PHYSICAL REVIEW D VOLUME 28, NUMBER 9 1 NOVEMBER 1983

Spectra of τ-leptonic atoms

George L. Strobel
Physics Department, University of Georgia, Athens, Georgia 30602

Edward L. Wills
Physics Department, University of Alabama at Birmingham, Birmingham, Alabama 35294
(Received 9 May 1983)

The low-lying atomic spectra are calculated nonrelativistically for a τ lepton interacting Coulombically with various finite-sized nuclei. The proton is taken as a point charge in these calculations. The nonspherical charge distribution of the deuteron nucleus is included; its effects are small but not negligible. The magnetic-moment interaction is calculated in perturbation theory and is negligible. The transition lifetimes are small compared to the τ-lepton lifetime. An experimental measurement of the γ-ray energies emitted by the τ-leptonic atom while cascading to the ground state would lead to an independent measure for the τ-lepton mass.

INTRODUCTION

τ-leptonic atoms consist of a nucleus and a τ lepton. The τ lepton is here assumed to have a mass of twice the proton mass, consistent with the experimental value,[1] and we calculate the atomic levels of this system treating the τ lepton as stable. Its experimental lifetime is 4.6×10^{-13} sec.[2] The Bohr radius of such a system is 28 fm or smaller, depending on the nucleus, so we ignore the presence of any electrons. τ-leptonic hydrogen is treated as though the proton were a point charge.

τ-leptonic deuterium is an intrinsically interesting three-body system to study. The τ is supposed to be just like an electron, but heavy, with a mass of 1.8 GeV, and also unstable, with a lifetime of 4.6×10^{-13} sec. The three-body (bound) system of proton, neutron, and τ should be well described nonrelativistically. Further, to some high degree of approximation, the neutron and the τ do not interact with each other. The assumed Coulombic interaction betweeen the τ and the proton and the presumed known neutron-proton force result in a possibly tractable three-body problem. However, here we neglect the presence of the τ lepton in determining the neutron-proton relative wave function, and instead determine the τ-lepton atomic wave function with a given charge distribution for the nucleus. This approximation is least accurate for the ground state of the system, where the τ is closest to the nucleus, but is still a reasonable approximation for that case. For other than s waves, the spectrum calculated is almost identical to that of a point nucleus charge. The nonspherical deuteron charge distribution splits the $2p_{3/2}$ states from the $2p_{1/2}$ state. This effect is larger than the relativistic point-charge splitting of the Dirac equation for the $2p_{3/2}$ from the $2p_{1/2}$ by three orders of magnitude. For s waves, the bound states have less binding energy than for a point nuclear charge, due to the assumed nuclear charge distribution. The Bohr radius of the τ deuteron system is 28.8 fm, which is about 10 times the nuclear radius.[3] The nuclear binding energy of 2.226 MeV is about 100 times larger than the atomic binding energy of order 20 keV. A comparison of the order of magnitude of these sizes and energies tends to justify the neglect of the τ lepton on the nuclear charge distribution.

For the $A\geq 3$ nuclei, the nuclear charge distribution is assumed to be a uniform sphere of radius R, where R was determined from the rms radius from electron scattering.[4] Of the nuclei considered, only ^{6}Li has a spin as large as one $\hbar$, and as a consequence, a possible quadrupole moment, but its experimental quadrupole moment[5] is only -0.1 fm^2, which we neglect. The largest nuclei considered is ^{24}Mg as the Bohr radius is exceeded by the size of this nucleus.

It is the purpose of this paper to ascertain the γ-ray energies that would result from a τ lepton cascading down the hydrogenic levels in a τ-leptonic atom. Estimates of the lifetime for these transitions are provided as well. The slowing-down time must be small compared to the τ-lepton lifetime if the transitions to the ground state can be expected to be detected experimentally.

The slowing-down time has been calculated by Fermi and Teller[6] for muons in carbon or iron, and by Wightman[7] for muons in liquid hydrogen. The experimental problems of slowing down and capturing muons have been discussed by Devons and Duerdoth.[8] The slowing-down time for muons was calculated as 10^{-13} to 10^{-9} sec, depending on the initial muon energy assumed or moderator used, but in both cases,[6,7] the slowing-down time was small compared to the muon lifetime of about 2×10^{-6} sec. Both the Wightman and the Fermi-Teller calculations can be scaled via the mass to apply to the slowing down of τ leptons. The slowing-down time for a 20-MeV τ lepton in liquid hydrogen is found to be on the order of 2×10^{-10} sec, which is long compared to the τ lifetime. Thus one must use τ leptons produced nearly at rest in the laboratory frame and use a moderator of larger Z. The τ slowing-down time from 20 keV until cascading into the ground state is estimated as 9×10^{-13} sec for a carbon moderator. This slowing-down time is still twice the τ lifetime. Thus the $2P$-to-$1S$ γ yield would be reduced by a factor of order 10 compared to the number of τ leptons produced.

