev. Lett. 48, 379 (1982).

[18]M. D. Scadron, Rep. Prog. Phys. 44, 213 (1981).

[19]V. A. Novikov *et al.*, Phys. Lett. 86B, 347 (1979).

[20]N. K. Pak, Phys. Lett. 109B, 397 (1982); S. Narison, IAPP Report No. TH-63, 1982 (unpublished).

[21]H. D. Politzer, Nucl. Phys. B117, 397 (1976); H. Pagels, Phys. Rev. D 19, 3080 (1979).

[22]P. Langacker and H. Pagels, Phys. Rev. D 8, 4595 (1973).

[23]D. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, *ibid.* 30, 1346 (1973).

[24]R. Miller and B. McKellar, J. Phys. G 7, L247 (1981); Los Alamos Report No. LA-UR-82-2145 (unpublished).

[25]M. Creutz, Phys. Rev. Lett. 45, 313 (1980); Phys. Rev. 21, 2308 (1980).

[26]M. D. Scadron, Ann. Phys. (N.Y.) 148, 257 (1983).

[27]N. H. Fuchs, Phys. Lett. 97B, 151 (1980).