THEORY

For point charges the Dirac-equation energies are shown[9] in Table I. In Table I, m is the reduced mass of the atom, and is equal to $2m_p/3$ for a proton nucleus. For a proton nucleus, the $2P_{3/2}$-$2P_{1/2}$ energy splitting is 0.0554 eV. For a deuteron nucleus, this relativistic splitting is 0.0831 eV, for a point nuclear charge distribution.

For the deuteron nucleus, we treat the neutron and proton masses as equal. We introduce relative coordinates r for the neutron-proton separation and x, and τ-lepton—deuteron-center-of-mass relative separation (see Fig. 1). We take the neutron and proton masses as equal. Then the τ-lepton—proton separation y can be written as

$$y=|\vec{x}+\vec{r}/2| \ . \tag{1}$$

The nonrelativistic Schrödinger equation for the system can be written as

$$\left[\frac{-\hbar^2}{2M_r}\nabla_x{}^2-\frac{\hbar^2}{m_p}\nabla_r{}^2-\frac{e^2}{y}+V(\vec{r})+B\right]\Psi(\vec{x},\vec{y})=0 \ , \tag{2}$$

where B is the positive binding energy for the three-body system, M_r is the reduced mass of the nucleus and the τ lepton, and m_p is the proton mass. Neglecting the effect of the τ on the nuclear charge distribution, we can write the wave function Ψ as

$$\Psi(\vec{x},\vec{y})\cong\sum_{mM_d}\psi_1^{M_d}(\vec{r})\frac{\chi_{lj}(x)}{x}\mathscr{Y}^m_{l(1/2)j}(\hat{x})\begin{bmatrix}F & M_F\\ 1 & M_d\\ j & m\end{bmatrix} \ , \tag{3}$$

where

$$\psi_1^{M_d}(\vec{r})=\sum_{l_d=0}^{2}\phi_{l_d}(r)\mathscr{Y}^{M_d}_{l_d11}(\hat{r}) \tag{4}$$

and

$$\left[\frac{-\hbar^2}{m_p}\nabla_r{}^2+V(\vec{r})+B_0\right]\psi_1^{M_d}(\vec{r})=0 \ . \tag{5}$$

In Eq. (3), the quantity in square brackets is a Clebsch-Gordan coefficient and the $\mathscr{Y}$ is an angular momentum eigenfunction with z component m. M_d is the z component of angular momentum for the deuteron nuclear wave function. The deuteron spin 1 is coupled to the lepton angular momenta j to form the atomic angular momenta F.

The sum over l_d in Eq. (4) is necessary as the neutron-proton potential $V(r)$ is not spherically symmetric. The radial eigenfunctions for each orbital state ($l_d=0$ and $l_d=2$) are labeled $\phi_{l_d}(r)$. The nuclear binding energy is $B_0=2.226$ MeV.

We substitute Eq. (3) into Eq. (2), and use Eq. (5) to eliminate the nuclear potential and the nuclear Laplacian in favor of the nuclear binding energy B_0. Multiplying by $\psi_1^{*\bar{M}}$ and integrating we have

$$0=\int d\vec{r}\,\psi_1^{*\bar{M}}(\vec{r})\left[B-B_0-\frac{\hbar^2}{2M_r}\nabla_x^2-\frac{Ze^2}{y}\right]\times\sum_{Mm}\psi_1^M(\vec{r})\frac{\chi_{lj}(x)}{x}\mathscr{Y}^m_{l(1/2)j}(\hat{x})\begin{bmatrix}F & M_F\\ 1 & M\\ j & m\end{bmatrix} \ . \tag{6}$$

Using Eq. (1) in Eq. (6), and expanding in spherical harmonics, we obtain the following equation for $\chi_{lj}(x)$:

$$0=\left[B-B_0-\frac{\hbar^2}{2M_r}\left[\frac{d^2}{dx^2}-\frac{l(l+1)}{x^2}\right]-Ze^2\langle V\rangle\right]\chi_{lj}(x) \ , \tag{7}$$

where

$$\langle V\rangle=\sum_{\lambda l_d\bar{l}_d m\text{'s}}\begin{bmatrix}1 & \bar{M}_d\\ 1 & m_2\\ \bar{l}_d & m_{\bar{l}_d}\end{bmatrix}\begin{bmatrix}1 & M_d\\ 1 & m_2\\ l_d & m_{l_d}\end{bmatrix}\begin{bmatrix}\bar{l}_d & 0\\ l_d & 0\\ \lambda & 0\end{bmatrix}\begin{bmatrix}\bar{l}_d & m_{\bar{l}_d}\\ l_d & m_{l_d}\\ \lambda & \lambda\end{bmatrix}\begin{bmatrix}l & 0\\ l & 0\\ \lambda & 0\end{bmatrix}\begin{bmatrix}l & m_l\\ l & \bar{m}_l\\ \lambda & m_\lambda\end{bmatrix}$$
$$\times\begin{bmatrix}j & \bar{m}\\ \frac{1}{2} & m_s\\ l & \bar{m}_l\end{bmatrix}\begin{bmatrix}j & m\\ \frac{1}{2} & m_s\\ l & m_l\end{bmatrix}\begin{bmatrix}F & M_F\\ j & \bar{m}\\ 1 & \bar{M}_d\end{bmatrix}\begin{bmatrix}F & M_F\\ j & m\\ 1 & M_d\end{bmatrix}\left[\frac{2l_d+1}{2\bar{l}_d+1}\right]^{1/2}\int_0^\infty r^2dr\,\phi_{\bar{l}_d}(r)\phi_{l_d}(r)\frac{r_<{}^{\lambda}}{r_>{}^{\lambda+1}} \ . \tag{8}$$

Here $r_<$ ($r_>$) is the lesser (greater) of x and $r/2$, coming from the expansion of $1/y$. For the case $l=0$, $\langle V\rangle$ simplifies to

$$\langle V\rangle=\sum_{l_d=0}^{2}\int_0^\infty\frac{r^2dr\,\phi_{l_d}{}^2(r)}{r_>} \ . \tag{9}$$

In Eq. (7), the difference $B-B_0$ is the atomic binding energy. This atomic binding energy varies for different states because Eq. (8) for the potential $\langle V\rangle$ depends on F, l, and j, in addition to the angular momentum barrier appearing in Eq. (7) of $l(l+1)\hbar^2/2Mx^2$. M_r is the reduced mass of the τ-lepton—nucleus system in Eq. (7). We assume an analytic Hulthén form for the deuteron wave function

$$\phi_0=N\cos w\,(e^{-\alpha r}-e^{-\beta r})/r \ ,$$
$$\phi_2=N\sin w\,(e^{-\alpha r}-e^{-\beta r})/r \ , \tag{10}$$

TABLE I. Point Coulombic energy levels.

Level	Dirac energy	Nonrelativistic binding energy
$1S_{1/2}$	$m(1-Z^2\alpha^2)^{1/2}$	$mZ^2\alpha^2/2$
$2S_{1/2}$	$m\left[\dfrac{1+(1-Z^2\alpha^2)^{1/2}}{2}\right]^{1/2}$	$mZ^2\alpha^2(\frac{1}{8}+5Z^2\alpha^2/128)$
$2P_{1/2}$	$m\left[\dfrac{1+(1-Z^2\alpha^2)^{1/2}}{2}\right]^{1/2}$	$mZ^2\alpha^2(\frac{1}{8}+5Z^2\alpha^2/128)$
$2P_{3/2}$	$\dfrac{m}{2}(4-Z^2\alpha^2)^{1/2}$	$mZ^2\alpha^2(\frac{1}{8}+Z^2\alpha^2/128)$

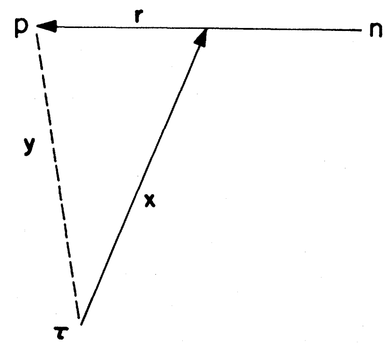

FIG. 1. Geometry for τ-leptonic deuterium. The nuclear charge distribution is somewhat cigar shaped.

where α reproduces the deuteron binding energy, $\beta=7\alpha$, and $\sin w=0.15$. These parameters will reproduce the deuteron quadrupole moment, and the analytic wave function chosen permits the integrals in Eq. (9) to be evaluated analytically.

The summation over m values in Eq. (8) is over all values save M_F, where the limits of the m sums are from the respective minus L value to the plus L value. The summations over m in Eq. (8) can be reduced to a sum over 6-j symbols; the general result is not given here. The quantities in the square brackets in Eq. (8) are Clebsch-Gordan coefficients.

For nuclei with $A \geq 3$, we take

$$V=-Ze^2/r \text{ if } r>R \text{ ,}$$

and (11)

$$V=\frac{-Ze^2}{2R}(3-r^2/R^2) \text{ if } r<R \text{ .}$$

The parameter R is given in Table II for the various nuclei considered. We ignore any possible nonspherical charge distributions, and solve Eq. (7) for various l values. The difference $B-B_0$ is the atomic binding energy, as before. It is always much smaller, 1% or less, than the nuclear binding energy, tending to justify the approximation of the τ-lepton presence having no effect on the nuclear charge distributions.

RESULTS

The spectra have been calculated for the levels corresponding to the hydrogenic $1S,2P,3D$ levels (see Table II). For comparison, the levels for a point nucleus charge are shown. Only the S levels are noticeably shifted upwards in energy compared to the point nucleus results. For carbon and magnesium, the $2P$ levels are also shifted upwards from that of a point charge. The Bohr radius scales as n^2/mZ^2, so even the $l=1$ levels have Bohr radii comparable to the nuclear size for the carbon and magnesium nuclei. The $3D$ energies are all nearly the same as if the nucleus were a point charge.

The effect of the nonspherical charge distribution of the deuteron nucleus is shown in Fig. 2. Because of the deuteron quadrupole moment, the spectra associated with the $2P_{3/2}$ leptonic state splits, depending on $F=\vec{j}+\vec{l}$. Energy splittings of 56, 14, and 70 eV from the $2P_{1/2}$ state are seen for the $F=\frac{3}{2}$, $\frac{5}{2}$, and $\frac{1}{2}$ states, respectively.

Including only the Coulomb interactions, all the $j=\frac{1}{2}$ levels are degenerate since $F=\frac{1}{2}$ or $\frac{3}{2}$ both have the same energy. Including the fine structure from the magnetic field of the deuteron interacting with the assumed Dirac magnetic moment of the τ lepton removes this degeneracy. The $F=\frac{1}{2}$ state is lowest, shifted downward 0.669 eV in a perturbative calculation,[10] while the $F=\frac{3}{2}$ level is shifted upwards by 0.515 eV. We now discuss transition lifetimes.

Using classical field theory, the probability[11] for spontaneous emission due to the dipole emission is

$$P=4e^2W^3\langle r\rangle^2/3\hbar c^3 \text{ ,} \tag{12}$$

where $\langle r\rangle$ is the transition dipole moment and W is the transition energy. For the $2P$-to-$1S$ transition this leads to

TABLE II. τ-leptonic energy levels for various nuclei.

Nucleus	Point-charge results E_{1S}	Point-charge results E_{2P}	Nuclear charge radius	Extended-charge results E_{1S}	Extended-charge results $E_{2P_{1/2}}$	Extended-charge results E_{3D}
$^1_1$H	16.646	4.161	0	16.646	4.161	1.749
$^2_1$H	24.969	6.242		22.592	6.238	2.774
$^4_2$He	133.168	33.292	2.08	122.051	33.300	14.796
$^6_3$Li	337.082	84.270	3.59	279.915	83.974	37.453
$^7_3$Li	349.566	87.391	3.50	290.223	87.103	38.840
$^9_4$Be	653.733	163.433	3.92	495.604	162.114	72.637
$^{12}_6$C	1540.944	385.236	3.04	1107.807	382.447	171.216
$^{24}_{12}$Mg	6637.914	1659.478	3.84	3089.414	1534.069	737.

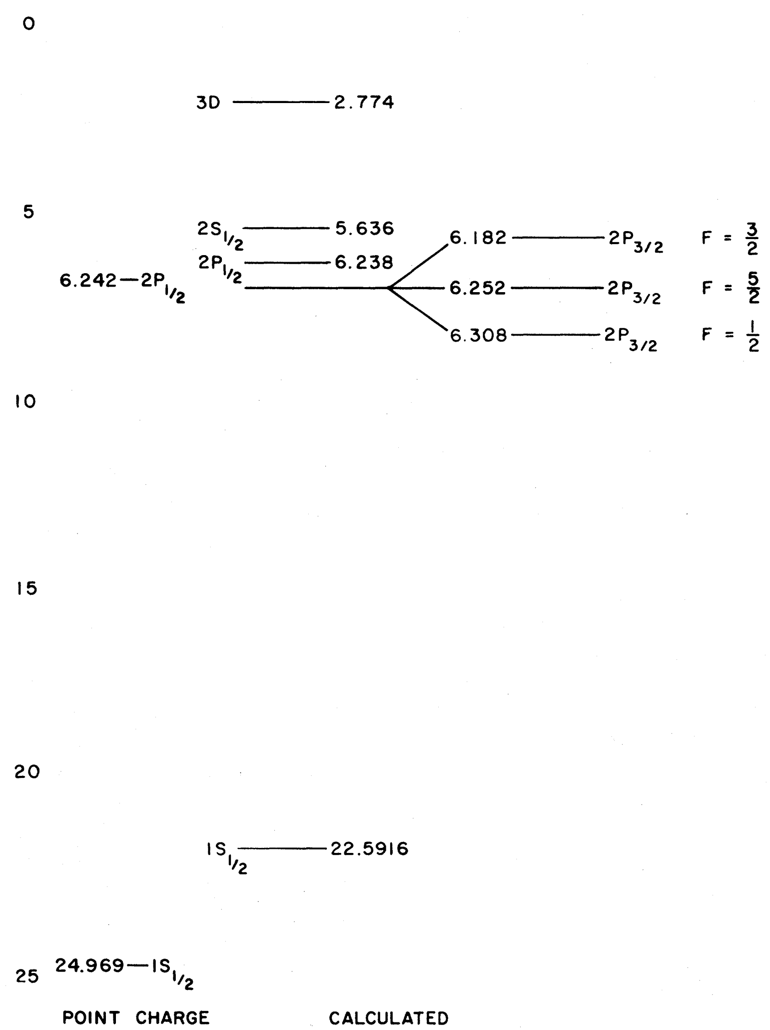

FIG. 2. τ-leptonic deuterium spectra in keV.

a transition lifetime of 4.8×10^{-14} sec in deuterium. Assuming the wave functions are Coulombic bound-state wave functions, the dipole transition matrix element is proportional to $1/MZ$, where M is the reduced mass and Z is the nuclear charge. The energy of the transition varies as MZ^2 so the probability of the transition due to spontaneous emission varies as MZ^4. This dependence certainly breaks down when the Bohr radius of the atom approaches the nuclear radius. This happens for a nuclear atomic weight of about 9 or 10 for atomic ground states.

TABLE III. τ-leptonic atom transition energies and lifetimes.

Nucleus	$E_{2P\text{-}1S}$ (keV)	$E_{2D\text{-}2P}$ (keV)	$t_{2P\text{-}1S}$ (sec)
${}^1_1$H	12.485	2.412	5.05×10^{-14}
${}^2_1$H	16.354	3.464	4.80×10^{-14}
${}^4_2$He	88.751	18.504	2.15×10^{-15}
${}^6_3$Li	195.941	46.521	5.69×10^{-16}
${}^7_3$Li	203.120	48.263	5.49×10^{-16}
${}^9_4$Be	333.490	89.477	2.40×10^{-16}
${}^{12}_6$C	725.360	211.2	5.86×10^{-17}
${}^{24}_{12}$Mg	1555.345	796.0	2.75×10^{-17}

In Table III is shown the energy for the 2P-to-1S and for the 3D-to-2P transition, as well as the calculated transition lifetime for dipole radiation. These lifetimes are all small compared to the 4.6×10^{-13} sec measured[2] for the τ-lepton lifetime. Providing the τ could be captured directly into a low-lying atomic orbit, the measurement of the γ energy emitted as the orbit decays to the 1s ground state will provide an independent measure of the τ-lepton mass. The transition energies are proportional to the reduced mass of the τ lepton and the nucleus for a point charge distribution. The above transition energies are based on a τ-lepton mass of twice the proton mass and on extended charge distributions. The slowing-down times for τ leptons, as well as the cascade time must be less than or comparable to the τ lifetime before experimental detection of the transition γ's would be feasible.

ACKNOWLEDGMENT

One of the authors (G.L.S.) expresses his gratitude for the hospitality shown him at the Livermore National Laboratory while completing this paper.

[1]M. L. Perl *et al.*, Phys. Lett. 70B, 487 (1977).
[2]G. J. Feldman, Phys. Rev. Lett. 48, 66 (1982).
[3]R. R. Roy and B. P. Nigam, *Nuclear Physics* (Wiley, New York, 1967).
[4]M. A. Preston, *Physics of the Nucleus* (Addison-Wesley, Reading, Mass., 1962).
[5]Harold A. Enge, *Nuclear Physics* (Addison-Wesley, Reading, Mass., 1966).
[6]E. Fermi and E. Teller, Phys. Rev. 72, 399 (1947).
[7]A. S. Wightman, Phys. Rev. 77, 521 (1949).
[8]S. Devons and I. Duerdoth, Adv. Nucl. Phys. 2, 295 (1969).
[9]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 56.
[10]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (Ref. 9), p. 58.
[11]L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955), p. 261.